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Conference Structures and Fair Allocation Rules

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Abstract:

To describe how the outcome of a cooperative game might depend on which groups of players hold cooperative planning conferences, we study allocation rules, which are functions mapping conference structures to payoff allocations. An allocation rule is fair if every conference always gives equal benefits to all its members. Any characteristic function game without sidepayments has a unique fair allocation rule. The fair allocation rule also satisfies a balanced-contributions formula, and is closely related to Harsanyi's generalized Shapley value for games without sidepayments. If the game is superadditive, then the fair allocation rule also satisfies a stability condition.

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0. INTRODUCTION

We expect that players in a cooperative game should meet together in a series of conferences to discuss possible cooperative plans and to sign jointly binding agreements. The goal of cooperative game theory is to help us understand how these conferences determine the ultimate outcome of the game. To accomplish this goal, one could try to model these conferences as dynamic processes or as noncooperative games in their own right; but such models are hard to construct without being ad hoc or unrealistic, because negotiation conferences are in fact very complex phenomena. In this paper, we will try to avoid this difficulty by taking conferences as black boxes. That is, we will study allocation rules, which tell us how the players' payoffs ought to depend on which conferences occur, but we will not try to describe the internal workings of these conferences. We will only assume that the net effect of a conference should be fair, in that it benefits all of its members equally. Our main result is that every game has a unique fair allocation rule satisfying a natural Pareto-efficiency property.

The results of this paper generalize earlier results in [7], by dropping the sidepayments assumption and by allowing for conferences of more than two players. For some other approaches to the question of how final payofs should depend on the structure of cooperation in a game, see Luce and Raiffa [5] (Chapter 10), Maschler [6], Aumann and Drèze [1], Owen [10], and Shenoy [13].

1. Basic Definitions

Throughout this paper, we shall let V be a characteristic function game without sidepayments, and we let

(1.0) $\mathbb{N} = \{1,2,\ldots,n\}$

represent the set of players in V. That is, we assume that V is a function which maps each set of players $S \subseteq N$ onto a set $V(S)$ such that:

(1.1) $V(S)$ is a closed subset of $\mathbb{R}^n$;

(1.2) $\emptyset \neq V(S) \neq \mathbb{R}^n$ if $S \neq \emptyset$ $(V(\emptyset) = \mathbb{R}^n)$;

(1.3) if $x \in V(S)$, $y \in \mathbb{R}^n$, and $y_i \leq x_i$ $\forall i \in S$ then $y \in V(S)$.

(Notice that this is somewhat weaker than the usual definition of a characteristic function game without sidepayments, as in [2]. For example, we shall not need convexity of $V(S)$ in this paper).

The set $V(S)$ is interpreted as the set of all payoff allocations which give the members of S a combination of payoffs which they could guarantee themselves together, without cooperating with the other players. Condition (1.3) asserts that free disposal of utility payoffs is always possible for any coalition S, so that $V(S)$ must be a comprehensive subset of $\mathbb{R}^n$.

For any set $S \subseteq N$, let $\partial V(S)$ be the weakly Pareto-efficient frontier of $V(S)$. That is:

$$(1.4) \qquad \partial V(S) = \left\{ x \in V(S) \;\middle|\; \begin{array}{l} \text{if } y_i > x_i \text{ for all } i \in S, \\ \text{then } y \notin V(S) \end{array} \right\}$$

To describe how the players organize their cooperation, we must specify which groups of players are willing and able to confer together for the purpose of planning cooperative actions. It may be that some players will not talk to each other directly, or that some players can only communicate with each other in the presence of many other players, as in a convention or a committee meeting.

We shall use the term <u>conference</u> to refer to any set of two or more players who might meet together to discuss their cooperative plans. A <u>conference structure</u> is then any collection of conferences. We let CS denote the set of all possible conference structures, so that:

$$(1.5) \qquad CS = \{Q \mid \forall S \in Q,\ S \subseteq N \text{ and } |S| \geq 2\}.$$

Given a conference structure $Q \in CS$, players i and j are <u>connected</u> by Q if $i=j$ or there exists some sequence of conferences $(S_1,\ldots,S_m)$ such that:

$$i \in S_1,\ j \in S_m,\ \{S_1,\ldots,S_m\} \subseteq Q, \text{ and}$$
$$S_k \cap S_{k+1} \neq \emptyset \text{ for every } k=1,\ldots,m-1.$$

That is, two players are connected by Q if they can be coordinated either by meeting together in some permissable conference to which they both belong ($m=1$), or by meeting in separate conferences which have some members in common to serve as intermediaries ($m=2$), or by some longer sequence of overlapping conferences ($m \geq 3$). We let N/Q denote the partition of N defined by this connectedness relation, so that:

(1.6) $$N/Q = \{\{j | i \text{ and } j \text{ are connected by } Q\} | i \in N\}$$

That is, the sets in N/Q are the maximal connected coalitions which can be coordinated if players only communicate by meeting in the conferences of Q.

Given $Q \in CS$, $S \subseteq N$, and $i \in N$, we define conference structures $Q-S$, $Q \cap^* S$ and $Q-^* i$ by the following formulas:

(1.7) $$Q-S = \{T | T \in Q \text{ and } T \neq S\},$$

(1.8) $$Q \cap^* S = \{T | T \in Q \text{ and } T \subseteq S\},$$

(1.9) $$Q-^* i = \{T | T \in Q \text{ and } i \notin T\}.$$

So $Q-S$ is the conference structure differing from Q in that S is dropped from the list of permissable conferences. $Q \cap^* S$ differs from Q in that all conferences containing players outside S are eliminated. $Q-^* i$ is the conferences structue differing from Q in that all conferences containing player i are eliminated.

2. ALLOCATION RULES

One would expect that the outcome of game V ought to depend on how the players meet to organize their cooperation. That is, each player's payoff should be a function of the cooperation structure. Thus, we are interested in studying funtions of the form $X: CS \to \mathbb{R}^n$, mapping each conference structure Q onto a payoff allocation $X(Q) = (X_1(Q),\ldots,X_n(Q))$. Our goal is to find such a function for which each $X_i(Q)$ could be reasonably interpreted as how much player i could expect to get in game V if Q were the set of conferences held by the players.

We formally define an <u>allocation rule</u> for the game V to be any function $X:CS \to \mathbb{R}^n$ such that

(2.1) $$X(Q) \varepsilon \partial V(S), \forall Q \varepsilon CS, \forall S \varepsilon N/Q.$$

This condition (2.1) asserts that, if S is a maximal connected coalition for the conference structure Q, then the members of S ought to coordinate themselves so as to achieve a (weakly) Pareto-efficient allocation among those allocations available to them. Thus, we are assuming that the players will effectively form the largest possible coalitions which can be coordinated with the given conference structure.

There are infinitely many allocation rules for the game V, because any point in $\partial V(S)$ can be chosen for $X(Q)$, and this set is always infinite (provided $Q \neq \emptyset$). To find a narrower range of interesting allocation rules, we must consider some additional conditions which a reasonable allocation rule might also satisfy.

When people cooperate with each other, it is often suggested that any two players should enjoy the same gains from their cooperation together, relative to what they would get without cooperation. This intuitive notion of fairness in cooperation is what we shall call the <u>equal-gains principle.</u> This principle clearly involves some interpersonal comparison of utility, so it cannot be based purely on the concepts of individual Bayesian decision theory. However, it has been argued elsewhere ([4], [8]) that there may be strong theoretical reasons why we should expect bargaining to be conducted with reference to the equal-gains principle (or some version of it), even when utility is not linearly transferable between individuals. In any case, the equal-gains principle is a widely familiar common sense notion. ("You should do this for me because I have done more for you already.") Experimental data has confirmed its importance even when utility is not transferable [9]. In Section 6, we will relax this assumption that utility is interpersonally comparable.

To apply the equal-gains principle to our alllocation rules, we say that an allocation rule $X:CS \to \mathbb{R}^n$ is <u>fair</u> if and only if:

$$(2.2) \qquad X_i(Q) - X_i(Q-S) = X_j(Q) - X_j(Q-S), \quad \forall Q \in CS, \forall S \in Q, \forall i \in S, \forall j \in S.$$

So $X(.)$ is a fair allocation rule if every conference always gives equal benefits to all of its members. That is, if the members of S decided not to meet together, then this change in the conference structure (from Q to $Q-S$) should affect all members of S equally.

It turns out that there is a unique fair allocation rule for any game. The main task of this paper is to prove this fact and describe the fair allocation rule.

3. BALANCED CONTRIBUTIONS AND THE SHAPLEY VALUE

We may assume that a set S can be included in the actual conference structure of a game only if all members of S agree that they want to meet together. So the fairness condition (2.2) asserts that, by refusing to participate in a conference no player can hurt any other member of the conference any more than he would hurt himself. But a player may expect to participate in many conferences, and he might also threaten to withdraw his support from all of them, if some of his demands are not recognized.

If player j withdrew his support from all conferences in Q, then the conference structure would have to change from Q to $Q-^{*}j$. (Recall (1.9)). The allocation for player i would then change from $X_i(Q)$ to $X_i(Q-^{*}j)$ as a result of j dropping out. The difference, $X_i(Q) - X_i(Q-^{*}j)$, may be called j's <u>contribution</u> to i in Q.

We say that an allocation rule $X:CS \to \mathbb{R}^n$ has <u>balanced contributions</u> if and only if:

$$(3.1) \qquad X_i(Q)-X_i(Q-^{*}j) = X_j(Q) - X_j(Q-^{*}i), \quad \forall Q \in CS,\ \forall i \in N,\ \forall j \in N.$$

That is, X(.) has balanced contributions if j's contribution to i always equals i's contribution to j, in any conference structure. We shall see that fairness and balanced contributions are closely related properties.

The Shapley value [12] was orginally defined only for games with sidepayments, but it is also closely related to our fair allocation rules. To show this relationship, let us say that $X: CS \to R^n$ satisfies the <u>Shapley formula</u> if and only if:

$$(3.2) \qquad X_i(Q) - X_i(\emptyset) = \sum_{\substack{S \subseteq N \\ (i \in S)}} \frac{(|S|-1)!\,(n-|S|)!}{n!} \left(Z(Q \cap^* S) - Z(Q \cap^* S -^* i)\right),$$

for all Q in CS and all i in N, where:

$$(3.3) \qquad Z(Q) = \sum_{j \in N} X_j(Q), \ \forall Q \in CS.$$

This Shapley formula asserts that each player's total gain from a conference structure should be a weighted average of his contributions to all players in smaller conference structures.

4. EXISTENCE AND UNIQUENESS

We can now state and prove our main result, characterizing the fair allocation rules.

<u>Theorem 1</u> There exists a unique fair allocation rule for the game V. This allocation rule also has balanced contributions and satisfies the Shapley formula, and no other allocation rule for V satisfies either of these properties.

We prove Theorem 1 in a series of four lemmas.

Lemma 1. There exists an allocation rule satisfying the Shapley formula.

Proof of Lemma 1 Let $X_i(\emptyset) = \text{maximum } \{x_i | x \in V(\{i\})\}$ and let $Z(\emptyset) = \sum_{i \in N} X_i(\emptyset)$. Then (2.1) and (3.2) are trivially satisfied for $Q = \emptyset$. (Notice $N/\emptyset = \{\{i\} | i \in N\}$.)

We say that a conference structure Q is simple if there exists some $T \in N/Q$ such that $Q \cap^* T = Q$. That is, Q is simple if all players who belong to any conferences are connected together. We can now proceed to construct X(Q) and Z(Q) inductively, in order of increasing $|Q|$.

Suppose first that Q is simple, with T being the only non-trivial coalition in N/Q. Then (3.2) is equivalent to

$$(4.1) \qquad X_i(Q) - X_i(\emptyset) = \begin{cases} \frac{1}{|T|} Z(Q) + A_i(Q), & \text{if } i \in T, \\ 0, & \text{if } i \notin T, \end{cases}$$

where

$$(4.2) \qquad A_i(Q) = \frac{-Z(Q -^* i)}{|T|} + \sum_{\substack{R \subset T \\ (i \in R)}} \frac{(|R|-1)!\ (|T|-|R|)!}{|T|!} (Z(Q \cap^* R) - Z(Q \cap^* R -^* i)).$$

The coeficients in (4.2) are derived from the combinatorial fact that:

$$\sum_{\substack{S \subseteq N \\ (S \cap T = R)}} \frac{(|S|-1)!\ (|N| - |S|)!}{|N|!} = \frac{(|R|-1)!\ (|T| - |R|)!}{|T|!}$$

There are two important implications of (4.2): that $A_i(Q)$ depends only on $Z(.)$ evaluated at conference structures strictly smaller than Q, and that $\sum_{i\varepsilon T} A_i(Q) = -Z(\emptyset)$.

So we can evaluate $A_i(Q)$ assuming inductively that $Z(Q')$ and $X(Q')$ have already been constructed for all $Q' \subset Q$. Then, since $V(T)$ is closed and comprehensive, we can choose $Z(Q)$ and $X(Q)$ so that (4.1) holds and $X(Q) \varepsilon \partial V(T)$. $X(Q) \varepsilon \partial V(\{i\})$ for all $i \notin T$ also follows from (4.1), since $X_i(Q)$ then equals $X_i(\emptyset)$. (3.3) holds for Q when we construct $Z(Q)$ and $X(Q)$ in this way because

$$\sum_{i\varepsilon N} X_i(Q) = Z(Q) + \sum_{i\varepsilon T} A_i(Q) + \sum_{i\varepsilon N} X_i(\emptyset) = Z(Q)$$

Suppose now that Q is not simple. For each $T \varepsilon N/Q$ and each $i \varepsilon T$, let $X_i(Q) = X_i(Q\cap^* T)$ (where $Q\cap^* T$ will be simple). Let $Z(Q) = \sum_{i\varepsilon N} X_i(Q)$. Then for each $T \varepsilon N/Q$, we have $X(Q) \varepsilon \partial V(T)$, because $X(Q\cap^* T) \varepsilon \partial V(T)$ (by the construction in the simple case), and because being in $\partial V(T)$ depends only on the payoffs to members of T.

We must now check that (3.2) holds for any Q. By construction, a player's allocation does not depend on conferences among players with whom he is not connected. So, for any $Q\varepsilon CS$, $T\varepsilon N/Q$, $i\varepsilon T$, $j\varepsilon T$, and $k\notin T$:

$$X_j(Q\cap^* S)=X_j(Q\cap^* T\cap^* S),X_j(Q\cap^* S-^* i)=X_j(Q\cap^* T\cap^* S-^* i),$$
$$X_k(Q\cap^* S)=X_k(Q\cap^* S-^* i),X_k(Q\cap^* T\cap^* S)=X_k(\emptyset)=X_k(Q\cap^* T\cap^* S-^* i),$$

and thus

$$Z(Q\cap^* S)-Z(Q\cap^* S-^* i)=Z(Q\cap^* T\cap^* S)-Z(Q\cap^* T\cap^* S-^* i).$$

Then (3.2) for Q follows from the fact that (3.2) holds for the simple conference structure $Q \cap^* T$ when $i \varepsilon T \varepsilon N/Q$.

<u>Lemma 2.</u> If X(.) satisfies the Shapley formula, then X(.) has balanced contributions.

<u>Proof of Lemma 2.</u> For any $Q \varepsilon CS$, $i \varepsilon N$, $j \varepsilon N$, (3.2) implies

$$X_i(Q) - X_i(Q-^*j) =$$

$$= \sum_{\substack{S \subseteq N \\ S \neq \emptyset}} \frac{(|S|-1)!\,(n-|S|)!}{n!} (Z(Q \cap^* S) - Z(Q \cap^* S -^* i) - Z(Q \cap^* S -^* j) + Z(Q \cap^* S -^* i -^* j)).$$

But this expression is symmetric with respect to i and j, so $X_j(Q)-X_j(Q-^*i)$ could also be expressed in this way. So (3.1) follows.

<u>Lemma 3.</u> If X(.) has balanced contributions, then X(.) is a fair allocation rule.

<u>Proof of Lemma 3.</u> Suppose that $\{i,j\} \subseteq S \varepsilon Q$. Then

$(Q-S)-^*i = Q-^*i$ and $(Q-S)-^*j = Q-^*j$. So

$$X_i(Q) - X_i(Q-S) =$$

$$= X_j(Q)-X_j(Q-^*i)+X_i(Q-^*j)-X_j(Q-S)+X_j(Q-S-^*i)-X_i(Q-S-^*j)$$

$$= X_j(Q)-X_j(Q-S).$$

<u>Lemma 4.</u> There is at most one fair allocation rule for V.

<u>Proof of Lemma 4.</u> Suppose that there were two such fair allocation rules $X:CS \rightarrow \mathbb{R}^n$ and $Y:CS \rightarrow \mathbb{R}^n$. Let Q be a minimal conference structure such that $X_k(Q) \neq Y_k(Q)$ for some $k \varepsilon N$. (that is, $X(Q') = Y(Q')$ for all $Q' \subset Q$.) We may assume that $X_k(Q) > Y_k(Q)$,

with no loss of generality. Suppose $\{i,j\} \subseteq S \in Q$. Then:

$$X_i(Q)-X_j(Q) = X_i(Q-S)-X_j(Q-S)$$
$$= Y_i(Q-S)-Y_j(Q-S) = Y_i(Q)-Y_j(Q).$$

So $X_i(Q)-Y_i(Q) = X_j(Q)-Y_j(Q)$ for all i and j who belong to any common conference in Q. Then, arguing along the connecting sequence, we must have $X_i(Q)-Y_i(Q) = X_k(Q)-Y_k(Q) > 0$ for any player i who is connected to k in Q. Let $S \in N/Q$ be the connected coalition containing player k. So $X_i(Q) > Y_i(Q)$ for all i in S. Then $X(Q) \in \partial V(S)$ implies $Y(Q) \notin \partial V(S)$, so X and Y cannot both satisfy (2.1).

Theorem 1 follows immediately from these four lemmas.

5. STABILITY

We have described a fair allocation rule as one in which every conference always gives equal benefits to its members. However, we have not actually shown that the effect of adding a conference need be properly beneficial for its members. We now address this question.

We say that an allocation rule $X:CS \to \mathbb{R}^n$ is <u>stable</u> if and only if

(5.1) $\quad X_i(Q) \geq X_i(Q-S), \forall Q \in CS, \forall S \in Q, \forall i \in S.$

That is, X(.) is stable if adding any set S to the conference structure will never hurt the members of S. (So stability obviously imples individual rationality.)

A characteristic function game V without sidepayments is <u>superadditive</u> if and only if:

(5.2) $V(S) \cap V(T) \subseteq V(S \cup T)$,

for all S and T such that $S \cap T = \emptyset$.

That is, V is superadditive if a merger of two disjoint coalitions can always achieve all allocations which were feasible for them both apart.

Theorem 2. Let $X: CS \to \mathbb{R}^n$ be the fair allocation rule for V. If V is superadditive, then $X(.)$ is stable.

Proof of Theorem 2. Suppose that the theorem were false. Then we could choose $Q \in CS$, $S \in Q$, and $k \in S$ so that: $X_k(Q) < X_k(Q-S)$ but

$$X_k(Q') \geqslant X_k(Q' - S) \text{ for all } Q' \subset Q.$$

(That is, Q is minimal among the conference structures which violate the theorem.) Let $T \in N/Q$ be the connected coalition containing k.

For any $i \in T$, balanced contributions implies that:

$$X_i(Q) = X_k(Q) - X_k(Q-{}^*i) + X_i(Q-{}^*k)$$
$$< X_k(Q-S) - X_k(Q-S{}^*i) + X_i(Q-S-{}^*k) = X_i(Q-S).$$

Thus $X_i(Q) < X_i(Q-S)$ for all $i \in T$.

But $X(Q) \in \partial V(T)$, by (2.1). So $X(Q-S) \notin V(T)$, which contradicts superadditivity, since $X(Q-S) \in \bigcap_{R \in N/(Q-S)} V(R)$, and since T is a union of some coalitions in $N/(Q-S)$. ($N/(Q-S)$ is a refinement of N/Q.) This contradiction proves the theorem.

6. Harsanyi's bargaining solutions and fair allocation rules

Harsanyi's bargaining solutions for n-person cooperative games [3] are a generalization of the Shapley value to games without sidepayments, and are closely related to our fair allocation rules. To show this relationship, we must first make some definiitions and prove two lemmas, which may be of interest in their own right.

Thus, far, we have assumed that the players' utility scales are interpersonally comparable, so that the equal-gains condition (2.2) is appropriate as a characterization of fairness or equity in cooperation. However, any affine transformations of the players' utility scales would preserve all their decision-theoretic and game-theoretic properties of the utility scales (since we have not assumed transferable utility). So it may be more appropriate to consider a broader class of fairness conditions, equalizing weighted utility. Given any vector $\lambda = (\lambda_1,\ldots,\lambda_n)$, with all $\lambda_i > 0$, we say that an allocation rule $X:CS \to R^n$ is <u>λ-fair</u> if and only if

(6.1) $$\lambda_i(X_i(Q) - X_i(Q-S)) = \lambda_j(X_j(Q) - X_j(Q-S)),$$
$$\forall Q\varepsilon CS,\ \forall S\varepsilon Q,\ \forall i\varepsilon S,\ \forall j\varepsilon S.$$

<u>Lemma 5</u>. For any positive n-vector λ, there exists exactly one λ-fair allocation rule X^λ: $CS \to \mathbb{R}^n$ for the game V.

<u>Proof</u> Given V and λ, we define the game $\hat{V}^{\lambda}$ so that

(6.2) $$\hat{V}^{\lambda}(S) = \{(\lambda_1 x_1,\ldots,\lambda_n x_n) | (x_1,\ldots,x_n) \ \varepsilon\ V(S)\}.$$

Then X^λ is a λ-fair allocation rule for V if and only if Y^λ is a fair allocation rule for $\hat{V}^\lambda$, where

(6.3) $\qquad Y_i^\lambda(Q) = \lambda_i X_i^\lambda(Q), \quad \forall Q \varepsilon CS, \ \forall i \varepsilon N.$

Thus Lemma 5 follows from the fact that $\hat{V}^\lambda$ has a unique fair allocation rule.

In this paper, we have assumed that the players' payoff allocation should depend on the structure of conferences which the players may form. In [3], however, Harsanyi has assumed a simpler functional dependence, in which conflict payoffs depend only on the formation of a single coalition. Thus, let us define a <u>coalitional allocation rule</u> for V to be any function $U: 2^N \to \mathbb{R}^n$, mapping each coalition $S \subseteq N$ to some allocation vector u(S) satisfying

(6.4) $\qquad U(S) \ \varepsilon \ \partial V(S)$

where we may arbitarily set $U_i(S) = 0$ if $i \notin S$. Generalizing (3.1), we say that U has <u>λ-balanced contributions</u> if and only if

(6.5) $\qquad \lambda_i(U_i(S)-U_i(S-j)) = \lambda_j(U_j(S)-U_j(S-i)), \ \forall S \subseteq N, \ \forall i \varepsilon S, \ \forall_j \varepsilon S$

$\qquad$ (where $S - i = \{k | k \ \varepsilon \ S \text{ and } k \neq i\}$).

For any coalition $S \subseteq N$, we let $\bar{Q}^S$ be the set of all conferences involving only players in S. That is,

(6.6) $\qquad \bar{Q}^S = \{T | T \subseteq S \text{ and } |T| \geq 2\}.$

<u>Lemma 6.</u> Given any positive n-vector λ, the unique coalitional allocation rule for V with λ-balanced contributions is

(6.7) $\qquad U_i(S) = X_i^\lambda(\bar{Q}^S) \quad \forall S \subseteq N, \ \forall i \varepsilon S,$

where X^λ is the λ-fair allocation rule for V.

<u>Proof.</u> To check that (6.7) does define a coalitional allocation rule with λ-balanced contributions, define Y^λ as in (6.3). Then use the fact that Y^λ is a fair allocation rule (for $\hat{V}^\lambda$) to get

$$\lambda_i(U_i(S) - U_i(S-j)) = Y_i^\lambda(\bar{Q}^S) - Y_i^\lambda(\bar{Q}^{S-j})$$

$$= Y_i^\lambda(\bar{Q}^S) - Y_i^\lambda(\bar{Q}^S -^* j) = Y_j^\lambda(\bar{Q}^S) - Y_j^\lambda(\bar{Q}^S -^* i) = \lambda_j(U_j(S) - U_j(S-i)).$$

To check uniqueness, suppose that U and $\hat{U}$ are two different coalitional allocation functions for V with λ-balanced contributions, and let S be one of the smallest coalitions with $U(S) \neq \hat{U}(S)$. For any i and j in S, (6.5) implies that

$$\lambda_j(U_j(S) - \hat{U}_j(S)) = \lambda_i(U_i(S) - U_i(S-j)) + \lambda_j U_j(S-i)$$

$$- \lambda_i(\hat{U}_i(S) - \hat{U}_i(S-j)) - \lambda_j \hat{U}_j(S-i)$$

$$= \lambda_i(U_i(S) - \hat{U}_i(S)),$$

because $U(T) = \hat{U}(T)$ for all T smaller than S. But these equalities, together with $U(S) \in \partial V(S)$ and $\hat{U}(S) \in \partial V(S)$, imply that $U(S) = \hat{U}(S)$. This contradiction proves uniqueness, so the lemma is proven.

In [7], graphical cooperation structures were studied, where a <u>graphical cooperation structure</u> is just a conference structure in which each conference has exactly two members (and so can be represented graphically by a link between the two players). For example, "complete cooperation within the coalition S" would be represented by

$$\bar{G}^S = \{\{i,j\} | i \in S, j \in S, \text{ and } i \neq j\}$$

in the context of the graphical cooperation structures from [7], where we implicitly assume that only bilateral conferences among the players are allowed. In contrast, the more complicated conference structure $\bar{Q}^S$ from (6.6) represents "complete cooperation within the coalition S" in the more general context of conference structures studied in this paper. As an easy application of Lemma 6, we can show that the fair allocations for the members of S when there is "complete cooperation within S" is the same for both notions of "complete cooperation". That is, for any $S \subseteq N$,

$$X^\lambda(\overline{G}^S) = X^\lambda(\overline{Q}^S),$$

where $X^\lambda(\cdot)$ is the λ-fair allocation rule for V. To prove this, simply observe that, as a function of S, $X^\lambda(\overline{G}^S)$ is a coalitional allocation rule with λ-balanced contributions (the argument is the same as for $X^\lambda(\overline{Q}^S)$) and apply Lemma 6.

We can now characterize Harsanyi's bargaining solutions in terms of our fair allocation rules.

<u>Theorem 3.</u> Let λ be any positive n-vector, and let $u^N = (u_1^N,\ldots,u_n^N)$ satisfy

$$u^N \in V(N) \text{ and } \sum_{i\in N} \lambda_i u_i^N = \underset{x \in V(N)}{\text{maximum}} \sum_{i\in N} \lambda_i x_i,$$

Then u^N is a bargaining solution for the game V, (in the sense of Harsanyi [3]) if and only if

$$u^N = X^\lambda(\overline{Q}^N),$$

where X^λ is the λ-fair allocation rule for V and $\overline{Q}^N$ is the complete cooperation structure on N (see (6.6)).

<u>Proof.</u> In Harsanyi's bargaining model, each coalition S promises a dividend w_i^S to each player i, where $w_i^S = 0$ if $i \notin S$. These dividends must be constructed so that the allocation vector u^S is in $\partial V(S)$, where

$$(6.8) \qquad u_i^S = \sum_{\substack{T \subseteq S \\ T \supseteq \{i\}}} w_i^T, \qquad \forall S \subseteq N,\ \forall i \in S.$$

If λ can be chosen so that

$$(6.9) \qquad \lambda_i w_i^S = \lambda_j w_j^S, \qquad \forall S \subseteq N, \ \forall i \in S, \ \forall j \in S,$$

and

$$\sum_{i \in N} \lambda_i u_i^N = \underset{x \in V(N)}{\text{maximum}} \sum_i \lambda_i x_i,$$

then $u^N = (u_1^N, \ldots, u_n^N)$ is a bargaining solution (in the sense of Harsanyi) for our game V, with weights λ.

Harsanyi (page 210 in [3]) has pointed out that (6.8) is equivalent to

$$(6.10) \qquad w_i^S = \sum_{\substack{T \subseteq S \\ T \supseteq \{i\}}} (-1)^{|S|-|T|} u_i^T \qquad \forall S \subseteq N, \quad \forall i \in S.$$

If i and j are distinct players in S, then (6.10) can be rewritten as

$$(6.11) \qquad w_i^S = \sum_{\substack{T \subseteq S \\ T \supseteq \{i,j\}}} (-1)^{|S|-|T|} (u_i^T - u_i^{T-j})$$

Using (6.8) and (6.11), it is then straightforward to show that (6.9) holds if and only if

$$(6.12) \qquad \lambda_i u_i^S - \lambda_i u_i^{S-j} = \lambda_j u_j^S - \lambda_j u_j^{S-i}, \qquad \forall S \subseteq N, \ \forall i \in S, \ \forall j \in S.$$

But by Lemma 6, (6.12) can hold if and only if

$$u_i^S = X_i^\lambda(\bar{Q}^S), \quad \forall S \subseteq N, \quad \forall i \in S.$$

So $u_i^N = X_i^\lambda(\bar{Q}^N)$, which proves the theorem.

<u>Corollary</u> Suppose that V is equivalent to a game v with sidepayments, in the sense that

$$V(S) = \{x \in \mathbb{R}^n \mid \sum_{i \in S} x_i \leq v(S)\}, \quad \forall S \subseteq N.$$

Then $X(\bar{Q}^N)$ equals the Shapley value of v, when X is the fair allocation rule for V and $\bar{Q}^N$ is the complete cooperation structure.

<u>Proof</u>. It is well known that, for games with sidepayments, Harsanyi's bargaining solution coincides with the Shapley value, with all scale factors $\lambda_i=1$.

7. <u>Example</u>

Let us consider the following three-person game suggested by Roth [11]:

$$V(\{i\}) = \{(x_1, x_2, x_3) \mid x_i \leq 0\}, \text{ for each } i$$

$$V(\{12\}) = \{(x_1, x_2, x_3) \mid x_1 \leq \tfrac{1}{2}, x_2 \leq \tfrac{1}{2}\}$$

$$V(\{13\}) = \{(x_1, x_2, x_3) \mid x_1 \leq \tfrac{1}{4}, x_3 \leq \tfrac{3}{4}\}$$

$$V(\{23\}) = \{(x_1, x_2, x_3) \mid x_2 \leq \tfrac{1}{4}, x_3 \leq \tfrac{3}{4}\}$$

$$V(\{123\}) = \left\{(x_1, x_2, x_3) \;\middle|\; \begin{array}{l} \text{there is some } (y_1, y_2, y_3) \text{ in the} \\ \text{convex hull of } \{(\tfrac{1}{2}, \tfrac{1}{2}, 0), (\tfrac{1}{4}, 0, \tfrac{3}{4}) \ (0, \tfrac{1}{4}, \tfrac{3}{4})\} \\ \text{such that } x_i \leq y_i \text{ for all } i \end{array}\right\}$$

(This game corresponds to Roth's V_p for $p=\frac{1}{4}$.) The fair allocation rule for this game is as follows.

Q	$X(Q)$
$\emptyset$	$(0, 0, 0)$
$\{\{23\}\}$	$(0, \frac{1}{4}, \frac{1}{4})$
$\{\{13\}\}$	$(\frac{1}{4}, 0, \frac{1}{4})$
$\{\{12\}\}$	$(\frac{1}{2}, \frac{1}{2}, 0)$
$\{\{12\}, \{13\}\}$	$(\frac{1}{2}, \frac{1}{4}, 0)$
$\{\{12\}, \{23\}\}$	$(\frac{1}{4}, \frac{1}{2}, 0)$
$\{\{23\}, \{13\}\}$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$
$\{\{12\}, \{13\}, \{23\}\}$	$(\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$
$\{\{123\}\}$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
$\{\{23\}, \{123\}\}$	$(\frac{1}{8}, \frac{3}{8}, \frac{3}{8})$
$\{\{13\}, \{123\}\}$	$(\frac{3}{8}, \frac{1}{8}, \frac{3}{8})$
$\{\{12\}, \{123\}\}$	$(\frac{1}{2}, \frac{1}{2}, 0)$
$\{\{12\}, \{13\}, \{123\}\}$	$(\frac{1}{2}, \frac{1}{4}, 0)$
$\{\{12\}, \{23\}, \{123\}\}$	$(\frac{1}{4}, \frac{1}{2}, 0)$
$\{\{23\}, \{13\}, \{123\}\}$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$
$\{\{12\}, \{13\}, \{23\}, \{123\}\}$	$(\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$

If we let $\lambda=(1,1,1)$, then this is also the λ-fair allocation rule, and we have

$$\sum_{i\varepsilon N} \lambda_i X_i(\bar{Q}^N) = \frac{5}{12} + \frac{5}{12} + \frac{2}{12} = \max_{x\varepsilon V(N)} \sum_{i\varepsilon N} \lambda_i x_i .$$

Thus, $X(\bar{Q}^N) = (\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$ is a Harsanyi bargaining solution for this game.

Roth [11] has argued that the outcome of this game should be $(\frac{1}{2}, \frac{1}{2}, 0)$ because this is the best feasible allocation for players 1 and 2, and they can achieve it by themselves. However, when viewed in the context of the fair allocation rule, the solution $(\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$ is seen to have the stability property of Theorem 2, since this game is superadditive. For example, player 2 would certainly prefer to get $\frac{1}{2}$, which he gets from $\{\{12\}\}$, over $\frac{5}{12}$, which he gets from the complete cooperation structure $\bar{Q}^N$. However, if 2 refused to participate in conferences with player 3, while player 1 continued to confer with 3, then the cooperation structure would change from $\bar{Q}^N$ to $\{\{12\}, \{13\}\}$, so that player 2 would lose $\frac{2}{12} = \frac{5}{12} - \frac{1}{4}$. (Notice that, as fairness requires, player 3 also loses $\frac{2}{12}$ from this change of conference structure.) Thus, player 2 would not want to be first to break off relations with player 3. If players must make their conference participation plans noncooperatively, then both players 1 and 2 will want to confer with 3, and so the complete conference structure $\bar{Q}^N$ will form.

To check balanced contributions, for example, notice that

$$X_2(\bar{Q}^N) - X_2(\bar{Q}^N - {}^*3) = 5/12 - 1/2 = -1/12$$

$$X_3(\bar{Q}^N) - X_3(\bar{Q}^N - {}^*2) = 2/12 - 1/4 = -1/12$$

since $\bar{Q}^N - {}^*3 = \{\{12\}\}$ and $\bar{Q}^N - {}^*2 = \{\{13\}\}$. In this case, the contribution of 3 to 2 is negative, so we see that players may make negative contributions to each other in superadditive games.

One objection to our analysis might be that, for some conference structures, the fair allocation is only weakly Pareto-efficient. For example, the allocation $(\frac{1}{4}, 0, \frac{1}{4})$ for $Q=\{\{13\}\}$ is weakly dominated by $(\frac{1}{4}, 0, \frac{3}{4})$, which is feasible for the coalition $\{13\}$. However, the distinction between weak Pareto-efficiency and strong Pareto-efficiency is not robust with respect to small changes in the game, For example, if $V(\{13\})$ were enlarged to the comprehensive convex hull of

$$\{(\frac{1}{4}, 0, \frac{3}{4}), (\frac{1}{4}+\varepsilon, 0, 0), (0, 0, \frac{3}{4}+\varepsilon)\},$$

then $X(\{\{13\}\})$ would shift to $(\frac{1}{4} + \frac{2\varepsilon}{3+4\varepsilon}, 0, \frac{1}{4} + \frac{2\varepsilon}{3+4\varepsilon})$, which is on the strongly Pareto-efficient frontier of the new $V(\{13\})$ set, and which converges to $(\frac{1}{4}, 0, \frac{1}{4})$ as ε goes to zero. That is, if player 3's sacrifice below $\frac{3}{4}$ could make possible any infinitesimal gains for player 1 above $\frac{1}{4}$, then the fair allocation $X(\{\{13\}\})$ would change infinitesimally, but would become strongly efficient for $\{13\}$. Thus, the distinction between strong and weak efficiency is significant only if we can assume that there is really nothing which player 3 can offer player 1 above the $x_1=\frac{1}{4}$ level, no matter how much player 3 sacrifices.

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cooperation together, relative to what they would get without cooperation. This intuitive notion of fairness in cooperation is what we shall call the equal-gains principle. This principle clearly involves some interpersonal comparison of utility, so it cannot be based purely on the concepts of individual Bayesian decision theory. However, it has been argued elsewhere ([4], [8]) that there may be strong theoretical reasons why we should expect bargaining to be conducted with reference to the equal-gains principle (or some version of it), even when utility is not linearly transferable between individuals. In any case, the equal-gains principle is a widely familiar common sense notion. ("You should do this for me because I have done more for you already.") Experimental data has confirmed its importance even when utility is not transferable [9]. In Section 6, we will relax this assumption that utility is interpersonally comparable.

To apply the equal-gains principle to our alllocation rules, we say that an allocation rule $X: CS \to \mathbb{R}^n$ is fair if and only if:

$$X_i(Q) - X_i(Q-S) = X_j(Q) - X_j(Q-S), \quad \forall Q \in CS,\ \forall S \in Q,\ \forall i \in S,\ \forall j \in S. \tag{2.2}$$

So $X(.)$ is a fair allocation rule if every conference always gives equal benefits to all of its members. That is, if the members of S decided not to meet together, then this change in the conference structure (from Q to Q-S) should affect all members of S equally.

It turns out that there is a unique fair allocation rule for any game. The main task of this paper is to prove this fact and

$$\lambda_i(U_i(S) - U_i(S-j)) = Y_i^\lambda(\bar{Q}^S) - Y_i^\lambda(\bar{Q}^{S-j})$$

$$= Y_i^\lambda(\bar{Q}^S) - Y_i^\lambda(\bar{Q}^S -^* j) = Y_j^\lambda(\bar{Q}^S) - Y_j^\lambda(\bar{Q}^S -^* i) = \lambda_j(U_j(S)-U_j(S-i)).$$

To check uniqueness, suppose that U and $\hat{U}$ are two different coalitional allocation functions for V with λ-balanced contributions, and let S be one of the smallest coalitions with $U(S) \neq \hat{U}(S)$. For any i and j in S, (6.5) implies that

$$\lambda_j(U_j(S) - \hat{U}_j(S)) = \lambda_i(U_i(S) - U_i(S-j)) + \lambda_j U_j(S-i)$$

$$- \lambda_i(\hat{U}_i(S) - \hat{U}_i(S-j)) - \lambda_j \hat{U}_j(S-i)$$

$$= \lambda_i(U_i(S) - \hat{U}_i(S)),$$

because $U(T) = \hat{U}(T)$ for all T smaller than S. But these equalities, together with $U(S) \varepsilon \partial V(S)$ and $\hat{U}(S) \varepsilon \partial V(S)$, imply that $U(S) = \hat{U}(S)$. This contradiction proves uniqueness, so the lemma is proven.

In [7], graphical cooperation structures were studied, where a <u>graphical cooperation structure</u> is just a conference structure in which each conference has exactly two members (and so can be represented graphically by a link between the two players). For example, "complete cooperation within the coalition S" would be represented by

$$\bar{G}^S = \{\{i,j\} | i\varepsilon S, j\varepsilon S, \text{ and } i \neq j\}$$

in the context of the graphical cooperation structures from [7], where we implicitly assume that only bilateral conferences among the players are allowed. In contrast, the more complicated conference structure $\bar{Q}^S$ from (6.6) represents "complete cooperation within the coalition S" in the more general context of conference structures studied in this paper. As an easy application of Lemma 6, we can show that the fair allocations for the members of S when there is "complete cooperation within S" is the same for both notions of "complete cooperation". That is, for any $S \subseteq N$,

$$X^{\lambda}(\bar{G}^S) = X^{\lambda}(\bar{Q}^S),$$

where $X^{\lambda}(\cdot)$ is the λ-fair allocation rule for V. To prove this, simply observe that, as a function of S, $X^{\lambda}(\bar{G}^S)$ is a coalitional allocation rule with λ-balanced contributions (the argument is the same as for $X^{\lambda}(\bar{Q}^S)$) and apply Lemma 6.

We can now characterize Harsanyi's bargaining solutions in terms of our fair allocation rules.

Theorem 3. Let λ be any positive n-vector, and let $u^N = (u_1^N,\ldots,u_n^N)$ satisfy

$$u^N \in V(N) \text{ and } \sum_{i\in N} \lambda_i u_i^N = \underset{x \in V(N)}{\text{maximum}} \sum_{i\in N} \lambda_i x_i,$$

Then u^N is a bargaining solution for the game V, (in the sense of Harsanyi [3]) if and only if

$$u^N = X^{\lambda}(\bar{Q}^N),$$

where X^{λ} is the λ-fair allocation rule for V and $\bar{Q}^N$ is the complete cooperation structure on N (see (6.6)).

Proof. In Harsanyi's bargaining model, each coalition S promises a dividend w_i^S to each player i, where $w_i^S = 0$ if $i \notin S$. These dividends must be constructed so that the allocation vector u^S is in $\partial V(S)$, where

$$u_i^S = \sum_{\substack{T\subseteq S \\ T\supseteq\{i\}}} w_i^T, \qquad \forall S\subseteq N,\ \forall i\in S. \tag{6.8}$$

If λ can be chosen so that

$$(6.9) \qquad \lambda_i w_i^S = \lambda_j w_j^S, \quad \forall S \subseteq N,\ \forall i\varepsilon S,\ \forall j\varepsilon S,$$

and

$$\sum_{i\varepsilon N} \lambda_i u_i^N = \underset{x\varepsilon V(N)}{\text{maximum}} \sum_i \lambda_i x_i,$$

then $u^N = (u_1^N,\ldots,u_n^N)$ is a bargaining solution (in the sense of Harsanyi) for our game V, with weights λ.

Harsanyi (page 210 in [3]) has pointed out that (6.8) is equivalent to

$$(6.10) \qquad w_i^S = \sum_{\substack{T \subseteq S \\ T \supseteq \{i\}}} (-1)^{|S|-|T|} u_i^T \qquad \forall S \subseteq N,\ \forall i\varepsilon S.$$

If i and j are distinct players in S, then (6.10) can be rewritten as

$$(6.11) \qquad w_i^S = \sum_{\substack{T \subseteq S \\ T \supseteq \{i,j\}}} (-1)^{|S|-|T|}(u_i^T - u_i^{T-j})$$

Using (6.8) and (6.11), it is then straightforward to show that (6.9) holds if and only if

$$(6.12) \qquad \lambda_i u_i^S - \lambda_i u_i^{S-j} = \lambda_j u_j^S - \lambda_j u_j^{S-i}, \qquad \forall S\subseteq N,\ \forall i\varepsilon S,\ \forall j\varepsilon S.$$

But by Lemma 6, (6.12) can hold if and only if

$$u_i^S = X_i^\lambda(\bar{Q}^S), \quad \forall S\subseteq N, \quad \forall i\varepsilon S.$$

So $u_i^N = X_i^\lambda(\bar{Q}^N)$, which proves the theorem.

Corollary Suppose that V is equivalent to a game v with sidepayments, in the sense that

$$V(S) = \{x \in \mathbb{R}^n \mid \sum_{i \in S} x_i \le v(S)\}, \quad \forall S \subseteq N.$$

Then $X(\bar{Q}^N)$ equals the Shapley value of v, when X is the fair allocation rule for V and $\bar{Q}^N$ is the complete cooperation structure.

Proof. It is well known that, for games with sidepayments, Harsanyi's bargaining solution coincides with the Shapley value, with all scale factors $\lambda_i=1$.

7. Example

Let us consider the following three-person game suggested by Roth [11]:

$$V(\{i\}) = \{(x_1, x_2, x_3) \mid x_i \le 0\}, \text{ for each } i$$

$$V(\{12\}) = \{(x_1, x_2, x_3) \mid x_1 \le \tfrac{1}{2}, x_2 \le \tfrac{1}{2}\}$$

$$V(\{13\}) = \{(x_1, x_2, x_3) \mid x_1 \le \tfrac{1}{4}, x_3 \le \tfrac{3}{4}\}$$

$$V(\{23\}) = \{(x_1, x_2, x_3) \mid x_2 \le \tfrac{1}{4}, x_3 \le \tfrac{3}{4}\}$$

$$V(\{123\}) = \left\{(x_1, x_2, x_3) \,\middle|\, \begin{array}{l} \text{there is some } (y_1, y_2, y_3) \text{ in the} \\ \text{convex hull of } \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{4}, 0, \frac{3}{4}) \ (0, \frac{1}{4}, \frac{3}{4})\} \\ \text{such that } x_i \le y_i \text{ for all } i \end{array}\right\}$$

(This game corresponds to Roth's V_p for $p=\frac{1}{4}$.) The fair allocation rule for this game is as follows.

Q	X(Q)
$\emptyset$	(0, 0, 0)
{{23}}	$(0, \frac{1}{4}, \frac{1}{4})$
{{13}}	$(\frac{1}{4}, 0, \frac{1}{4})$
{{12}}	$(\frac{1}{2}, \frac{1}{2}, 0)$
{{12}, {13}}	$(\frac{1}{2}, \frac{1}{4}, 0)$
{{12}, {23}}	$(\frac{1}{4}, \frac{1}{2}, 0)$
{{23}, {13}}	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$
{{12}, {13}, {23}}	$(\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$
{{123}}	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
{{23}, {123}}	$(\frac{1}{8}, \frac{3}{8}, \frac{3}{8})$
{{13}, {123}}	$(\frac{3}{8}, \frac{1}{8}, \frac{3}{8})$
{{12}, {123}}	$(\frac{1}{2}, \frac{1}{2}, 0)$
{{12}, {13}, {123}}	$(\frac{1}{2}, \frac{1}{4}, 0)$
{{12}, {23}, {123}}	$(\frac{1}{4}, \frac{1}{2}, 0)$
{{23}, {13}, {123}}	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$
{{12}, {13}, {23}, {123}}	$(\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$

If we let $\lambda=(1,1,1)$, then this is also the λ-fair allocation rule, and we have

$$\sum_{i\varepsilon N} \lambda_i X_i(\bar{Q}^N) = \frac{5}{12} + \frac{5}{12} + \frac{2}{12} = \max_{x\varepsilon V(N)} \sum_{i\varepsilon N} \lambda_i x_i .$$

Thus, $X(\bar{Q}^N) = (\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$ is a Harsanyi bargaining solution for this game.

Roth [11] has argued that the outcome of this game should be $(\frac{1}{2}, \frac{1}{2}, 0)$ because this is the best feasible allocation for players 1 and 2, and they can achieve it by themselves. However, when viewed in the context of the fair allocation rule, the solution $(\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$ is seen to have the stability property of Theorem 2, since this game is superadditive. For example, player 2 would certainly prefer to get $\frac{1}{2}$, which he gets from $\{\{12\}\}$, over $\frac{5}{12}$, which he gets from the complete cooperation structure $\bar{Q}^N$. However, if 2 refused to participate in conferences with player 3, while player 1 continued to confer with 3, then the cooperation structure would change from $\bar{Q}^N$ to $\{\{12\}, \{13\}\}$, so that player 2 would lose $\frac{2}{12} = \frac{5}{12} - \frac{1}{4}$. (Notice that, as fairness requires, player 3 also loses $\frac{2}{12}$ from this change of conference structure.) Thus, player 2 would not want to be first to break off relations with player 3. If players must make their conference participation plans noncooperatively, then both players 1 and 2 will want to confer with 3, and so the complete conference structure $\bar{Q}^N$ will form.

To check balanced contributions, for example, notice that

$$X_2(\bar{Q}^N) - X_2(\bar{Q}^N - {}^*3) = 5/12 - 1/2 = -1/12$$

$$X_3(\bar{Q}^N) - X_3(\bar{Q}^N - {}^*2) = 2/12 - 1/4 = -1/12$$

since $\bar{Q}^N - {}^*3 = \{\{12\}\}$ and $\bar{Q}^N - {}^*2 = \{\{13\}\}$. In this case, the contribution of 3 to 2 is negative, so we see that players may make negative contributions to each other in superadditive games.

One objection to our analysis might be that, for some conference structures, the fair allocation is only weakly Pareto-efficient. For example, the allocation $(\frac{1}{4}, 0, \frac{1}{4})$ for $Q=\{\{13\}\}$ is weakly dominated by $(\frac{1}{4}, 0, \frac{3}{4})$, which is feasible for the coalition $\{13\}$. However, the distinction between weak Pareto-efficiency and strong Pareto-efficiency is not robust with respect to small changes in the game, For example, if $V(\{13\})$ were enlarged to the comprehensive convex hull of

$$\{(\tfrac{1}{4}, 0, \tfrac{3}{4}), (\tfrac{1}{4}+\varepsilon, 0, 0), (0, 0, \tfrac{3}{4}+\varepsilon)\},$$

then $X(\{\{13\}\})$ would shift to $(\frac{1}{4} + \frac{2\varepsilon}{3+4\varepsilon}, 0, \frac{1}{4} + \frac{2\varepsilon}{3+4\varepsilon})$, which is on the strongly Pareto-efficient frontier of the new $V(\{13\})$ set, and which converges to $(\frac{1}{4}, 0, \frac{1}{4})$ as ε goes to zero. That is, if player 3's sacrifice below $\frac{3}{4}$ could make possible any infinitesimal gains for player 1 above $\frac{1}{4}$, then the fair allocation $X(\{\{13\}\})$ would change infinitesimally, but would become strongly efficient for $\{13\}$. Thus, the distinction between strong and weak efficiency is significant only if we can assume that there is really nothing which player 3 can offer player 1 above the $x_1=\frac{1}{4}$ level, no matter how much player 3 sacrifices.

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