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A BIDDING MODEL OF PRICE
FORMATION UNDER UNCERTAINTY

by

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Comments are invited.

For a graphical treatment of this subject and for additional results, see Discussion Paper No. 406, "Rational Expectations, Information Acquisition, and Competitive Bidding."

A BIDDING MODEL OF PRICE FORMATION UNDER UNCERTAINTY

The theory of competitive equilibrium under conditions of certainty is well understood. Early treatments of equilibrium under uncertainty proceeded by applying the certainty theory to markets for state-contingent claims. In the important case when markets are incomplete, these treatments are not satisfactory because the exchange process itself can reveal information which may alter traders' preferences.

The theory of rational expectations equilibria (REE) has evolved to cope with the complexities of information being transmitted through prices and trades. Among the REE models most often studied are some which involve equilibrium prices (called fully revealing prices) that convey all of the information available to traders in the economy (Green [1973], Khilstrom and Mirman [1975], Grossman [1976], Radner [1977]). In these models, traders equilibrium demands do not depend on their private information; they depend only on the prevailing prices.

This conclusion raises a variety of questions. How do prices come to reflect information if no trader takes any action which reveals his information? Why does any trader bother to collect information which he will never use? In an investigation of the first question, Beja [1977] concludes that informative prices cannot be the outcome of any process resembling tatonnement. Grossman and Stiglitz [1978], studying the second question, conclude that there cannot be incentives to collect costly information when markets are informationally efficient.

The questions raised above are, I believe, most fruitfully addressed by specifying mechanisms for price formation and then examining the mechanisms' properties. This paper studies the mechanisms introduced by Vickrey [1961].

Vickrey's investigation focused on one aspect of uncertainty--traders were uncertain about other traders' tastes. He showed that, in such a context, his mechanisms lead to Pareto optimal allocations. They also lead to "prices" resembling competitive prices.

Rational expectations models frequently focus on another aspect of uncertainty--traders are uncertain about the utility which they may eventually realize from any particular allocation of the mechanism. In such models, the various traders' preferences among commodities and risk attitudes are assumed to be known, but some parameter affecting the consumption value of the commodity is unknown. The models studied in this paper allow traders to be uncertain about both the consumption value of the commodity being traded and the tastes of the other traders.

The properties of Vickrey mechanisms are most easily developed for the simple case in which a single, indivisible object is to be sold to one of n possible buyers (bidders). Section 1 presents a formulation of the single object Vickrey auction with risk-neutral bidders and characterizes the bidders' optimal strategies. Section 2 studies a family of models for which explicit equilibrium strategies can be derived. One model in this family exhibits a fully revealing equilibrium.

Bidders' incentives to gather information are analyzed in Section 3. It is shown that bidders generally have a positive incentive to collect information, even when information is costly and prices are fully revealing.

The robustness of the results developed for the auctions of sections 2 and 3 is examined in Section 4. Sections 5 and 6 extend some of the results to auctions involving multiple items. In Section 7, I present some concluding remarks. The existence of equilibria for bidding games is discussed in an appendix.

1. A GENERAL VICKREY AUCTION

The four primitive objects of the auction model are

1. The set of players, $N = \{1, \dots, n\}$, where $n \geq 2$.
2. A real-valued parameter θ which is unknown to the bidders but which influences the value of the object being auctioned (OBA). (Some authors call θ the payoff relevant variable.) The prior distribution of θ is denoted by G .
3. Signals X_1, \dots, X_n which are observed by players 1 through n , respectively. These signals are taken to be real-valued random variables and, given θ , their conditional joint distribution is denoted by \bar{F}_θ .
4. Bounded functions H_1, \dots, H_n mapping \mathbb{R}^2 into \mathbb{R}_+ . For the generic bidder i , the quantity V_i defined by $V_i = H_i(\theta, X_i)$ is the private value of the OBA.

A strategy for player i is a measurable function $p_i: \mathbb{R} \rightarrow \mathbb{R}_+$ which specifies a bid $p_i(x)$ for each possible value x of X_i . When the strategies p_1, \dots, p_n are held fixed, one can define the following random variables:

$$(1) \quad W = \max_{j \in N} p_j(X_j) .$$

$$(2) \quad W_i = \max_{\substack{j \in N \\ j \neq i}} p_j(X_j) .$$

In a Vickrey auction, the player submitting the highest bid wins the auction and obtains the OBA for a price equal to the highest bid among his competitors. In case of ties, the winner is selected "at random" from among the high bidders. Let i^* denote the winner of the auction. When the strategies are held fixed, i^* is a random variable. The notation

$\{i^* = i\}$ denotes either the event that $i^* = i$ or the indicator variable for that event, as appropriate for the context. Then the generic bidder's expected payoff from the auction is

$$(3) \quad E[(V_i - W_i)\{i^* = i\}].$$

Implicit in this expression are the assumptions that players' utilities are additive separable in money and that players are risk-neutral regarding gambles involving money.

I shall assume that the \bar{F}_θ 's have continuous uniformly bounded densities \bar{f}_θ on \mathbb{R}^n . With this assumption, all conditional expectations given signal information can and will be defined using Bayes' Theorem.

Given strategies p_1, \dots, p_n , define $i^*(b)$ to be the identity of the winning bidder when p_i is replaced by $\hat{p}_i \equiv b$. A strategy p_i is called an optimal response^[1] to the opposing strategies if for every x in the range of X_i

$$(4) \quad p_i(x) \in \arg \max_b E[(V_i - W_i)\{i^*(b) = i\} | X_i = x].$$

An equilibrium is an n -tuple of strategies (p_1, \dots, p_n) such that each is an optimal response to the others.

The reader should interpret W_i in this set-up as the price which player i faces. Different players face different prices, but no player can influence the price he faces. Notice that W_i does not depend on X_i . This key fact distinguishes Vickrey prices from REE prices and leads eventually to the sensible conclusion that players can profit from their private information.

The first formal problem is to characterize the role of ties in the equilibria of the Vickrey auction. Fix strategies p_2, \dots, p_n and con-

sider player 1's problem in locating an optimal response.

Proposition 1. Let x be in the range of X_1 . Then

$$(5) \quad \bar{b} \in \arg \max_b E[(V_1 - W_1)\{i^*(b) = 1\} | X_1 = x]$$

if and only if both (6) and (7) hold,

$$(6) \quad \bar{b} \in \arg \max_b E[(V_1 - W_1)\{W_1 < b\} | X_1 = x]$$

$$(7) \quad E[(V_1 - W_1)\{W_1 = \bar{b}\} | X_1 = x] = 0.$$

Proof.

To show (5) \Rightarrow (7), suppose that (7) does not hold. One case is that the left hand side of (7) is positive:

$$E[(V_1 - W_1)\{W_1 = \bar{b}\} | X_1 = x] = 2\alpha > 0.$$

Then by the specified treatment of ties,

$$E[(V_1 - W_1)\{W_1 = \bar{b}, i^*(\bar{b}) \neq 1\} | X_1 = x] \geq \alpha > 0.$$

Hence, for $\delta > 0$,

$$\begin{aligned} & E[(V_1 - W_1)\{i^*(\bar{b} + \delta) = 1\} | X_1 = x] \\ & - E[(V_1 - W_1)\{i^*(\bar{b}) = 1\} | X_1 = x] \\ & = E[(V_1 - W_1)(\{W_1 = \bar{b}, i^*(\bar{b}) \neq 1\} + \{\bar{b} < W_1 < \bar{b} + \delta\} \\ & \quad + \{W_1 = \bar{b} + \delta, i^*(\bar{b} + \delta) = 1\}) | X_1 = x] \\ & \geq \alpha - \bar{b}P\{\bar{b} < W_1 \leq \bar{b} + \delta | X_1 = x\} \rightarrow \alpha > 0 \end{aligned}$$

as $\delta \rightarrow 0$. If the left-hand side of (7) is negative, a similar argument shows that $\bar{b} - \delta$ is better than \bar{b} for some small positive δ .

To show that (5) \Rightarrow (6), suppose (6) does not hold and choose b^* so that

$$E[(V_1 - W_1)\{W_1 < \bar{b}\} | X_1 = x] < E[(V_1 - W_1)\{W_1 < b^*\} | X_1 = x]$$

Since the last expression is left-continuous in b^* , there is some $\delta > 0$ such that both

$$(8) \quad E[(V_1 - W_1)\{W_1 < \bar{b}\} | X_1 = x] < E[(V_1 - W_1)\{W_1 < b^* - \delta\} | X_1 = x]$$

and $P\{W_1 = b^* - \delta | X_1 = x\} = 0$. This latter fact implies that

$$(9) \quad E[(V_1 - W_1)\{i^*(b^* - \delta) = 1\} | X_1 = x] = E[(V_1 - W_1)\{W_1 < b^* - \delta\} | X_1 = x].$$

Since (5) \Rightarrow (7), (5) also implies that

$$(10) \quad E[(V_1 - W_1)\{i^*(\bar{b}) = 1\} | X_1 = x] = E[(V_1 - W_1)\{W_1 < \bar{b}\} | X_1 = x].$$

Combining (8), (9) and (10) yields

$$E[(V_1 - W_1)\{i^*(\bar{b}) = 1\} | X_1 = x] < E[(V_1 - W_1)\{i^*(b^* - \delta) = 1\} | X_1 = x]$$

in contradiction to (5).

The foregoing argument proceeded by (5) \Rightarrow [(5) and (7)] \Rightarrow (6). The argument that [(6) and (7)] \Rightarrow [(5)] is similar to the argument that [(5) and (7)] \Rightarrow (6).

O.E.D.

In view of Proposition 1, the equilibria of the auction game can be found by solving the problems (6) for each player simultaneously and then checking to see that (7) holds. Pursuing this idea, consider player 1's problem when he has observed $\{X_1 = x\}$. The bid b satisfies (6) if and only if for every positive number δ ,

$$(11) \quad E[(V_1 - W_1)\{W_1 < b\} | X_1 = x] \geq E[(V_1 - W_1)\{W_1 < b + \delta\} | X_1 = x]$$

and

$$(12) \quad E[(V_1 - W_1)\{W_1 < b\} | X_1 = x] \geq E[(V_1 - W_1)\{W_1 < b - \delta\} | X_1 = x].$$

Equations (11) and (12) reduce to

$$(13) \quad 0 \geq E[(V_1 - W_1)\{b \leq W_1 < b + \delta\} | X_1 = x]$$

and

$$(14) \quad 0 \leq E[(V_1 - W_1)\{b - \delta \leq W_1 < b\} | X_1 = x].$$

Vickrey analyzed the special case in which $V_i = X_i$. He showed that in this case each bidder has a dominant strategy, namely, $p(x) = x$. Unfortunately, in certain degenerate cases, there are other less plausible equilibria, as displayed in Table 1.

Table 1

Equilibria When V_1 and V_2 are Degenerate Variables				
Case	V_1	V_2	Player 1's Bid	Player 2's Bid
1	10	20	10	20
2	10	20	18	25
3	10	20	5	12
4	20	20	20	20
5	20	20	0	20
6	20	20	40	20

Pathological cases such as 2, 3, 5, and 6 do not occur when the optimal responses always satisfy the two positivity conditions:

$$(15) \quad \forall \delta > 0 \quad P\{b \leq W_1 < b + \delta | X_1\} > 0.$$

$$(16) \quad \forall \delta > 0 \quad P\{b - \delta < W_1 \leq b | X_1\} > 0.$$

When (15) is satisfied, (13) is equivalent to

$$(17) \quad 0 \geq E[V_1 - W_1 | X_1 = x, b \leq W_1 < b + \delta].$$

Similarly, when (16) is satisfied, (14) is equivalent to

$$(18) \quad 0 \leq E[V_1 - W_1 | X_1 = x, b - \delta \leq W_1 < b].$$

Any optimal response satisfying (17) and (18) is called a regular optimal response and any equilibrium in regular optimal responses is called a regular equilibrium.

Define the function \hat{g}_1 by

$$(19) \quad \hat{g}_1(s,t) = E[V_1 | X_1 = s, W_1 = t].$$

In the coming sections, models are developed for which \hat{g}_1 is continuous in t . Then letting $\delta \rightarrow 0$ in (17) and (18) yields the "first-order condition"

$$0 = \hat{g}_1(s,b) - b$$

or, equivalently,

$$(20) \quad b = E[V_1 | X_1 = s, W_1 = b].$$

Equation (20) is highly suggestive of the rational expectations effect. It is also the intuitive foundation of much of the following analysis.

2. A FAMILY OF MODELS WITH EXPLICIT SOLUTIONS

Suppose that the value functions are identical ($H_1 = \dots = H_n = H$) and that H is continuous and increasing in both arguments. Suppose too, that conditional on θ , the signals are independent and identically distributed with common distribution function F_θ .

$$(21) \quad \bar{F}_\theta(t_1, \dots, t_n) = F_\theta(t_1) \dots F_\theta(t_n) .$$

Let f_θ be the corresponding density function and suppose that f_θ has the monotone likelihood property (MLP):

Monotone Likelihood Property--For any $\theta_1 > \theta_2$, $f_{\theta_2}(x)/f_{\theta_1}(x)$ is a decreasing function of x on its range of definition (where any positive number divided by zero is defined to be $+\infty$). [2]

Define the variables Y_i and the function g by

$$(22) \quad Y_i = \max_{\substack{j \in N \\ j \neq i}} X_j$$

$$(23) \quad g(t, s) = E[H(\theta, X_1) | X_1 = t, Y_1 = s].$$

Applying Bayes' Theorem,

$$(24) \quad g(t, s) = \frac{\int H(r, t) f_r(t) F_r^{n-2}(s) f_r(s) dG(r)}{\int f_r(t) F_r^{n-2}(s) f_r(s) dG(r)}$$

Proposition 2. Under the foregoing assumptions, the strategies given by $p_1(t) = \dots = p_n(t) = g(t, t)$ form a regular equilibrium.

The proof of Proposition 2 relies heavily on the MLP through the following lemma.

Lemma 1: Under the foregoing assumptions,

- (i) Y_1 has the monotone likelihood property and
- (ii) g is continuous and increasing in both arguments.

Proof.

To see that Y_1 has the MLP, note that its conditional distribution (given θ) is $F_{\theta}^{n-1}(-)$.

Take $\theta_1 > \theta_2$. Then the relevant likelihood ratio is

$$[(n-1)F_{\theta_2}^{n-2}(z) f_{\theta_2}(z)] / [(n-1)F_{\theta_1}^{n-2}(z) f_{\theta_1}(z)].$$

Since $f_{\theta_1}(z)/f_{\theta_2}(z)$ is decreasing by assumption, it suffices to show that $F_{\theta_2}(z)/F_{\theta_1}(z)$ is decreasing. Taking the derivative, I need to show that

$$0 < f_{\theta_1}(z)F_{\theta_2}(z) - f_{\theta_2}(z)F_{\theta_1}(z)$$

or that

$$(25) \quad \frac{f_{\theta_2}(z)}{f_{\theta_1}(z)} < \frac{F_{\theta_2}(z)}{F_{\theta_1}(z)} = \frac{\int_{-\infty}^z f_{\theta_2}(r) dr}{\int_{-\infty}^z f_{\theta_1}(r) dr} .$$

By the monotone likelihood property for X_1 ,

$$(26) \quad f_{\theta_2}(z)/f_{\theta_1}(z) < f_{\theta_2}(r)/f_{\theta_1}(r)$$

for $r < z$. Then (25) follows from (26).

Since X_1 and Y_1 are conditionally independent and share the monotone likelihood property, the conditional distribution of θ given $X_1 = s_1$ and $Y_1 = t_1$ stochastically dominates the conditional distribution given $X_1 = s_2$, $Y_1 = t_2$ if $s_1 \geq s_2$ and $t_1 \geq t_2$. From this fact and the assumption that H is increasing in both arguments, it follows from (23) that g is increasing in both arguments.

Finally, using (24) and the boundedness and continuity of f , it can be shown (using the Dominated Convergence Theorem [Royden, 1968]) that $g(-,-)$ is continuous.

Proof of Proposition 2.

It is direct from lemma 1 that p_1 is continuous and increasing. Continuity and the distributional assumptions ensure that the positivity conditions (15) and (16) always hold. Hence, if the specified strategies form an equilibrium, then they form a regular equilibrium. Take $\delta > 0$, $X_1 = x$, and $b = p_1(x)$. Since $p_1 = \dots = p_n$ and since p_1 is increasing, it follows from (2) that $W_1 = p_1(Y_1)$. Since p_1 is increasing, $b \leq W_1$ if and only if $x \leq Y_1$. From these facts and the definition of g ,

$$\begin{aligned} & E[V_1 - W_1 | X_1 = x, b \leq W_1 < b + \delta] \\ &= E[V_1 - W_1 | X_1 = x, x \leq Y_1 \leq p_1^{-1}(b + \delta)] \\ &= E[g(x, Y_1) - g(Y_1, Y_1) | X_1 = x, x \leq Y_1 \leq p_1^{-1}(b + \delta)] \\ &\leq 0 \end{aligned}$$

since g is increasing and $x \leq Y_1$. Similarly,

$$0 \leq E[V_1 - W_1 | X_1 = x, b - \delta < W_1 \leq b].$$

Hence (17) and (18) hold and these are sufficient for the optimality condition (6).

To check that (7) holds, note that

$$\begin{aligned} & E[(V_1 - W_1)\{W_1 = p_1(x)\} | X_1 = x] \\ &= P\{W_1 = p_1(x) | X_1 = x\} \cdot E[V_1 - W_1 | X_1 = x, W_1 = p_1(x)] \\ &= 0 \cdot 0 = 0. \end{aligned}$$

Then Proposition 2 follows from Proposition 1.

The similarity between these equilibria and rational expectations equilibria is emphasized by Proposition 3 below. Informally, the proposition asserts that if a player were told the price W_1 and given an opportunity to revise his bid, he would not choose to make any revision. Thus, each player has chosen his most preferred allocation at the prevailing prices, given both private information and price information.

Proposition 3. In the equilibrium of Proposition 2,

$$(27) \quad p_1(X_1) \in \arg \max_b E[(V_1 - W_1)\{W_1 < b\} | X_1, W_1].$$

Proof:

Observe that if $p_1(X_1) \leq W_1$ then $X_1 \leq Y_1$, since $W_1 = p_1(Y_1)$ and p_1 is increasing. Choose any $\delta > 0$. Then

$$\begin{aligned} (28) \quad & E[(V_1 - W_1)\{W_1 < p_1(X_1) + \delta\} | W_1, X_1] \\ & - E[(V_1 - W_1)\{W_1 < p_1(X_1)\} | W_1, X_1] \\ & = E[(V_1 - W_1)\{p_1(X_1) \leq W_1 < p_1(X_1) + \delta\} | W_1, X_1] \\ & = (E[V_1 | W_1, X_1] - W_1)\{p_1(X_1) \leq W_1 < p_1(X_1) + \delta\} \\ & = [g(X_1, Y_1) - g(Y_1, Y_1)]\{p_1(X_1) \leq W_1 < p_1(X_1) + \delta\} \\ & \leq 0 \end{aligned}$$

since $X_1 \leq Y_1$ on $\{p_1(X_1) \leq W_1\}$ and since g is increasing. Hence, no improvement is ever possible by increasing $p_1(X_1)$ to $p_1(X_1) + \delta$. A similar argument applies to reductions from $p_1(X_1)$ to $p_1(X_1) - \delta$.

An interesting feature of the equilibrium of Proposition 2 is that W_1 itself has the monotone likelihood property. To see this, note that since p_1 is increasing $W_1 = p_1(Y_1)$. Then since Y_1 has the MLP by lemma 1 and since p_1 is increasing, it follows that W_1 has the MLP.

Next, consider the special case in which all values of θ are positive and ^[3]

$$(29) \quad F_{\theta}(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ t/\theta & \text{for } 0 \leq t \leq \theta \\ 1 & \text{for } \theta \leq t \end{cases}$$

that is, X_1 is uniformly distributed on $[0, \theta]$.

Since $W_1 = p_1(Y_1)$ for the equilibria in this section and since (with this choice of F_{θ} 's) Y_1 is a sufficient statistic for X_2, \dots, X_n , W_1 may be regarded as a fully revealing price. In view of Proposition 3, the uniform likelihood model yields equilibria resembling fully revealing rational expectations equilibria.

When, in addition to assuming that (29) holds, the prior is assumed to be

$$(30) \quad G(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ t/M & \text{for } 0 \leq t \leq M \\ 1 & \text{for } M \leq t \end{cases}$$

for some $M > 0$ and when $H(\theta, t) \equiv \theta$, the equilibrium strategies become

$$(31) \quad p_1(z) = \frac{(n-1)(z^{2-n} - M^{2-n})}{(n-2)(z^{1-n} - M^{1-n})}.$$

For n large, $p_1(z) \approx z \cdot (n-1)/(n-2)$. Also $E[Y_{i^*} | \theta] = \theta \cdot (n-1)/(n+1)$ so the winner's expected payoff satisfies

$$(32) \quad E[(\theta - W_{i^*}) | \theta] = \theta \cdot \left(1 - \frac{(n-1)^2}{(n-2)(n+1)}\right) > 0.$$

The winning bidder has a positive expected payoff but this payoff approaches zero as n grows large. In fact $Y_{i^*}^*$ approaches θ almost surely as $n \rightarrow \infty$ and hence W_{i^*} also approaches θ almost surely.

The interpretation of this result is that prices can reflect quite a lot of information. Wilson [1978] has proven a result showing that this sort of convergence is common in first price auctions. Milgrom [1979] has extended Wilson's result. The analogue of these theorems for the Vickrey auction of this section is given below.

Proposition 4. Let $B(s)$ be the supremum of the support of X_1 when $\theta = s$. Then one may always take $H(s, B(s)) = \lim_{z \rightarrow B(s)} H(s, z)$,

Let W_{1*}^n be the price paid by the winning bidder in an n -bidder auction. Then W_{1*}^n converges in probability to $H(\theta, B(\theta))$ as $n \rightarrow \infty$ if and only if for all $\theta_1 > \theta_2$

$$(33) \quad \lim_{z \rightarrow B(\theta_1)} \frac{f_{\theta_2}(z)}{f_{\theta_1}(z)} = 0.$$

The proof of this result is lengthy and analogous to the proof of Theorem 3 in Milgrom [1979].

3. INCENTIVES TO COLLECT INFORMATION

Consider the player set $N = \{1, \dots, n\}$ and the equilibrium given by Proposition 2 for the model in the preceding section. Consider the problem of an additional player $n+1$ with signal X_{n+1} . Instead of assuming that $V_{n+1} = H(\theta, X_{n+1})$ and that X_{n+1} is distributed according to the F_θ 's, assume that (i) $V_{n+1} \leq V_1$ and (ii) X_{n+1} is a garbling of X_1 , that is, the conditional joint distribution of θ, X_2, \dots, X_n given X_1 and X_{n+1} does not depend on X_{n+1} .

Proposition 5.

$$(34) \quad \sup_b E[(V_{n+1} - W)\{W < b\} | X_{n+1}] = 0 \quad \text{a.s.}$$

Accordingly, an equilibrium of the $n+1$ -player game is given by

$$p_1(t) = \dots = p_n(t) = g(t, t)$$

$$p_{n+1}(t) \equiv 0.$$

Proof.

The supremum in (34) is non-negative since the choice $b=0$ is allowed. It therefore suffices to show that it is also non-positive.

Since p_1 is an equilibrium strategy, for any real number b

$$(35) \quad \begin{aligned} 0 &\geq E[(V_1 - W_1)\{p_1(X_1) \leq W_1 < b\} | X_1] \\ &= E[(V_1 - W_1)\{p_1(X_1) \leq W_1 < b\} | X_1, X_{n+1}] \end{aligned}$$

since X_{n+1} is a garbling of X_1 . Taking the conditional expectation of (35) given X_{n+1} yields

$$(36) \quad \begin{aligned} 0 &\geq E[E[(V_1 - W_1)\{p_1(X_1) \leq W_1 < b\} | X_1, X_{n+1}] | X_{n+1}] \\ &= E[(V_1 - W_1)\{p_1(X_1) \leq W_1 < b\} | X_{n+1}]. \end{aligned}$$

Note as before that $W_1 < p_1(X_1)$ implies $Y_1 < X_1$. So,

$$\begin{aligned}
 (37) \quad & E[(V_1 - p_1(X_1))\{W_1 < p_1(X_1) < b\} | X_1, X_{n+1}] \\
 & = E[(g(Y_1, X_1) - g(X_1, X_1))\{W_1 < p_1(X_1) < b\} | X_1, X_{n+1}] \\
 & \leq 0.
 \end{aligned}$$

It follows that

$$(38) \quad E[(V_1 - p_1(X_1))\{W_1 < p_1(X_1) < b\} | X_{n+1}] \leq 0.$$

Since $V_{n+1} \leq V_1$,

$$\begin{aligned}
 (39) \quad & E[(V_{n+1} - W)\{W < b\} | X_{n+1}] \\
 & \leq E[(V_1 - W)\{W < b\} | X_{n+1}] \\
 & = E[(V_1 - W)\{p_1(X_1) \leq W < b\} | X_{n+1}] \\
 & \quad + E[(V_1 - W)\{W_1 < p_1(X_1) < b\} | X_{n+1}] \\
 & = E[(V_1 - W)\{p_1(X_1) \leq W < b\} | X_{n+1}] \\
 & \quad + E[(V_1 - p_1(X_1))\{W_1 \leq p_1(X_1) < b\} | X_{n+1}] \\
 & \leq 0.
 \end{aligned}$$

where the last inequality follows from (36) and (38). Since (39) holds for all real numbers b , the supremum is also non-positive, as claimed.

The thrust of Proposition 5 is that (at least for a class of models) poorly informed bidders cannot earn a positive expected payoff. For these models, there is an incentive to collect information.

It is not difficult to formally incorporate the possibility of gathering costly information into the ambient model. Suppose, for example, that each bidder may choose whether to observe his signal at a cost of c . Let $\pi(n)$ be the expected payoff per bidder in the n -bidder model and suppose $c < \pi(2)$. Since payoffs are bounded, $\pi(n)$ must tend to zero as n grows

large. Hence, there exists a largest integer k for which $c \leq \pi(k)$.

Suppose there are m bidders in all. An equilibrium of this auction with costly information occurs when (i) the first $n(= \text{MIN}(m,k))$ bidders pay to observe their signals and then bid according to the equilibrium strategies of Proposition 2 and (ii) the remaining bidders choose not to observe their signals and then bid zero. In the case the F_{θ} 's satisfy (25), this model yields a fully revealing equilibrium with costly information in which only bidders who buy information have positive expected payoffs.

4. ROBUSTNESS OF RESULTS

Table 2 lists the significant assumptions used in the developments of the preceding sections. In this section I demonstrate that the first five assumptions can be substantially weakened or eliminated without impairing the results of the earlier sections.

When separability and risk-neutrality are eliminated, player 1's optimization problem becomes

$$(40) \quad \text{MAX}_b E[u(\theta, X_1, W_1)\{W_1 < b\} | X_1 = x]$$

where $u(\theta_0, x, b)$ is the payoff to player 1 when he wins with a bid of b in the event $\{\theta = \theta_0, X_1 = x\}$.

Assume that the functions $u(\theta_0, x, -): \mathbb{R}_+ \rightarrow \mathbb{R}$ are uniformly bounded, decreasing and equi-continuous in b . Assume, too that for all pairs (θ_0, x) , $u(\theta_0, x, 0) \geq 0$. Finally, assume that for all (θ_0, x) there is some b such that $u(\theta_0, x, b) < 0$. Note that it is implicit in (40) that utilities are normalized so that a losing bidder's utility is zero.

Table 2

List of Significant Assumptions

- | | |
|--------------------------|---|
| 1. Separability: | Players preferences over money and the OBA are additive separable. |
| 2. Risk-neutrality: | Players are risk-neutral regarding money gambles. |
| 3. Symmetry: | Players have identical preferences and prior beliefs and equally informative signals. |
| 4. Independence: | Conditional on θ , the signals are independent. |
| 5. Real-valued signals: | Signals are real-valued random variables |
| 6. Monotone likelihoods: | The signals have the monotone likelihood property. |

With these assumptions, there exists a unique b^* such that

$$(41) \quad E[u(\theta, X_1, b^*) | X_1 = s, Y_1 = t] = 0.$$

Define $g(s, t) = b^*$. Then one can mimic the proofs of lemma 1 and Proposition 2 to show that these hold. Also, Proposition 3 holds when (27) is replaced by the analogous statement

$$(27') \quad p_1(X_1) \in \arg \max_b E[u(\theta, X_1, b) \{W_1 < b\} | X_1, W_1].$$

Proposition 4 holds by an argument resembling that given by Milgrom [1979].

Proposition 5 is difficult to state without the separability assumption. When one can write $u(V_1 - b)$ for $u(\theta, X_1, b)$, the proposition is true with (34) replaced by

$$(34') \quad \sup_b E[u(V_{n+1} - W) \{W < b\} | X_{n+1}] = 0.$$

The proof is a minor variation on the proof given for the original Proposition 5.

The independence assumption interacts with the monotone likelihood assumption to yield the following:

Conditional Monotone Likelihood Property (CMLP)

Let B_1, \dots, B_n be Borel sets and define corresponding events A_i by

$$(42) \quad A_i = \bigcap_{j \neq i} \{X_j \in B_j\}$$

Suppose that for every i and every non-null event A_i of the specified form, the family of distributions $\{F_\theta^i(-|A_i)\}$ indexed by θ has the monotone likelihood property. Then X_1, \dots, X_n are said to have the conditional monotone likelihood property.

In proving lemma 1, a key fact is that assumptions 4 and 6 combine to yield the CMLP. When the CMLP is directly assumed, one can dispense with the independence assumption and still have that g is monotone in its first argument.

As an example where independence fails but the CMLP holds, suppose that, given θ , X_1, \dots, X_n have a joint normal distribution with $E[X_i | \theta] = \theta$ and $\text{Var}[X_i | \theta] = \sigma^2$ for each i and suppose that for $i \neq j$, $E[(X_i - \theta)(X_j - \theta) | \theta] / \sigma^2 = \rho$, where $0 \leq \rho < 1$. One can show that X_1, \dots, X_n then has the CMLP. Then Propositions 2 through 5 still hold.

The symmetry assumption plays a central role in Propositions 2 and 3. Propositions 4 and 5 can be rescued, but I shall focus attention on 5. Suppose that all assumptions except symmetry are satisfied and that p_1, \dots, p_n are equilibrium strategies with the following properties:

1. The range of each p_i and W_i coincide.
2. The range of X_i does not depend on $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n$.
3. Each p_i is increasing and satisfies the analogue of (20),

$$(43) \quad \begin{aligned} p_i(x) &= E[V_i | X_i = x, W_i = p_i(x)] \\ &= \hat{g}_i(x, p_i(x)). \end{aligned}$$

An immediate consequence of the CMLP is

Lemma 2. \hat{g}_i is increasing in its first argument.

Proposition 5 can be proven using (43) in place of the expression $p_i(t) = g(t, t)$ and using lemma 2 in place of lemma 1.

In the model of section 2, each bidder's information is represented by a real-valued signal (Assumption 5). Yet in many interesting cases, the variable observed may be much more complex, such as a weather map or a con-

sultant's report. The form of the signal, however, is essentially irrelevant to its ability to transmit information in this model.

Suppose X_1 is a variable whose values are consultant's reports and suppose that for distinct reports x and \bar{x} , it is always the case that $E[\theta|X_1 = x] \neq E[\theta|X_1 = \bar{x}]$. Then one can define a real-valued variable $X_1^* = E[\theta|X_1]$ which reveals "the same information" as X_1 .

Generally, the condition that the X_i 's have the MLP is not necessary to the development, since it is not even necessary that they be real-valued. A more basic condition is that $X_i^* = E[\theta|X_i]$ (or some similar variables) have the MLP. An interpretation of this condition is offered in section 7.

5. MULTIPLE ITEMS--ONE PER BIDDER

The theory developed in the preceding sections remains substantially the same when there are $k < n$ items to be distributed among the n bidders. Let \bar{W}_i be the k^{th} highest bid among the opponents of player i . If i submits one of the k highest bids, he wins one item and pays \bar{W}_i .

Suppose that $V_i = H(\theta, X_i)$ where H , θ , and the X_i 's have the properties assumed in section 2. Let \bar{Y}_i be the k^{th} highest signal among the opponents of player i . Set

$$(44) \quad \bar{g}(t, s) = E[H(\theta, x_1) | X_1 = t, \bar{Y}_1 = s]$$

$$= \frac{\int H(r, t) f_r(t) f_r(s) F_r^{n-k}(s) (1 - F_r(s))^{k-1} dG(r)}{\int f_r(t) f_r(s) F_r^{n-k}(s) (1 - F_r(s))^{k-1} dG(r)}$$

Proceeding analogously to lemma 1, one can show

Lemma 3. Under the assumptions of this section,

- (i) \bar{Y}_1 has the monotone likelihood property
- (ii) \bar{g} is continuous and increasing in both arguments.

Arguing as in Proposition 2, one can show:

Proposition 6. Under the assumptions of this section, the strategies given by $p_1(t) = \dots = p_n(t) = \bar{g}(t, t)$ form an equilibrium.

Analogous to Proposition 3, one has:

Proposition 7. In the equilibrium of Proposition 6,

$$(45) \quad p_1(X_1) \in \arg \max_b E[(V_1 - \bar{W}_1) \{ \bar{W}_1 < b \} | X_1, \bar{W}_1].$$

A statement of Proposition 4 in this context requires the introduction of new notation (since there are now several winning bidders i^*) and is therefore omitted here. Proposition 5 becomes:

Proposition 8. Suppose $V_{n+1} \leq V_j$ and X_{n+1} is garbling of X_j for all $j \leq k$. Let \bar{W} be the k^{th} highest bidder among players 1 through n . Then

$$(46) \quad \sup_b E[(V_{n+1} - \bar{W})\{\bar{W} < b\} | X_{n+1}] = 0 \quad \text{a.s.}$$

6. MULTIPLE ITEMS - UNRESTRICTED

Suppose that there are k identical items to be distributed and that any bidder may be allocated any number of items from zero through k . Let $v_i(m)$ be the value to player i of an allocation of m items and define the variables.

$$(47) \quad v_i^l = v_i(l) - v_i(l-1)$$

for $l \geq 1$. Normalize v_i so that $v_i(0) = 0$. Each player's preferences are assumed to be convex and monotone, that is,

$$(48) \quad v_i^1 \geq v_i^2 \geq \dots \geq v_i^k \geq 0.$$

Each player submits k bids and the k highest bids each win one of the OBA's. If player i has submitted m of the k highest bids, he pays a price equal to the sum of the m highest unsuccessful bids among his competitors.

In this set up, a strategy for player i is a function $p_i: \mathbb{R} \rightarrow \mathbb{R}_+^k$ specifying the k bids to be submitted by i as a function of his signal X_i . Since the labelling of the bids is irrelevant in this auction, there is no loss of generality in ordering the individual bids $p_i^l(x_i)$ according to

$$(49) \quad p_i^1 \geq p_i^2 \geq \dots \geq p_i^k.$$

Define w_i^q to be the q^{th} highest bid among the opponents of player i . If player i wins m of the OBA's, his payoff is

$$(50) \quad v_i^1 + \dots + v_i^m - w_i^{k+1-m} - w_i^{k-m} - \dots - w_i^k.$$

The marginal impact on i 's payoff of his bid $p_i^m(x_i)$ is therefore

$$(51) \quad (v_i^m - w_i^{k+1-m}) \{w_i^{k+1-m} < p_i^m(x_i)\}.$$

It follows that $p_i^m(x_i)$ is a solution of

$$(52) \quad \text{Max}_{b \in [p_i^{m+1}(x_i), p_i^{m-1}(x_i)]} E[(V_i^m - W_i^{k+1-m})\{W_i^{k+1-m} < b\} | x_i].$$

with the convention that $p_i^0 \equiv 0$.

Suppose that each V_i^m has the form $V_i^m = H_m(\theta, x_i)$ where θ is a real-valued payoff relevant variable and H_m is bounded, continuous, and increasing. Suppose the X_i 's satisfy the assumptions of section 2. Consider a function \hat{p}_i^m such that $\hat{p}_i^m(x_i)$ solves the problem

$$(53) \quad \text{Max}_{b \geq 0} E[(V_i^m - W_i^{k+1-m})\{W_i^{k+1-m} < b\} | x_i].$$

Proposition 9. If for each m , W_i^{k+1-m} has the monotone likelihood property, and if $P\{W_i^{k+1-m} = W_i^{k+2-m}\} = 0$, then

$$(54) \quad \hat{p}_i^1 \geq \hat{p}_i^2 \geq \dots \geq \hat{p}_i^k.$$

Proof.

Since $W_i^{k+2-m} \leq W_i^{k+1-m}$ and since the latter has the MLP, one can show that (for a version of conditional probability) the conditional distribution of θ given $W_i^{k+2-m} = s$ stochastically dominates the conditional distribution of θ given $W_i^{k+1-m} = s$. Define

$$(55) \quad e_i^m(s, t) = E[V_i^m | x_i = s, W_i^{k+1-m} = t].$$

From the previous observation, $e_i^{m-1}(s, t) > e_i^m(s, t)$ for all s and t . By the monotone likelihood property, each e_i^m is increasing in t .

Using (55) and the inequalities $V_i^m \leq V_i^{m-1}$ and $W_i^{k+2-m} \leq W_i^{k+1-m}$, one may infer that on the set $\{\hat{p}_i^m(x_i) > \hat{p}_i^{m-1}(x_i)\}$,

$$\begin{aligned}
 (56) \quad & E[(V_i^m - W_i^{k+1-m})\{\hat{p}_i^{m-1}(X_i) \leq W_i^{k+1-m} < \hat{p}_i^m(X_i)\} | X_i] \\
 & = E[(e_i^m(X_i, W_i^{k+1-m}) - W_i^{k+1-m})\{\hat{p}_i^{m-1}(X_i) \leq W_i^{k+1-m} < \hat{p}_i^m(X_i)\} | X_i] \\
 & < E[(e_i^{m-1}(X_i, W_i^{k+2-m}) - W_i^{k+1-m})\{\hat{p}_i^{m-1}(X_i) \leq W_i^{k+1-m} < \hat{p}_i^m(X_i)\} | X_i] \\
 & \leq E[(e_i^{m-1}(X_i, W_i^{k+2-m}) - W_i^{k+2-m})\{\hat{p}_i^{m-1}(X_i) \leq W_i^{k+2-m} < \hat{p}_i^m(X_i)\} | X_i] \\
 & \leq 0
 \end{aligned}$$

where the last inequality follows from the optimality of \hat{p}_i^{m-1} (see (13)).

Since \hat{p}_i^m is optimal, it follows that $\{\hat{p}_i^m(X_i) > \hat{p}_i^{m-1}(X_i)\}$ is empty.

When prices have the monotone likelihood property, one can find player i 's optimal response by solving the problems (53). These problems are identical in form to the maximization problem (6) in the single object auction, and I have shown for that case that for a class of models the prices have the MLP. The present model is complex, but one may expect conclusions which are qualitatively similar to the conclusions in the single object auction.

7. CONCLUDING REMARKS

In the standard rational expectations formulation, traders are price-takers and all traders face the same prices. If, in addition, prices are informative and depend on the traders' private information, the conclusion is inescapable that some player sees his private information already reflected in prices. This conclusion leads to the question: Why does the trader bother to collect information which is already revealed by prices when he assumes he cannot influence prices?

There appears to be two principal approaches to resolving this information paradox. First, one may assume that players know that their actions influence prices. A bidding model with this feature has been developed by Wilson [1978]. The second approach is to assume that traders are price-takers, but that different traders face different prices. I have taken this second approach in this paper.

A defect of my model as a model of price formation is that it considers only one side of the market--the buyers' side. To incorporate sellers into the model, one might assume that each seller makes available for sale some number of objects and specifies a reservation price for each. If both buyer and seller are to be price takers, then the price π paid by the buyer i to the seller j can depend only on the prices named by the other traders l ($l \neq i, l \neq j$).

The qualitative conclusions reached in the models of this paper are suggestive of the conclusions one might expect from any more complete model following the second approach:

- (i) Prices are informative, and higher prices are indicative of greater value.

- (ii) There are positive incentives to gather information, even when it is costly and prices are "fully revealing".
- (iii) "Inside information", (that is, information which is not widely available) tends to lead to greater advantage than "public information".
- (iv) A trader who is told the marginal price of the last object he has successfully purchased would not choose to change his bids, even if he had the opportunity to do so.

In addition, one might well expect the following property.

- (v) For all players i , $E[|W_i^1 - W_i^{m(i)+1}|] = O(1/n)$ where $m(i)$ is the number of objects purchased by player i in the n bidder auction. Thus, for large n , the players perceive an essentially fixed price per unit in equilibrium.

It seems clear that the monotone likelihood property has played a major role in this development, so it is appropriate to discuss its economic meaning. The following two lemmas clarify the Bayesian interpretation of the MLP.

Lemma 4. Suppose X has the monotone likelihood property. Then if $x_1 > x_2$ and G is any non-degenerate prior distribution for θ , $G(-|X = x_1)$ stochastically dominates $G(-|X = x_2)$.

Lemma 5. Suppose that for every non-degenerate prior distribution G for θ and every $x_1 > x_2$, $G(-|X = x_1)$ stochastically dominates $G(-|X = x_2)$. Then $\{F_\theta\}$ has the monotone likelihood property.

The essence of the MLP, applied to traders' signals, is that it is possible to arrange the various posterior distributions $G(-|X)$ into order according to stochastic dominance. Suppose, for example, a trader's signal is a research report X and that such reports can be put on a scale from

"least favorable" to "most favorable", in the sense that the posteriors are ordered by stochastic dominance. Then X is informationally equivalent to the signal $X^* = E[\theta|X]$ and the latter has the MLP. Thus, the MLP assumption in the model of section 2 can be weakened to an assumption that $E[\theta|X_1]$ has the MLP.

When the MLP is applied to prices, no such weakening is satisfactory. It is necessary not only that price information can be put on the "least favorable" to "most favorable" scale, but also that higher prices are more favorable indicators than lower prices.

8. APPENDIX:

Existence of Equilibria in Bidding Games

Two issues arise in the standard fixed point approach to proving equilibria of bidding games. First, the payoff functions in these games are typically discontinuous because a small change in one's bid can change one's role from loser to winner. This problem is related to the issue of non-regular equilibria, described in section 1.

The second issue, which is addressed in this appendix, concerns compactness in the space of bidder's strategies.

Suppose that bids must be denominated in discrete units, e.g., pennies. Since each V_i is bounded, each bidder can make only a finite number of bids. Suppose each V_i is of the form $V_i = H_i(\theta, X_i)$ and that the joint distribution functions \bar{F}_θ of $(X_1, \dots, X_n) \in \mathbb{R}^n$ have uniformly bounded equi-continuous density functions \bar{f}_θ . A pure strategy for player i is a function $p_i: \mathbb{R} \rightarrow \mathbb{B} = \{b_1, \dots, b_m\}$ where \mathbb{B} is the set of possible bids. Let ψ^m be the unit simplex in \mathbb{R}^m .

$$(57) \quad \psi^m = \{(c_1, \dots, c_m) \in \mathbb{R}^m : \forall i \ c_i \geq 0 \text{ and } \sum c_i = 1\}.$$

A behavioral strategy for player i is a function $\hat{p}_i: \mathbb{R} \rightarrow \psi^m$ whose j^{th} component is \hat{p}_i^j .

When the players adopt pure strategies $p = (p_1, \dots, p_n)$, player i 's payoff in any realization $(\theta, x_1, \dots, x_n)$ is $\pi_i(\theta, p_i(x_1), \dots, p_n(x_n))$. Assume that each π_i is bounded and measurable.

When the behavioral strategies $\hat{p} = (\hat{p}_1, \dots, \hat{p}_n)$ are adopted, the payoff for $(\theta, x_1, \dots, x_n)$ to player i is

$$(58) \quad \hat{\pi}_i(\theta, \hat{p}_1(x_1), \dots, \hat{p}_n(x_n)) = \sum_{j_1=1}^m \dots \sum_{j_n=1}^m \pi_i(\theta, b_{j_1}, \dots, b_{j_n}) \hat{p}_1^{j_1}(x_1) \dots \hat{p}_n^{j_n}(x_n).$$

Proposition 10. A behavioral strategy equilibrium for this game exists.

Proof:

Let G be the prior distribution for θ . Under the ambient assumptions, for any fixed strategies $\hat{p} = (\hat{p}_1, \dots, \hat{p}_n)$, the functions defined by

$$(59) \quad \begin{aligned} \bar{\pi}_i(z|\hat{p}) &= E[\hat{\pi}_i | X_i = z] \\ &= \frac{\int \hat{\pi}(\theta, \hat{p}_1(x_1), \dots, \hat{p}_n(x_n)) f_\theta(x_1, \dots, z, \dots, x_n) dG(\theta)}{\int f_\theta(x_1, \dots, z, \dots, x_n) dG(\theta)} \end{aligned}$$

are continuous in z . By a measurable selection argument which I shall omit, this continuity ensures the existence of a measurable optimal pure response p_i for player i to the strategies \hat{p}_j of the other players. The set of optimal behavioral responses \hat{p}_i is clearly convex. It is also relatively compact in the product topology. Moreover, in the product topology, each $\bar{\pi}_i$ is continuous in \hat{p} . It follows that the optimal behavioral response correspondence is closed and its values are non-empty, convex and compact. So by Glicksberg's [1952] fixed point theorem, the correspondence has a fixed point \hat{p} which is the required equilibrium.

Footnotes

- [1] This is not the standard definition. Traditionally, p_i is called an optimal response if it maximizes (3). Any response which I call optimal is also optimal under the traditional criterion.
- [2] I shall use phrases like " f_θ has the MLP," " F_θ has the MLP" and " X_1 has the MLP" interchangeably.
- [3] Although these F_θ 's do not satisfy our assumptions because their densities are neither continuous nor uniformly bounded, they do lead to continuous strategies if G is continuous. One can check that the violated assumptions were used only to establish the continuity of p_1 . Hence, Proposition 2 also applies to these F_θ 's.

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