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Semisimple Games and Generalized Power Indices

by

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SEMISIMPLE GAMES AND GENERALIZED POWER INDICES

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I. Introduction

A variety of indices have been developed for measuring power on the class $\mathcal{C}$ of simple n-person games. While it seems clear that no one index will serve to model power effectively over all of $\mathcal{C}$, various indices have been important for their practical and theoretical contributions. However, the fact that simple games cannot take individual indifference or abstention into account limits both the scope of such games and our ability to define power in many common game-type situations.

In this paper we utilize the ideas and notation from [1] to define a more general class of games $\mathcal{S}$, the semisimple n-person games. Using this extension, and any given index of individual power on the simple games $\mathcal{C}$, we obtain a generalized and richer notion of a player's power on $\mathcal{S}$ in the form of a vector (with $2^n - 1$ components). If probabilities of abstention are known for each player, we can recapture a single numerical value, which agrees with the given index of power on $\mathcal{C}$ when abstention has probability zero.

Returning to the domain of simple games, we make use of a definition of collective power first introduced by Coleman [2]. We refer to this as the enactment power associated with a simple game and we present a simple set of axioms which characterize our measure of enactment power. The veto power of a game can then be defined as the enactment power of its dual, from which it follows that these two competing notions of power sum to unity. These measures of power enable us to compare, for example, the power of absolute majority rule and two-thirds rule for varying sized voting bodies.

By combining the ideas described in the above two paragraphs, we then obtain for any given power index on $\mathcal{C}$ a nonnormalized index for players in a semisimple game. This generalized index enables us to compare individual power in a wide variety of games as well as to compare the power of different semisimple games. Finally, we discuss the extent to which the generalized power indices enable us to "resolve" some of the well-known paradoxes associated with standard power indices.

II. Semisimple Games and Generalized Individual Power

Let N be a set of players with $n = |N|$. A simple game is a collection Γ_N of subsets of N such that $\phi \notin \Gamma_N$. We say that Γ_N is a monotone simple game if, in addition,

$$M: A \in \Gamma_N \text{ and } A \subseteq B \Rightarrow B \in \Gamma_N.$$

We let $\mathcal{C}$ denote the class of monotone simple games. Given $\Gamma_N \in \mathcal{C}$, the subsets in Γ_N are the winning coalitions. To allow flexibility for later developments (namely, the dual game) we do not insist that games in $\mathcal{C}$ be proper $(A \in \Gamma_N \Rightarrow N \setminus A \notin \Gamma_N)$

To motivate what follows consider the absolute majority rule game on N defined by

$$\Gamma_n = \{S \subseteq N \mid |S| \geq [\tfrac{n+2}{2}]\},$$

where $[x]$ denotes the largest integer which does not exceed x. In contrast to Γ_n, simple majority rule Σ_n allows a subset S of "yes" voters to do its will if and only if $|S| > |T|$, where T is the set of "no" voters with $N \setminus (S \cup T)$ being the abstainers. Note that Σ_n has too much structure to belong to $\mathcal{C}$ despite its elementary and ubiquitous nature. To deal with situations where indifference or abstention may be present, we utilize an idea introduced in [1]. Similar ideas, with different notation, appear in [3] and [4].

We define a monotone semisimple game Γ on N as an indexed family $\{\Gamma_J\}_{J \in 2^N \setminus \{\phi\}}$ of simple games Γ_J on J such that

M1: $I \in \Gamma_K$ and $I \subseteq J \subseteq K \Rightarrow I \in \Gamma_J$.

M2: $I \in \Gamma_J$ and $I \subseteq J \subseteq K \Rightarrow I \cup (K \setminus J) \in \Gamma_K$.

The idea behind this definition is that for a set N of players, each of whom is capable of abstaining, we must specify the winning coalitions for each set J of nonabstaining voters. Conditions M1 and M2 are the two natural monotonicity requirements corresponding to

1) a fixed set of "yes" voters with increased abstention; and

2) an increased set of "yes" voters taken from the set of abstainers.

We let $\mathcal{J}$ denote the class of monotone semisimple games.

It is easy to see that any $\Gamma_N \in \mathcal{C}$ can be expressed as a member of $\mathcal{J}$ by identifying Γ_N with $\{\Gamma_K\}$, where $\Gamma_K = \Gamma_N \cap 2^K$. The simple majority rule game Σ_n (which is not a simple game) belongs to $\mathcal{J}$ since $\Sigma_n = \{\Gamma_K\}$ where $\Gamma_K = \Gamma_{|K|}$, the absolute majority rule game with $|K|$ players. The class of (monotone) semisimple games thus provides a natural means of describing a variety of common voting rules and game situations where abstention or indifference is a meaningful option.

We assume in what follows, unless otherwise noted, the existence of a fixed universal player set N for every simple game. This assumption is plausible since we may always adjoin dummy players to a smaller game. Formally, a player i is a dummy in a game $\Gamma_N \in \mathcal{C}$ if $A \in \Gamma_N \Rightarrow A \setminus \{i\} \in \Gamma_N$. This definition extends to $\mathcal{J}$ in obvious fashion by calling i a dummy in $\Gamma = \{\Gamma_K\}$ if i is a dummy in Γ_K for all $K \in 2^N \setminus \{\phi\}$ for which $i \in K$.

A power index on $\mathcal{C}$ is a function $\rho: \mathcal{C} \to \mathbb{R}^n$ which assigns to each $\Gamma_N \in \mathcal{C}$ an n-tuple $\rho(\Gamma_N)$ of individual "powers." Thus $\rho_i(\Gamma_N)$ measures, in some fashion, the power of player i in the game Γ_N. While a variety of such indices have been proposed, we focus in this paper on those having an axiomatic characterization of a certain common form. Specifically, we recognize indices which are uniquely determined by a set of axioms of the following type:

i) If i is a dummy in Γ_N, then $\rho_i(\Gamma_N) = 0$.

ii) If i and j are "symmetric" players in Γ_N, then $\rho_i(\Gamma_N) = \rho_j(\Gamma_N)$.

iii) ρ is normalized in some systematic fashion (usually $\sum_{i \in N} \rho_i(\Gamma_N) = 1$).

iv) $\rho(\Gamma_1 \cup \Gamma_2)$, $\rho(\Gamma_1 \cap \Gamma_2)$ and $\rho(\Gamma_1) + \rho(\Gamma_2)$ are related in some "interesting" fashion.

The well-known power indices of Shapley and Shubik [5] and Banzhaf [6] satisfy the above conditions (see Dubey [7] for axiomatic developments). Deegan and Packel [8] develop another such index and a generalization [9] that provides a broad family of axiomatically based power indices.

Let $\rho: \mathcal{C} \to \mathbb{R}^n$ be any power index of the form described above and normalized to unity. We can "extend" ρ to $\mathcal{S}$ by defining, for $\Gamma = \{\Gamma_K\} \in \mathcal{S}$, $\rho(\Gamma) = \{\rho(\Gamma_K)\}_{K \in 2^N \setminus \{\phi\}}$. If we think of a player $i \notin K$ as having $\rho_i(\Gamma_K) = 0$ and if we enumerate the $2^n - 1$ nonempty subsets of N in some systematic and constant fashion, we see that the <u>generalized power</u> of a player i in a semisimple game is a $2^n - 1$ tuple which we denote by $\rho_i(\Gamma)$. In the next section we develop a notion of the overall power of a game. This will lead to a modification of an index ρ on $\mathcal{S}$. We postpone specific examples until these ideas have been presented.

III. The Power of a Game and its Dual

It is not uncommon to regard certain voting bodies or game situations as ineffectual in the sense that their inherent structuremakes changes in the status quo very difficult. We now use an idea introduced by Coleman [2] to define what we call the enactment power of a simple game. By using the notion of the dual of a game, we can then obtain a complementary blocking power for a game. At the individual player level, distinctions between "power to act" and "power to block" have been made in [2] and [9].

Consider any $\Gamma \in \mathcal{C}$. If we assume that each of the 2^n possible coalitions of "yes" voters is equally likely to form, then a motion to alter the status quo will pass with probability $|\Gamma| / 2^n$. As with individual power, there are clearly other plausible formulations. Nevertheless, we find this approach to be pleasing in its simplicity, the axiomatization it gives rise to, and the specific results it provides.

Given $\Gamma \in \mathcal{C}$, the enactment power of Γ is defined by $p(\Gamma) = |\Gamma| / 2^n$. If, abandoning the assumption of a 'fixed universe' N temporarily, the game Γ has r players, then we would define $p(\Gamma) = |\Gamma| / 2^r$. Let $\Gamma \setminus i$ denote the simple game on $N \setminus \{i\}$ defined by $A \in \Gamma \setminus i \Leftrightarrow A \in \Gamma \cap 2^{N \setminus \{i\}}$. The first axiom requires that the power of a game not depend on the presence or absence of dummy players.

A 1 : i a dummy in $\Gamma \Rightarrow p(\Gamma \setminus i) = p(\Gamma)$.

The second axiom fixes a power value for the games which require unanimous support of their nondummy players to "carry" a motion.

A 2 : Γ a unanimity game with r players $\Rightarrow p(\Gamma) = 1 / 2^r$

The final axiom describes how power in a combined game depends upon power in the constituent games. It turns out to be identical with one of the characterizing axioms for both the Shapley-Shubik and the Banzhaf power indices (see [7]).

$$A3 : p(\Gamma_1 \cup \Gamma_2) = p(\Gamma_1) + p(\Gamma_2) - p(\Gamma_1 \cap \Gamma_2).$$

Theorem 1 The unique function $p : \mathcal{C} \to R$ satisfying A1, A2, and A3 is the enactment power $p(\Gamma) = |\Gamma| / 2^n$.

Proof. It is easily checked that the function $p(\Gamma) = |\Gamma| / 2^n$ satisfies A1, A2, and A3. Indeed, A2 is immediate, A1 follows since adding a dummy player to a game doubles the number of winning coalitions, and A3 results from the set theoretic identity $|S \cup T| = |S| + |T| - |S \cap T|$. Conversely, let p be any function satisfying A1, A2, and A3. The set theoretic identity mentioned above can be extended inductively as follows. Let $Q(k,m)$ denote the set of increasing sequences of length k chosen from the first m positive integers. Then

$$(1) \quad \left|\bigcup_{i=1}^{m} S_i\right| = \sum_{k=1}^{m} (-1)^{k+1} \sum_{\sigma \in Q(k,m)} \left|\bigcap_{j \in \sigma} S_j\right| .$$

By identical reasoning, A3 extends to

$$(2) \quad p\left(\bigcup_{i=1}^{m} \Gamma_i\right) = \sum_{k=1}^{m} (-1)^{k+1} \sum_{\sigma \in Q(k,m)} p\left(\bigcap_{j \in \sigma} \Gamma_j\right)$$

Given any $\Gamma \in \mathcal{C}$ with $\Gamma = \{S_i\}_{i=1}^{m}$ we can write $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$, where Γ_i is the game generated by (consisting of all supersets of) S_i . In this case $\bigcap_{j \in \sigma} \Gamma_j$ is the game generated by $S_\sigma = \bigcup_{j \in \sigma} S_j$, and can be viewed using A 1 as a unanimity game with $|S_\sigma|$ players. By A 2 and (2) it follows that $p(\Gamma)$ has a well-determined value and that this value is

$$(3) \quad p(\Gamma) = 1/2^n \left[\sum_{k=1}^{m} (-1)^{k+1} \sum_{\sigma \in Q(k,m)} 2^{n-|S_\sigma|} \right] .$$

This establishes that at most one function $p: \mathcal{C} \to \mathbb{R}$ can satisfy A 1 , A 2 , and A 3 , and the fact that $p(\Gamma) = |\Gamma| / 2^n$ is such a function completes the proof. For insurance we note that the bracketed quantity on the right hand side of (3) is indeed $|\Gamma|$ by a careful application of (1) to $\bigcup_{i=1}^{m} \Gamma_i$.

Q.E.D.

The dual Γ^* of a simple game Γ in $\mathcal{C}$ is defined by

$$\Gamma^* = \{S \in 2^N \setminus \{\emptyset\} \mid N \setminus S \notin \Gamma\} .$$

Thus, the winning coalitions in Γ^* are those which can "block" the formation of winning coalitions in Γ . We define the blocking power p^* on $\mathcal{C}$ by $p^*(\Gamma) = p(\Gamma^*)$.

Theorem 2. $\forall$ non-empty $\Gamma \in \mathcal{C}$, $p(\Gamma) + p^*(\Gamma) = 1$.

Proof For all $S \subseteq N$, either $S \in \Gamma$ or $N \setminus S \in \Gamma^*$, but not both. Thus $|\Gamma| + |\Gamma^*| = 2^n$. It follows directly that $p(\Gamma) + p^*(\Gamma) = 1$.

Q.E.D.

If we define $\Gamma \in \mathcal{C}$ to be <u>strong</u> if $S \notin \Gamma \Rightarrow N \setminus S \in \Gamma$ and to be <u>proper</u> if $S \in \Gamma \Rightarrow N \setminus S \notin \Gamma$, additional relationships readily emerge. The following theorem summarizes a variety of results connecting these ideas. The proofs are immediate and will be omitted.

<u>Theorem 3</u> Given $\Gamma \in \mathcal{C}$ with Γ nonempty :

i) $\Gamma^{**} = \Gamma$

ii) $(\Gamma_1 \cup \Gamma_2)^* = \Gamma_1^* \cap \Gamma_2^*$

iii) Γ strong $\Leftrightarrow$ Γ^* proper

iv) Γ strong $\Rightarrow p(\Gamma) \geq 1/2$

v) Γ proper $\Rightarrow p(\Gamma) \leq 1/2$

vi) $\Gamma = \Gamma^* \Leftrightarrow \Gamma$ strong and proper $\Rightarrow p(\Gamma) = 1/2$

vii) $p(\Gamma) = 1/2 \Rightarrow \Gamma$ is either both strong and proper or neither.

We close this section with a table of enactment powers for several well known classes of simple games. All results assume a player set of size n.

<u>TABLE I: Enactment Powers and Other Properties for Some Simple Games</u>

Γ	$p(\Gamma)$	Characteristics
Abs. Majority Rule, n odd	$1/2$	Proper, Strong, $\Gamma^* = \Gamma$
Abs. Majority Rule, n even	$1/2 - \binom{n}{n/2}/2^{n+1}$	Proper, $\Gamma^* = \{S \mid \lvert S\rvert \geq n/2\}$
$\{S \mid \lvert S\rvert \geq k\}$	$\sum_{j=k}^{n} \binom{n}{j} / 2^n$	Proper $\Leftrightarrow k > \frac{n}{2}$, $\Gamma^* = \{S \mid \lvert S\rvert > n-k\}$
Dictator (n - 1 dummies)	$1/2$	Proper, Strong, $\Gamma^* = \Gamma$
T - oligarchy: $\{S \mid S \supseteq T\}$	$1/2^{\lvert T\rvert}$	Proper, $\Gamma^* = \{S \mid S \cap T \neq \emptyset\}$

IV. Nonnormalized Power in Semisimple Games

Given a normalized power index $\rho : \mathcal{C} \to \mathbb{R}^n$ and a semisimple game $\Gamma = \{\Gamma_K\}_{K \in 2^N \setminus \{\emptyset\}} \in \mathcal{S}$, the previous sections provide a measure of power $p(\Gamma_K)$ for each simple game Γ_K and a $2^n - 1$ tuple $\rho(\Gamma) = \{\rho(\Gamma_K)\}$ of individual powers. It seems natural, for each Γ_K, to divide up the game power $p(\Gamma_K)$ in proportion to the individual powers $\rho(\Gamma_K) = \{\rho_i(\Gamma_K)\}_{i=1}^n$. Since $\sum_{i=1}^n \rho_i(\Gamma_K) = 1$, this is readily done by defining, for each $i \in N$, $\Pi_i(\Gamma_K) = p(\Gamma_K)\rho_i(\Gamma_K)$. Doing this for each nonempty subset K of N, we are able to define a <u>non-normalized power index</u> $\Pi : \mathcal{S} \to \mathbb{R}^n$ by

$$\Pi(\Gamma) = \{p(\Gamma_K)\rho(\Gamma_K)\}_{K \in 2^N \setminus \{\emptyset\}} .$$

We illustrate these ideas with the weighted voting game $\Gamma = [5;3,2,1,1,1]$, where the 5 players (we denote them by a, b, c, d, e respectively) have weights of 3, 2, 1, 1, 1 and a quota of 5 votes is needed to form a winning coalition (so abstention is effectively a vote against changing the <u>status quo</u>). The following table describes game and individual powers for the seven nonequivalent nonempty simple games which occur when individual abstention is allowed. We use the index developed in [8] for reasons of simplicity and nonobjectivity. This normalized-to-unity index assigns power to player i in Γ according to the formula

$$\rho_i(\Gamma) = \frac{1}{|\mathcal{M}|} \sum_{\substack{S \in \mathcal{M} \\ i \in S}} 1/|S| ,$$

where $\mathcal{M}$ denotes the set of minimal winning coalitions in Γ . To ease comparison, we round off the individual index values to two decimal places.

TABLE II: Enactment and Individual Power Values for the Game $\Gamma = [5;3,2,1,1,1]$

Player Set K	$p(\Gamma_K)$	$\rho(\Gamma_K)$	$\Pi(\Gamma_K)$
$\{a,b,c,d,e\}$	13/32	(.30,.15,.18,.18,.18)	(.12,.06,.07,.07,.07)
$\{a,b,c,d\}$	5/16	(.42,.25,.17,.17,.00)	(.13,.08,.05,.05,.00)
$\{a,c,d,e\}$	1/4	(.33,.00,.22,.22,.22)	(.08,.00,.05,.05,.05)
$\{b,c,d,e\}$	1/16	(.00,.25,.25,.25,.25)	(.00,.02,.02,.02,.02)
$\{a,b,c\}$	1/8	(.50,.50,.00,.00,.00)	(.06,.06,.00,.00,.00)
$\{a,c,d\}$	1/8	(.33,.00,.33,.33,.00)	(.04,.00,.04,.04,.00)
$\{a,b\}$	1/4	(.50,.50,.00,.00,.00)	(.12,.12,.00,.00,.00)

As a second illustration, we compare index values for absolute and simple majority rule. If Γ_n is absolute majority rule, then the results of Table I applied to Γ_n regarded as a member of $\mathcal{A}$ give $p(\Gamma_n) = \{p(\Gamma_K)\}$ where $k = |K|$ and

$$p(\Gamma_K) = \begin{cases} \sum_{j=[\frac{n+2}{2}]}^{k} \binom{k}{j} / 2^k & \text{if } k \geq [\frac{n+2}{2}] \\ 0 & \text{if } k < [\frac{n+2}{2}] \end{cases}$$

Since each individual in K has power $1/k$ in Γ_K by symmetry, we then have

$$\pi_i(\Gamma_K) = \begin{cases} 1/k\ p(\Gamma_K) & \text{if } i \in K \\ 0 & \text{if } i \notin K \end{cases}$$

In contrast, with Σ_n = simple majority rule we have $p(\Sigma_n) = \{p(\Gamma_{|K|})\}_{K \in 2^N \setminus \{\emptyset\}}$

where

$$p(\Gamma_{|K|}) = \begin{cases} 1/2 & \text{if } |K| \text{ is odd} \\ 1/2 - \binom{k}{k/2}/2^{k+1} & \text{if } |K| = k \text{ is even} \end{cases}$$

In this case, for $i \in K$,

$$\pi_i(\Gamma_{|K|}) = \begin{cases} 1/(2k) & k \text{ odd} \\ 1/k(1/2 - \binom{k}{k/2}/2^{k+1}) & k \text{ even} \end{cases}$$

Thus we see that simple majority rule affords more power to enact at each level of abstention, with absolute majority rule leading to zero individual power as abstentions increase past $[\frac{n}{2}]$. These conclusions are rather obvious from the structure of the games themselves, but our model enables the differentiations in power to be quantified.

The notion of generalized power as a 2^n-1 tuple seems natural in view of the complexity which is added when abstention is allowed. If we assume that the probabilities of abstention are known and independent for each voter, our index can once again be reduced to a single number by means of an "expected power" computation. Indeed, let v_i = the probability that voter i fails to abstain in the game $\Gamma = \{\Gamma_K\} \in \mathscr{S}$. Then the probability of ending up with a set K of nonabstaining voters is $v_K = \prod_{i \in K} v_i \prod_{i \notin K} (1 - v_i)$. We can then define the enactment power of Γ by $p^v(\Gamma) = \sum_{K \neq \emptyset} v_K p(\Gamma_K)$ and the individual powers by $\rho_i^v(\Gamma) = \sum_{K \neq \emptyset} v_K \rho_i(\Gamma_K)$.

and $\Pi_i^v(\Gamma) = \sum_{K \neq \emptyset} v_K \Pi_i(\Gamma_K)$.

Clearly if there is no abstention permitted or assumed, these index valued reduce to the original values determined by whatevever simple index ρ is being used.

If we assume that each individual votes with equal probability v, then $v_K = v^k(1-v)^{n-k}$ $(k = |K|)$. For the "democratic" games of absolute and simple majority rule we then obtain the following enactment powers:

$$p(\Gamma_n) = \sum_{k=[\frac{n+2}{2}]}^{n} [\binom{n}{k} v^k (1-v)^{n-k} \sum_{j=[\frac{n+2}{2}]}^{k} \binom{k}{j} / 2^k]$$

$$p(\Sigma_n) = \sum_{k=1}^{n} [\binom{n}{k} v^k (1-v)^{n-k} \sum_{j=[\frac{k+2}{2}]}^{k} \binom{k}{j} / 2^k] .$$

By democratic symmetry, individual powers Π_i are obtained by dividing by n. The following table lists enactment powers for Γ_n and Σ_n for various values of v and n.

[TABLE III about here]

TABLE III: Enactment Powers for Γ_n/Σ_n with Constant Voting Probability v

n/v	1	.9	.75	.6	.5	.4	.25	.1
1	.500/.500	.450/.450	.375/.375	.300/.300	.250/.250	.200/.200	.125/.125	.050/.050
2	.025/.025	.202/.292	.141/.328	.090/.330	.062/.312	.040/.280	.016/.203	.002/.092
3	.500/.500	.425/.439	.316/.387	.216/.360	.156/.344	.104/.320	.043/.254	.007/.129
4	.312/.312	.241/.365	.152/.386	.084/.376	.051/.363	.027/.344	.007/.288	.000/.160
5	.500/.500	.407/.436	.275/.403	.163/.389	.103/.377	.058/.360	.016/.313	.001/.186
10	.377/.377	.262/.429	.128/.429	.047/.420	.020/.412	.006/.401	.001/.371	.000/.275
11	.500/.500	.367/.443	.194/.432	.0782/.423	.034/.416	.012/.405	.001/.378	.000/.287
24	.419/.419	.242/.457	.072/.453	.011/.448	.002/.443	.000/.436	.000/.418	.000/.364
25	.500/.500	.306/.458	.010/.454	.017/.449	.003/.444	.000/.437	.000/.420	.000/.368
100	.460/.460	.134/.479	.004/.477	.000/.474	.000/.472	.000/.468	.000/.460	.000/.436
101	.500/.500	.156/.479	.005/.477	.000/.474	.000/.472	.000/.469	.000/.460	.000/.437

As seen from Table III, enactment power for absolute majority rule with abstention approaches 0 as n increases. For simple majority rule enactment powers tend to .5 with increasing n. This limiting behavior can be demonstrated analytically and serves to highlight the well known disparity in efficacy between these two common voting rules. The table also indicates that the distinction between even and odd sized groups becomes less and less significant as n increases. While these results are all in accordance with what one might expect, they provide a numerical framework for our intuition and add credence to our model of enactment power.

V. Concluding Comments

By extending the notion of a simple game to allow abstention, we are able to encompass a significantly increased class of real-world processes involving gaming behavior and social choice. Abstention is common practice in a wide variety of legislative and group decisionmaking bodies. The ideas of group and individual power which arise from our extension can reflect, in quantitative fashion, the consequences of abstention with respect to overall decisionmaking efficacy and how this efficacy is apportioned among the voters. Our approach does not provide a new definition of individual power, but rather a means of extending any given measure of power on simple games. Since such measures on simple games have been useful both practically and theoretically, our rather natural extensions should find similar application.

One intriguing characteristic of power indices on simple games has been the various paradoxes associated with them [8,10,11]. The weighted voting game $\Gamma = [5;3,2,1,1,1]$ considered in Table II suggests that our extension may "resolve" some or all of these paradoxes as a result of viewing power as a $2^n - 1$ vector. Specifically, it may be noted that the "1 - vote" players c, d, and e have more power when there is no abstention than the "2 - vote" player b. When abstention is allowed this surprising result does not continue to hold, and player b's overall power must be viewed as not directly comparable to that of c, d, and e. This weighted voting paradox arises only for the index developed in [8]. Other paradoxes such as the "new member" paradox where adding a new player can increase the power of an established player can also be seen to dissolve under abstention. We do not offer these observations as conclusive proof that paradoxes will no longer be present, but merely as an indication that allowing abstention may mitigate them significantly.

Just as there are a variety of indices measuring individual power, there should be alternative ways to define the enactment power of a game. The approach we used employs an aggregation axiom (A 3) compatible with the one used [7] to characterize the Shapley-Shubik and the Banzhaf index. If the approach of [8] were used, it would seem reasonable to define enactment power in proportion to the probability of a minimal winning coalition forming. In this case A 3 could be replaced by

$$A3' : p(\Gamma_1 \cup \Gamma_2) = \frac{|M(\Gamma_1)|p(\Gamma_1) + |M(\Gamma_2)|p(\Gamma_2)}{|M(\Gamma_1)| + |M(\Gamma_2)|} ,$$

where $M(\Gamma)$ denotes the family of minimal winning coalitions in Γ. This approach with a few qualifications provides a characterization of another enactment power measure, $p(\Gamma) = |M(\Gamma)|/2^n$. We omit the details.

Another consequence of our generalization to semisimple games is the ability to formulate two alternative monotonic social choice processes as such games. The power index ideas then allow us to discuss power both for this situation and, when neutrality of the choice process is assumed, for an arbitrary alternative Arrovian framework (cf. [1], [4], and [12]p. 54). This makes possible the specification of conditions on the power index values (such as equal power) as axioms to be used in attempting to characterize social decision functions and voting rules. See [13] for an initial result in this direction.

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