OPTIMAL AUCTION DESIGN[1]

by

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Abstract. This paper considers the problem faced by a seller who has a single object to sell to one of several possible buyers, when the seller has imperfect information about how much the buyers might be willing to pay for the object. The seller's problem is to design an auction game which has a Nash equilibrium giving him the highest possible expected utility. Optimal auctions are derived in this paper for a wide class of auction design problems.

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1. Introduction

Consider the problem faced by someone who has an object to sell, and who does not know how much his prospective buyers might be willing to pay for the object. This seller would like to find some auction procedure which can give him the highest expected revenue or utility among all the different kinds of auctions known (progressive auctions, Dutch auctions, sealed bid auctions, discriminatory auctions, etc.). In this paper, we will construct such optimal auctions for a wide class of sellers' auction design problems. Although these auctions generally sell the object at a discount below what the highest bidder is willing to pay, and sometimes they do not even sell to highest bidder, we shall prove that no other auction mechanism can give higher expected utility to the seller.

To analyze the potential performance of different kinds of auctions, we follow Vickrey [11] and study the auctions as noncooperative games with imperfect information. (See Harsanyi [3] for more on this subject.) Noncooperative equilibria of specific auctions have been studied in several papers, such as Griesmer, Levitan, and Shubik [1], Ortega-Reichert [7], Wilson [12],[13]. Wilson [14] and Milgrom [5] have shown asymptotic optimality properties for sealed-bid auctions as the number of bidders goes to infinity. Harris and Raviv [2] have found optimal auctions for a class of symmetric two-bidder auction problems. Independent work on optimal auctions has also been done by Riley and Samuelson [8] and Maskin and Riley [4].
A general bibliogaraphy of the literature on competitive bidding has been collected by Rothkopf and Stark [10].

The general plan of this paper is as follows. Section 2 presents the basic assumptions and notation needed to describe the class of auction design problems which we will study. In Section 3, we characterize the set of feasible auction mechanisms and show how to formulate the auction design problem as a mathematical optimization problem. Two lemmas, needed to analyze and solve the auction design problem, are presented in Section 4. Section 5 describes a class of optimal auctions for auction design problems satisfying a regulatory condition. This solution is then extended to the general case in Section 6. In Section 7, an example is presented to show the kinds of counter-intuitive auctions which may be optimal when bidders' value estimates are not stochastically independent. A few concluding comments about implementation are put forth in Section 8.

2. Basic definitions and assumptions.

To begin, we must develop our basic definitions and assumptions, to describe the class of auction design problems which this paper will consider. We assume that there is one seller who has a single object to sell. He faces n bidders, or potential buyers, numbered 1,2,...,n. We let N represent the set of bidders, so that

$$N = \{1,\ldots,n\}. \tag{2.1}$$

We will use i and j to represent typical bidders in N.

The seller's problem derives from the fact that he does not know how much the various bidders are willing to pay for the object. That is, for each bidder i, there is some quantity t_i which is i's value estimate for the object, and which represents the maximum amount which i would be willing to pay for the object given his current information about i.

We shall assume that the seller's uncertainty about the value estimate of bidder i can be described by a continuous probability distribution over a finite interval. Specifically, we let a_i represent the lowest possible value which i might assign to the object; we let b_i represent the highest possible value which i might assign to the object; and we let $f_i:[a_i,b_i]\to\mathbb{R}_+$ be the probability density function for i's value estimate t_i. We assume that:

$$-\infty<a_i<b_i<+\infty;$$

$$f_i(t_i)>0,\ \forall t_i\in[a_i,b_i];$$

and $f_i(.)$ is a continuous function on $[a_i,b_i]$.

$F_i:[a_i,b_i]\to[0,1]$ will denote the cumulative distribution fuction corresponding to the density $f_i(.)$, so that

$$F_i(t_i) = \int_{a_i}^{t_i} f_i(s_i)ds_i. \tag{2.2}$$

Thus $F_i(t_i)$ is the seller's assessment of the probability that bidder i has a value estimate of t_i or less.

We will let T denote the set of all possible combinations of bidders' value estimates; that is

$$T = [a_1,b_1] \times \ldots \times [a_n,b_n]. \tag{2.3}$$

For any bidder i, we let T_{-i} denote the set of all possible combinations of value estimates which might be held by bidders other than i, so that

$$T_{-i} = \underset{\substack{j\in N \\ j\neq i}}{X} [a_j,b_j] \tag{2.4}$$

Until Section 7, we will assume that the value estimates of the n bidders are stochastically independent random variables. Thus, the joint density function on T for the vector

$$t = (t_1,\ldots,t_n)$$

of individual value estimates is

$$f(t) = \underset{j\in N}{\Pi} f_j(t_j) \tag{2.5}$$

Of course, bidder i considers his own value estimate to be a known quantity, not a random variable. However, we assume that bidder i assesses the probability distributions for the other bidders' value estimates in the same ways as the seller does. That is, both the seller and bidder i assess the joint density function on T_{-i} for the vector

$$t_{-i} = (t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n)$$

of values for all bidders other than i to be

$$(2.6) \qquad f_{-i}(t_{-i}) = \prod_{\substack{j \in N \\ j \neq i}} f_j(t_j).$$

The seller's personal value estimate for the object, if he were to keep it and not sell it to any of the n bidders, will be denoted by t_0. We assume that the seller has no private information about the object, so that t_0 is known to all the bidders.

There are two general reasons why one bidder's value estimates may be unknown to the seller and the other bidders. First, the bidder's personal preferences might be unknown to the other agents (for example, if the object is a painting, the others might not know how much he really enjoys looking at the painting). Second, the bidder might have some special information about the intrinsic quality of the object (he might know if the painting is an old master or a copy). We may refer to these two factors as preference uncertainty and quality uncertainty[1]. This distinction is very important. If there are only preference uncertainties, then informing bidder i about bidder j's value estimate should not cause i to revise his valuation. (This does not mean that i might not revise his bidding strategy in an auction if he knew j's value estimate; this means only that i's honest preferences for having money versus having the object should not change.) However, if there are quality uncertainties, then bidder i might tend to revise his valuation of the object after learning about other bidders' value estimates. That is, if i learned that t_j was very low, suggesting that j had received discouraging information about the quality of the object, then i might honestly revise downward his assessment of how much he should be willing to pay for the object.

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1. I am indebted to Paul Milgrom for pointing out this distinction.

In much of the literature on auctions (see [11], for example), only the special case of pure preference uncertainty is considered. In this paper, we shall consider a more general class of problems, allowing for certain forms of quality uncertainty as well. Specifically, we shall assume that there exist n revision effect functions $e_j:[a_i,b_i] \to \mathbb{R}$ such that, if another bidder i learned that t_j was j's value estimate for the object, then i would revise his own valuation by $e_j(t_j)$. Thus, if bidder i learned that $t = (t_1,\ldots,t_n)$ was the vector of value estimates initially held by the n bidders, then i would revise his own valuation of the object to

$$v_i(t) = t_i + \sum_{\substack{j\in N \\ j\neq i}} e_j(t_j). \tag{2.7}$$

Similarly, we shall assume that the seller would reassess his personal valuation of the object to

$$v_o(t) = t_o + \sum_{j\in N} e_j(t_j) \tag{2.8}$$

if he learned that t was the vector of value estimates initially held by the bidders. In the case of pure preference uncertainty, we would simply have $e_j(t_j) \equiv 0$.

(To justify our interpretation of t_i as i's initial estimate of the value of the object, we should assume that these revision effects have expected-value zero, so that

$$\int_{a_j}^{b_j} e_j(t_j) f_j(t_j) dt_j = 0. \tag{2.9}$$

However, this assumption is not actually necessary for any of the results in this paper; without it, only the interpretation of the t_i would change.)

3. Feasible auction mechanisms

Given the density functions f_i and the revision effect functions e_i and v_i as above, the seller's problem is to select an auction mechanism to maximize his own expected utility. We must now develop the notation to describe the auction mechanisms wh which he might select. To begin, we shall restrict our attention to a special class of auction mechanisms: the direct revelation mechanisms.

In a direct revelation mechanism, the bidders simultaneously and confidentially announce their value estimates to the seller; and the seller then determines who gets the object and how much each bidder must pay, as some functions of the vector of announced value estimates $t = (t_1,\ldots,t_n)$. Thus, a direct revelation mechanism is described by a pair of outcome functions (p,x) (of the form $p:T \to \mathbb{R}^n$ and $x:T \to \mathbb{R}^n$) such that, if t is the vector of announced value estimates then $p_i(t)$ is the probability that i gets the object and $x_i(t)$ is the expected amount of money which bidder i must pay to the seller. (Notice that we allow for the possibility that a bidder might have to pay something even if he does not get the object.)

We shall assume throughout this paper that the seller and the bidders are risk neutral and have additively separable utility functions for money and the object being sold. Thus, if bidder i knows that his value estimate is t_i, then his expected utility from an auction mechanism described by (p,x) is

$$U_i(p,x,t_i) = \int_{T_{-i}} (v_i(t)p_i(t)-x_i(t))f_{-i}(t_{-i})dt_{-i} \tag{3.1}$$

where $dt_{-i} = dt_1 \cdots dt_{i-1} dt_{i+1} \cdots dt_n$.

Similarly, the expected utility for the seller from this auction mechanism is

$$(3.2) \qquad U_o(p,x) = \int_T \left(v_o(t)\left(1 - \sum_{j\in N} p_j(t)\right) + \sum_{j\in N} x_j(t)\right) f(t)\,dt$$

where $dt = dt_1 \ldots dt_n$.

Not every pair of functions (p,x) represents a feasible auction mechanism, however. There are three types of constraints which must be imposed on (p,x).

First, since there is only one object to be allocated, the function p must satisfy the following probability conditions:

$$(3.3) \qquad \sum_{j\in N} p_j(t) \leq 1 \text{ and } p_i(t) \geq 0, \quad \forall i\in N, \quad \forall t\in T.$$

Second, we assume that the seller cannot force a bidder to participate in an auction which offers him less expected utility than he could get on his own. If he did not participate in the auction, the bidder could not get the object, but also would not pay any money, so his utility payoff would be zero. Thus, to guarantee that the bidders will participate in the auction, the following <u>individual-rationality</u> conditions must be satisfied:

$$(3.4) \qquad U_i(p,x,t_i) \geq 0, \quad \forall i\in N, \quad \forall t_i\in[a_i,b_i],$$

Third, we assume that the seller could not prevent any bidder from lying about his value estimate, if the bidder expected to gain

from lying. Thus the revelation mechanism can be implemented only if no bidder ever expects to gain from lying. That is, honest responses must form a Nash equilibrium in the auction game. If bidder i claimed that s_i was his value estimate when t_i was his true value estimate, then his expected utility would be

$$\int_{T_{-i}} (v_i(t)p_i(t_{-i},s_i) - x_i(t_{-i},s_i))f_{-i}(t_{-i})dt_{-i},$$

where $(t_{-i},s_i) = (t_1,\ldots,t_{i-1},s_i,t_{i+1},\ldots,t_n)$.

Thus, to guarantee that no bidder has any incentive to lie about his value estimate, the following incentive-compatibility conditions must be satisfied:

$$U_i(p,x,t_i) \geq \int_{T_{-i}} (v_i(t)p_i(t_{-i},s_i) - x_i(t_{-i},s_i)f_{-i}(t_{-i})dt_{-i} \tag{3.5}$$

$$\forall i \in N, \forall t_i \in [a_i,b_i], \quad \forall s_i \in [a_i,b_i].$$

We say that (p,x) is feasible (or that (p,x) represents a feasible auction mechanism) iff (3.3), (3.4), and (3.5) are all satisfied. That is, if the seller plans to allocate the object according to p and to demand monetary payments from bidders according to x, then the scheme can be implemented, with all bidders willing to participate honestly, if and only if (3.3)-(3.5) are satisfied.

Thus far,we have only considered direct revelation mechanisms, in which the bidders are supposed to honestly reveal their value estimates. However, the seller could design other kinds of auction games. In a general auction game, each bidder has some set of strategy options Θ_i; and there are outcome functions

$$\hat{p}:\Theta_1\times\ldots\times\Theta_n \to \mathbb{R}^n \text{ and } \hat{x}:\Theta_1\times\ldots\times\Theta_n \to \mathbb{R}^n ,$$

which described how the allocation of the object and the bidders' fees depend on the bidders' strategies. (That is, if $\theta = (\theta_1,\ldots,\theta_n)$ were the vector of strategies used by the bidder in the auction game, then $\hat{p}_i(\theta)$ would be the probability of i getting the object and $\hat{x}_i(\theta)$ would be the expected mayment from i to the seller.)

An auction mechanism is any such auction game together with a description of the strategic plans which the bidders are expected to use in playing the game. Formally, a strategic plan can be represented by a function $\hat{\theta}_i:[a_i,b_i] \to \Theta_i$, such that $\hat{\theta}_i(t_i)$ is the strategy which i is expected to use in the auction game if his value estimate is t_i. In this general notation, our direct revelation mechanisms are simply those auction mechanisms in which $\Theta_i = [a_i,b_i]$ and $\hat{\theta}_i(t_i) \equiv t_i$.

In this general framework, a feasible auction mechanism must satisfy constraints which generalize (3.3)-(3.5). Since there is only one object, the probabilities $\hat{p}_i(\theta)$ must be nonnegative and sum to one or less, for any θ. The auction mechanism must offer nonnegative expected utility to each bidder, given any possible

value estimate, or else he would not participate in the auction. The strategic plans must form a Nash equilibrium in the auction game, or else some bidder would revise his plans.

It might seem that problem of optimal auction design must be quite unmanageable, because there is no bound on the size or complexity of the strategy spaces Θ_i which the seller may use in constructing the auction game. The basic insight which enables us to solve auction design problems is that there is really no loss of generality in considering only direct revelation mechanisms. This follows from the following fact.

Lemma 1. (The revelation principle.) Given any feasible auction mechanism, there exists an equivalent feasible direct revelation mechanism which gives to the seller and all bidders the same expected utilities as in the given mechanism.

This revelation principle has been proven in the more general context of Bayesian collective choice problems, as Theorem 2 in [5]. To see why it is true, suppose that we are given a feasible auction mechanism with arbitrary stragegy spaces Θ_i, with outcome functions $\hat{p}$ and $\hat{x}$, and with strategic plans $\hat{\theta}_i$, as above. Then consider the direct revelation mechanism represented by the functions $p:T \to \mathbb{R}^n$ and $x:T \to \mathbb{R}^n$ such that

$$p(t_1,\ldots,t_n) = \hat{p}(\hat{\theta}_1(t_1),\ldots, \hat{\theta}_n(t_n)),$$

$$x(t_1,\ldots,t_n) = \hat{x}(\hat{\theta}_1(t_1),\ldots, \hat{\theta}_n(t_n)).$$

That is, in the direct revelation mechanism (p,x), the seller first asks each bidder to announce his type, and then computes the strategy which the bidder would have used according to the strategic plans in the given auction mechanism, and finally implements the outcomes prescribed in the given auction game for these strategies. Thus, the direct revelation mechanism (p,x) always yields the same outcomes as the given auction mechanism, so all agents get the same expected utilites in both mechanisms. And (p,x) must satisfy the incentive-compatibility constraints (3.5), because the strategic plans formed an equilibrium in the given feasible mechanism. (If any bidder could gain by lying to the seller in the revelation game, then he could have gained by "lying to himself" or revising his strategic plan in the given mechanism.) Thus, (p,x) is feasible.

Using the revelation principle, we may assume, without loss of generality, that the seller only considers auction mechanisms in the class of feasible direct revelation mechanisms. That is, we may henceforth identify the set of feasible auction mechanisms with the set of all outcome functions (p,x) which satisfy the constraints (3.3) through (3.5). The seller's auction design problem is to choose these functions $p:T\to\mathbb{R}^n$ and $x:T\to\mathbb{R}^n$ so as to maximize $U_0(p, x)$ subject to (3.3) - (3.5).

Notice that we have not used (2.7) or (2.8) anywhere in this section. Thus, (3.3) - (3.5) characterize the set of all feasible auction mechanisms even when the bidders compute their revised valuations $v_i(t)$ using functions $v_i:T\to\mathbb{R}$, which are not of the special additve form (2.7). However, in the next three sections,

to derive an explicit solution to the problem of optimal auction design, we shall have to restrict our attention to the class of problems in which (2.7) and (2.8) hold.

4. Analysis of the problem.

Given an auction mechanism (p,x) we define

$$(4.1) \qquad Q_i(p,t_i) = \int_{T_{-i}} p_i(t) f_{-i}(t_{-i}) dt_{-i}$$

for any bidder i and any value estimate t_i. So $Q_i(p,t_i)$ is the conditional probability that bidder i will get the object from the auction mechanism (p,x) given that his value extimate is t_i.

Out first result is a simplified characterization of the feasible auction mechanisms.

Lemma 2. (p,x) is feasible if and only if the following conditions hold:

$$(4.2) \quad \text{if } s_i \leq t_i \text{ then } Q_i(p,s_i) \leq Q_i(p,t_i), \quad \forall i \in N, \forall s_i, t_i \in [a_i,b_i];$$

$$(4.3) \quad U_i(p,x,t_i) = U_i(p,x,a_i) + \int_{a_i}^{t_i} Q_i(p,s_i) ds_i, \quad \forall i \in N, \forall t_i \in [a_i,b_i];$$

$$(4.4) \quad U_i(p,x,a_i) \geq 0, \quad \forall i \in N; \quad \text{and}$$

$$(3.3) \quad \sum_{j \in N} p_j(t) \leq 1 \quad \text{and} \quad p_i(t) \geq 0, \quad \forall i \in N, \quad \forall t \in T.$$

Proof. Using (2.8), our special assumption about the form of $v_i(t)$, we get

$$\int_{T_{-i}} (v_i(t) p_i(t_{-i},s_i) - x_i(t_{-i},s_i)) f_{-i}(t_{-i}) dt_{-i}$$

$$= \int_{T_{-i}} ((v_i(t_{-i},s_i) + (t_i - s_i))(p_i(t_{-i},s_i) - x_i(t_{-i},s_i)) f_{-i}(t_{-i}) dt_{-i}$$

$$= U_i(p,x,s_i) + (t_i - s_i) Q_i(p,s_i).$$

Thus, the incentive-compatibility constraint (3.5) is equivalent to

$$(4.6)\quad U_i(p,x,t_i) \geq U_i(p,x,s_i) + (t_i-s_i)Q_i(p,s_i),\ \forall i\in N,\ \forall t_i,s_i\in[a_i,b_i].$$

Thus (p,x) is feasible if and only if (3.3), (3.4), and (4.6) hold. We will now show that (3.4) and (4.6) imply (4.2)-(4.4).

Using (4.6) twice (once with the roles of s_i and t_i switched), we get

$$(t_i-s_i)Q_i(p,s_i) \leq U_i(p,x,t_i) - U_i(p,x,s_i)$$

$$\leq (t_i-s_i)Q_i(p,t_i).$$

Then (4.2) follows, when $s_i \leq t_i$.

These inequalities can be rewritten for any $\delta > 0$

$$Q_i(p,s_i)\delta \leq U_i(p,x,s_i+\delta)-U_i(p,x,s_i)$$

$$\leq Q_i(p,s_i+\delta)\delta.$$

Since $Q_i(p,s_i)$ is increasing in s_i, it is Riemann integrable. So:

$$\int_{a_i}^{t_i} Q_i(p,s_i)ds_i = U_i(p,x,t_i)-U_i(p,x,a_i).$$

Which gives us (4.3).

Of course, (4.4) follows directly from (3.4), so all the conditions in Lemma 2 follow from feasibility.

New we must show that the conditions in Lemma 2 also imply (3.4) and (4.6).

Since $Q_i(p,s_i) \geq 0$ (by (3.3)), (3.4) follows from (4.3) and (4.4).

To show (4.6), suppose $s_i \leq t_i$; then (4.2) and (4.3) give us:

$$U_i(p,x,t_i) = U_i(p,x,s_i) + \int_{s_i}^{t_i} Q_i(p,r_i)\,dr_i$$

$$\geq U_i(p,x,s_i) + \int_{s_i}^{t_i} Q_i(p,s_i)\,dr_i$$

$$= U_i(p,x,s_i) + (t_i - s_i)Q_i(p,s_i).$$

Similarly, if $s_i > t_i$ then

$$U_i(p,x,t_i) = U_i(p,x,s_i) - \int_{t_i}^{s_i} Q_i(p,r_i)\,dr_i$$

$$\geq U_i(p,x,s_i) - \int_{t_i}^{s_i} Q_i(p,s_i)\,dr_i$$

$$= U_i(p,x,s_i) + (t_i - s_i)Q_i(p,s_i).$$

Thus (4.6) follows from (4.2) and (4.3). So the conditions in Lemma 2 also imply feasibility. This proves the lemma.

So (p,x) represents an optimal auction if and only if it maximizes $U_o(p,x)$ subject to (4.2)-(4.4) and (3.3). Our next lemma offers some simpler conditions for optimality.

Lemma 3. Suppose that $p: T \to \mathbb{R}^n$ maximizes

$$(4.7) \quad \int_T \Big(\sum_{i \in N} \Big(t_i - e_i(t_i) - \frac{1-F_i(t_i)}{f_i(t_i)} - t_o \Big) p_i(t) \Big) f(t)\,dt$$

subject to the constraints (4.2) and (3.3). Suppose also that

$$(4.8) \quad x_i(t) = p_i(t)v_i(t) - \int_{a_i}^{t_i} p_i(t_{-i}, s_i)\,ds_i, \quad \forall i \in N, \quad \forall t \in T.$$

Then (p,x) represents an optimal auction.

Proof. Recalling (3.2), we may write the seller's objective function as

$$(4.9)\quad U_o(p,x) = \int_T v_o(t)f(t)dt + \sum_{i\in N} \int_T p_i(t)(v_i(t)-v_o(t))f(t)dt$$
$$+ \sum_{i\in N} \int_T (x_i(t)-p_i(t)v_i(t))f(t)dt.$$

But, using Lemma 2, we know that for any feasible (p,x):

$$(4.10)\quad \int_T (x_i(t)-p_i(t)v_i(t))f(t)dt$$
$$= -\int_{a_i}^{b_i} U_i(p,x,t_i)f_i(t_i)dt_i$$
$$= -\int_{a_i}^{b_i} (U_i(p,x,a_i)+\int_{a_i}^{t_i} Q_i(p,s_i)ds_i)f_i(t_i)dt_i$$
$$= -U_i(p,x,a_i)-\int_{a_i}^{b_i}\int_{s_i}^{b_i} f_i(t_i)Q_i(p,s_i)dt_ids_i$$
$$= -U_i(p,x,a_i)-\int_{a_i}^{b_i} (1-F_i(s_i))Q_i(p,s_i)ds_i$$
$$= -U_i(p,x,a_i)-\int_T (1-F_i(t_i))p_i(t)f_{-i}(t_{-i})dt.$$

From (2.7) and (2.8) we get

$$(4.11)\quad v_i(t)-v_o(t) = t_i-t_o-e_i(t_i).$$

Substituting (4.10) and (4.11) into (4.9) gives us:

$$(4.12)\quad U_o(p,x) = \int_T (\sum_{i\in N}(t_i-t_o-e_i(t_i)-\frac{1-F_i(t_i)}{f_i(t_i)})p_i(t))f(t)dt$$
$$+ \int_T v_o(t)f(t)dt - \sum_{i\in N} U_i(p,x,a_i).$$

So the seller's problem is to maximize (4.12) subject to the constraints (4.2), (4.3), (4.4), and (3.3) from Lemma 2. In this formulation, x appears only in the last term of the objective function and in the constraints (4.3) and (4.4). These two constraints may be rewritten as

$$\int_{T_{-i}} (p_i(t)v_i(t) - \int_{a_i}^{t_i} p_i(t_{-i},s_i)ds_i - x_i(t))f_{-i}(t_{-i})dt_{-i}$$

$$= U_i(p,x,a_i) \geq 0, \ \forall i \in N, \ \forall t_i \in [a_i,b_i].$$

If the seller chooses x according to (4.8), then he satisfies both (4.3) and (4.4), and he gets

$$\sum_{i \in N} U_i(p,x,a_i) = 0,$$

which is the best possible value for this term in (4.12).

Thus using (4.8), we can drop x from the seller's problem entirely. Furthermore, the second term on the right side of (4.12) is a constant, independent of (p,x). So the objective function can be simplified to (4.7), and (4.2) and (3.3) are the only constraints left to be satisfied. This completes the proof of the lemma.

Equation (4.12) also has some important implications which are worth mentioning. This equation tells us that the seller's expected utility from an auction mechanism is completely determined by the probability function p (which tells us who gets the object in each possible situation) and by the numbers $U_i(p,x,a_i)$ (which tell us how much expected utility each bidder

would enjoy if his value estimate were at its lowest possible level). Thus, for example, the seller must get the same expected utility from any two auction mechanisms which have the properties that (1) the object always goes to the bidder with the highest value estimate above t_o and (2) every bidder would expect zero utility if his value estimate were at its lowest possible level. If the bidders are symmetric and all $e_i=0$ and $a_i=0$, then the Dutch auctions and progressive auctions studied in [11] both have these two properties, so Vickrey's equivalence results may be viewed as a corollary of our equation (4.12). However, we shall see that Vickrey's auctions are not in general optimal for the seller.

5. Optimal auctions in the regular case.

With a simple regularity assumption, we can compute optimal auction mechanisms directly from Lemma 3.

We may say that our problem is regular if the function

$$(5.1) \qquad c_i(t_i) = t_i - e_i(t_i) - \frac{1-F_i(t_i)}{f_i(t_i)}$$

is a monotone strictly increasing function of t_i, for every i in N. That is, the problem is regular if $c_i(s_i)<c_i(t_i)$ whenever $a_i \leq s_i < t_i \leq b_i$. (Recall that we are assuming $f_i(t_i)>0$ for all t_i in $[a_i,b_i]$, so that $c_i(t_i)$ is always well-defined and continuous.)

Now consider an auction mechanism in which the seller keeps the object if $t_o > \max_{i\in N} (c_i(t_i))$, and he gives it to the bidder with the highest $c_i(t_i)$ otherwise. If $c_i(t_i)=c_j(t_j) = \max_{k\in N} (c_k(t_k)) \geq t_o$, then the seller may break the tie by giving to the lower-numbered player, or by some other arbitrary rule. (Ties will only happen with probability zero in the regular case.) Thus, for this auction mechanism,

$$(5.2) \quad p_i(t)>0 \text{ implies } c_i(t_i) = \max_{j\in N} (c_j(t_j)) \geq t_o.$$

For all t in T, this mechanism maximizes the sum

$$\sum_{i\in N} (c_i(t_i)-t_o)p_i(t)$$

subject to the constraints that

$$\sum_{j\in N} p_j(t) \leq 1 \text{ and } p_i(t) \geq 0, \ \forall i.$$

Thus p maximizes (4.7) subject to the probability condition (3.3). To check that it also satisfies (4.2) we need to use regularity. Suppose $s_i<t_i$. Then $c_i(s_i)<c_i(t_i)$, and so whenever bidder i could win the object by submitting a value estimate of s_i, he could also win if he changed to t_i. That is, $p_i(t_{-i},s_i) \leq p_i(t_{-i},t_i)$, for all t_{-i}. So $Q_i(p,t_i)$, the probability of i winning the object given that t_i is his value estimate, is indeed an increasing function of t_i, as (4.2) requires. Thus p satisfies all the condition of Lemma 3.

To complete the construction of our optimal auction, we let x be as in (4.8):

$$x_i(t) = p_i(t)(t_i + \sum_{\substack{j\in N \\ j\neq i}} e_j(t_j)) - \int_{a_i}^{t_i} p_i(t_{-i},s_i)ds_i.$$

This formula may be rewritten more intuitively, as follows. For any vector t_{-i} of value estimates from bidders other than i, let

$$(5.3) \qquad z_i(t_{-i}) = \inf\{s_i \mid c_i(s_i)\geq t_o \text{ and } c_i(s_i)\geq c_j(t_j), \forall j\neq i\}.$$

Then $z_i(t_{-i})$ is the infimum of all winning bids for i against t_{-i}; so

$$(5.4) \qquad p_i(t_{-i},s_i) = \begin{cases} 1 & \text{if } s_i>z_i(t_{-i}), \\ 0 & \text{if } s_i<z_i(t_{-i}). \end{cases}$$

This gives us

$$(5.5)\qquad \int_{a_i}^{t_i} p_i(t_{-i},s_i)ds_i = \begin{cases} t_i - z_i(t_{-i}) & \text{if } t_i \geq z_i(t_{-i}), \\ 0 & \text{if } t_i < z_i(t_{-i}). \end{cases}$$

Finally, (4.8) becomes

$$(5.6)\qquad x_i(t) = \begin{cases} z_i(t_{-i}) + \sum\limits_{\substack{j\in N \\ j\neq i}} e_j(t_j) & \text{if } p_i(t)=1, \\ 0 & \text{if } p_i(t)=0. \end{cases}$$

That is, bidder i must pay only when he gets the object, and then he pays $v_i(t_{-i},z_i(t_{-i}))$, the amount which the object would have been worth to him if he had submitted his lowest possible winning bid.

If all the revision effect functions are identically zero (that is, $e_i(t_i)=0$), and if all bidders are symmetric ($a_i=a_j$, $b_i=b_j$, $f_i(.)=f_j(.)$) and regular, then we get

$$(5.6)\qquad z_i(t_{-i}) = \max\{c_i^{-1}(t_o), \max_{j\neq i} t_j\}.$$

That is, our optimal auction becomes a modified Vickrey auction [11], in which the seller himself submits a bid equal to $c_i^{-1}(t_o)$ (notice that all $c_i=c_j$ in this symmetric case, and regularity guarantees that c_i is invertible) and then sells the object to the highest bidder at the second highest price. This conclusion only holds, however, when the bidders are symmetric and the $c_i(.)$ functions are strictly increasing.

For example, suppose $t_o=0$, each $a_i=0$, $b_i=100$, $e_i(t_i)=0$, and $f_i(t_i) = \frac{1}{100}$, for every i and every t_i between 0 and 100. Then straightforward computations give us $c_i(t_i) = 2t_i-100$, which is increasing in t_i. So the seller should sell to the highest bidder at the second highest price, except that he himself should submit a bid of $c_i^{-1}(0) = \frac{0+100}{2} = 50$. By announcing a reservation price of 50, the seller risks a probability $(1/2)^n$ of keeping the object even though some bidder is willing to pay more than t_o for it; but the seller also increases his expected revenue, because he can command a higher price when the object is sold.

Thus the optimal auction may not be expost efficent. To see more clearly why this can happen, consider the example in the above paragraph, for the case when n=1. Then the seller has value estimate $t_o=0$, and the one bidder has a value estimate taken from a uniform distribution on $[0,100]$. Ex post efficiency would require that the bidder must always get the object, as long as his value estimate is positive. But then the bidder would never admit to more than an infinitesmal value estimate, since any positive bid would win the object. So the seller would have to expect zero revenue if he never kept the object. In fact, the seller's optimal policy is to refuse to sell the object for less than 50, which gives him expected revenue 25.

More generally, when the bidders are asymmetric, the optimal auction may sometimes even sell to a bidder whose value estimate is not the highest. For example, when $e_i(t_i)=0$ and $f_i(t_i)= \frac{1}{b_i-a_i}$ for all t_i between a_i and b_i, (the general uniform-distribution case with no revision effects) we get

$$c_i(t_i) = 2t_i-b_i,$$

which is increasing in t_i. So in the optimal auction, the bidder with the highest $c_i(t_i)$ will get the object. If $b_i<b_j$, then i may win the object even if $t_i<t_j$, as long as $2t_i-b_i>2t_j-b_j$. In effect, the optimal auction descriminates against bidders for whom the upper bounds on the value estimates are higher. This descrimination discourages such bidders from under-representing value estimates close to their high b_j bounds.

6. Optimal auctions in the general case.

Without regularity, the auction mechanism proposed in the preceeding section would not be feasible, since it would violate (4.2). To extend our solution to the general case, we need some carefully chosen definitions.

The cumulative distribution function $F_i:[a_i,b_i]\to[0,1]$ for bidder i is continuous and strictly increasing, since we assume that the density function f_i is always strictly positive. Thus $F_i(.)$ has an inverse $F_i^{-1}:[0,1]\to[a_i,b_i]$, which is also continuous and strictly increasing.

For each bidder i, we now define four functions which have the unit interval [0,1] as their domain. First, for any q in [0,1], let

$$(6.1)\qquad h_i(q) = F_i^{-1}(q) - e_i(F_i^{-1}(q)) - \frac{1-q}{f_i(F_i^{-1}(q))}$$
$$= c_i(F_i^{-1}(q)),$$

and let

$$(6.2)\qquad H_i(q) = \int_0^q h_i(r)\,dr.$$

Next let $G_i:[0,1]\to\mathbb{R}$ be the convex hull of the function $H_i(.)$; in the notation of Rockafellar ([9], page 36)

$$(6.3)\qquad G_i(q) = \operatorname{conv} H_i(q)$$
$$= \min\left\{\omega H_i(r_1)+(1-\omega)H_i(r_2) \;\middle|\; \begin{matrix}\{\omega,r_1,r_2\}\subset[0,1] \\ \text{and } \omega r_1+(1-\omega)r_2=q\end{matrix}\right\}$$

That is, $G_i(.)$ is the highest convex function on $[0,1]$ such that $G_i(q) \leq H_i(q)$ for every q.

As a convex function, G_i is continuously differentiable except at countably many points, and its derivative is monotone increasing. We define $g_i:[0,1] \to \mathbb{R}$ so that

$$(6.4) \qquad g_i(q) = G_i'(q)$$

whenever this derivative is defined, and we extend $g_i(.)$ to all of $[0,1]$ by right-continuity.

We define $\bar{c}_i:[a_i,b_i] \to \mathbb{R}$ so that

$$(6.5) \qquad \bar{c}_i(t_i) = g_i(F_i(t_i)).$$

(It is straightforward to check that, in the regular case when $c_i(.)$ is increasing, we get $G_i=H_i, g_i=h_i$, and $\bar{c}_i=c_i$).

Finally, for any vector of value estimates t, let $M(t)$ be the set of bidders for whom $\bar{c}_i(t_i)$ is maximal among all bidders and is higher than t_o.

$$(6.6) \qquad M(t) = \{i \mid t_o \leq \bar{c}_i(t_i) = \max_{j \in N} \bar{c}_j(t_j)\}.$$

We can now state our main result: that in an optimal auction, the object should always be sold to the bidder with the highest $\bar{c}_i(t_i)$, provided this is not less than t_o. Thus, we may think of $\bar{c}_i(t_i)$ as the <u>priority level</u> for bidder i when his value estimate is t_i, in the seller's optimal auction.

Theorem: Let $\bar{p}:T\to\mathbb{R}^n$ and $\bar{x}:T\to\mathbb{R}^n$ satisfy

$$(6.7)\quad \bar{p}_i(t) = \begin{cases} 1/|M(t)| & \text{if } i\in M(t), \\ 0 & \text{if } i\notin M(t), \end{cases}$$

and

$$(6.8)\quad \bar{x}_i(t) = \bar{p}_i(t)v_i(t) - \int_{a_i}^{t_i} \bar{p}_i(t_{-i},s_i)\,ds_i$$

for all i in N and t in T. Then $(\bar{p},\bar{x})$ represents an optimal auction mechanism.

Proof: First, using integration by parts, we derive the following equations.

$$(6.9)\quad \int_T (h_i(F_i(t_i))-g_i(F_i(t_i)))p_i(t)f(t)\,dt$$

$$= \int_{a_i}^{b_i} (h_i(F_i(t_i))-g_i(F_i(t_i)))Q_i(p,t_i)f_i(t_i)\,dt_i$$

$$= ((H_i(F_i(t_i))-G_i(F_i(t_i)))Q_i(p,t_i))\Big|_{t_i=a_i}^{b_i}$$

$$-\int_{t_i=a_i}^{b_i} (H_i(F_i(t_i))-G_i(F_i(t_i)))\,dQ_i(p,t_i).$$

But G_i is the convex hull of H_i on $[0,1]$ and H_i is continuous, so $G_i(0)=H_i(0)$ and $G_i(1)=H_i(1)$. Thus the endpoint terms in the last expression above are zero.

Now, recall the maximand (4.7) in Lemma 3. Using (6.9) we get:

$$(6.10)\qquad \int_T \sum_{i\in N} \left(t_i - e_i(t_i) - \frac{1-F_i(t_i)}{f_i(t_i)} - t_o\right) p_i(t) f(t)\,dt$$

$$= \int_T \sum_{i\in N} \left(h_i(F_i(t_i)) - t_o\right) p_i(t) f(t)\,dt$$

$$= \int_T \sum_{i\in N} \left(\bar{c}_i(t_i) - t_o\right) p_i(t) f(t)\,dt$$

$$+ \sum_{i\in N} \int_T \left(h_i(F_i(t_i)) - g_i(F_i(t_i))\right) p_i(t) f(t)\,dt$$

$$= \int_T \left(\sum_{i\in N} \left(\bar{c}_i(t_i) - t_o\right) p_i(t)\right) f(t)\,dt$$

$$- \sum_{i\in N} \int_{t_i=a_i}^{b_i} \left(H_i(F_i(t_i)) - G_i(F_i(t_i))\right) dQ_i(p,t_i).$$

Now consider $(\bar{p},\bar{x})$ as defined in the theorem. Observe that $\bar{p}$ always puts all probability on bidders for whom $(\bar{c}_i(t_i)-t_o)$ is nonnegative and maximal.
Thus, for any p satisfying (3.3):

$$(6.11)\qquad \int_T \left(\sum_{i\in N} \left(\bar{c}_i(t_i) - t_o\right) \bar{p}_i(t) f(t)\,dt \right.$$

$$\geq \int_T \left(\sum_{i\in N} \left(\bar{c}_i(t_i) - t_o\right) p_i(t) f(t)\,dt. \right.$$

Of course $\bar{p}$ itself does satisfy the probability condition (3.3).

For any p which satisfies (4.2) (that is, for which $Q_i(p,t_i)$ is an increasing function of t_i), we must have

$$(6.12) \qquad \int_{t_i=a_i}^{b_i} (H_i(F_i(t_i))-G_i(F_i(t_i)))dQ_i(p,t_i) \geq 0$$

since $H_i \geq G_i$.

To see that $\bar{p}$ satisfies (4.2), observe first that $\bar{c}_i(t_i)$ is an increasing function of t_i, because F_i and g_i are both increasing functions. Thus $\bar{p}_i(t)$ is increasing as a function of t_i, for any fixed t_{-i}, and so $Q_i(\bar{p},t_i)$ is also an increasing function of t_i. So $\bar{p}$ satisfies (4.2).

Since G is the convex hull of H, we know that G must be flat whenever $G < H$; that is,
if $G_i(r)<H_i(r)$ then $g_i'(r) = G_i''(r) = 0$.
So if $H_i(F_i(t_i))-G_i(F_i(t_i))>0$ then $\bar{c}_i(t_i)$ and $Q_i(\bar{p},t_i)$ are constant in some neighborhood of t_i. This implies that

$$(6.13) \qquad \int_{t_i=a_i}^{b_i} (H_i(F_i(t_i))-G_i(F_i(t_i)))dQ_i(\bar{p},t_i)=0.$$

Substituting (6.11), (6.12), and (6.13) back into (6.10), we can see that $\bar{p}$ maximizes (4.7) subject to (4.2) and (3.3). This fact, together with Lemma 3, proves the Theorem.

To get some practical interpretation for these important $\bar{c}_i$. functions, consider the special case of n=1; that is, suppose there is only one bidder. Then our optimal auction becomes:

$$\bar{p}_1(t_1) = \begin{cases} 1 & \text{if } \bar{c}_1(t_1) \geq t_o, \\ 0 & \text{if } \bar{c}_1(t_i) < t_o, \end{cases}$$

$$\bar{x}_1(t_1) = \bar{p}_1(t_1) \cdot \min\{s_1 | \bar{c}_1(s_1) \geq t_o\}.$$

That is, the seller should offer to sell the object at the price

$$\bar{c}_1^{-1}(t_o) = \min \{s_1 | \bar{c}_1(s_1) \geq t_o\};$$

and he should keep the object if the bidder is unwilling to pay this price.

Thus, if bidder i were the only bidder, then the seller would sell the object to i if and only if $\bar{c}_i(t_i)$ were greater than or equal to t_o. In other words, $\bar{c}_i(t_i)$ is the highest level of t_o, the seller's personal value estimate, such that the seller would sell the object to i at a price of t_i or lower, if all other bidders were removed.

7. The independence assumption

Throughout this paper we have asssumed that the bidders' value estimates are stochastically independent. Independence is a strong assumption, so we now consider an example to show what optimal actions may look like when value estimates are not independent.

For simplicity, we consider a discrete example. Suppose there are two bidders, each of whom may have a value estimate of $t_i=10$ or $t_i=100$ for the object. Let us assume that the joint probability distribution for value estimates (t_1,t_2) is:

$$\Pr(10,10) = \Pr(100,100) = \frac{1}{3},$$

$$\Pr(10,100) = \Pr(100,10) = \frac{1}{6}.$$

Obviously the two value estimates are not independent. Let us also assume that there are no revision effects ($e_i=0$), and $t_o=0$.

Now consider the following auction mechanism. If both bidders have high value estimates ($t_1=t_2=100$), then sell the object to one of them for price 100, randomizing equally to determine which bidder buys the object. If one bidder has a high value estimate (100) and the other has a low value estimate (10), then sell the object to the high bidder for 100, and charge the low bidder 30 (but give him nothing). If both bidders have low value estimates (10), then give 15 units of money to one of them, and give 5 units of money and the object to the other, again choosing the recipient of the object at random.

The outcome functions (p,x) of this auction mechanism are:

$$p(100,100) = (\tfrac{1}{2}, \tfrac{1}{2}) = p(10,10),$$

$$p(10,100) = (0,1),\ p(100,10) = (1,0),$$

$$x(100,100) = (50,50),\ x(10,10) = (-10,-10),$$

$$x(10,100) = (30,100),\ x(100,10) = (100,30).$$

This may seem like a very strange auction, but in fact it is optimal. It is straightforward to check that honesty is a Nash equilibrium in this auction game, in that neither bidder has any incentive to misrepresent his value estimate if he expects the other bidder to be honest. Furthermore, the object is always delivered to a bidder who values it most highly; and yet each bidders' expected utility from this auction mechanism is zero, whether his value is high or low. So this auction mechanism is feasible and it allows the seller to exploit the entire value of the object from the bidders. Thus this is an optimal auction mechanism, and it gives the seller expected revenue

$$U_o(p,x) = \tfrac{1}{3}(100) + \tfrac{1}{6}(130) + \tfrac{1}{6}(130) + \tfrac{1}{3}(-20) = 70.$$

To see why this auction mechanism works so well, observe that the seller is really doing two things. First, he is selling the object to one of the highest bidders at the highest bidders' value estimate. Second, if a bidder says his value estimate is equal to 10, then that bidder is forced to accept a side-bet of the following form: "pay 30 if the other bidder's value is 100, get 15 if the other bidder's value is 10". This side-bet

has expected value 0 to a bidder whose value estimate is truly 10, since then the conditional probability is 1/3 that the other has value 100 and 2/3 that the other has value 10. But if a bidder were to lie and claim to have value estimate 10, when 100 was his true value estimate, then this side-bet would have expected value $\frac{2}{3}(-30)+\frac{1}{3}(10) = -\frac{50}{3}$ for him (since he would now assess conditional probabilities $\frac{2}{3}$ and $\frac{1}{3}$ respectively for the events that his competitor had value estimate 100 and 10). This negative expected value of the side-bet for a lying bidder exactly counterbalances the temptation to misrepresent in order to buy the object at a lower price.

These side-bets were not possible in the independent case, because each bidders' condition probability distribution over the others' value estimates was constant. But in the general non-independent case, we may expect that this side-bet phenomenon will commonly arise. That is, the seller can exploit the full value of the object by always selling to the highest bidder at the highest bidders' valuation, and then by setting up side-bets which have zero expected value if a bidder is honest but have negative expected value if he lies. If the side-bets are carefully designed, they can counterbalance the incentive to lie to buy the object at a lower price.

Of course, we have made heavy use of the risk-neutrality assumption in this analysis. For risk-averse bidders, the optimal auctions might be somewhat less extreme. Also, the auction game suggested in our example has an unfortunate second equilibrium in which both bidders always claim to be of the low type, although other optimal auction mechanisms can be designed in which the honest equilibrium is unique. (E.g., change x to:

$x(100,100)=(100,100)$, $x(10,10)=(-15,-15)$,

$x(10,100)=(40,0)$, $x(100,10)=(0,40)$;

keeping p as above.)

One might ask whether there are any optimal auctions for our example which do not have this strange property of sometimes telling the seller to pay money to the bidders. The answer is No; if we add the constraint that the seller should never pay money to the bidders (that is, all $x_i(t) \geq 0$), then no feasible auction mechanism gives the seller expected utility higher than $66\frac{2}{3}$. To prove this fact, observe that the auction design problem is a linear programming problem when the number of possible value estimates is finite, as in this example. The objective function in the problem is $U_o(p,x)$, which is linear in p and x. As in Section 2, the feasibility constraints are of three types: probability constraints ($p_i(t) \geq 0, \sum_i p_i(t) \leq 1$), individual-rationality constraints ($U_i(p,x,t_i) \geq 0$), and incentive-compatibility constraints (that $U_i(p,x,t_i)$ must be greater than or or equal to the utility which i would expect from acting as if s_i were his value estimate when t_i was true). All of these constraints are linear in p and x. So we get a linear programming problem, and for our example its optimal value is 70, with the optimal solution shown above. But if we add the constraints $x_i(t) \geq 0$ for all i and t, then the optimal value drops to $66\frac{2}{3}$, for this example. To attain this "second-best" value of $66\frac{2}{3}$ with nonnegative x, the seller should keep the object if if $t_1 = t_2 = 10$, and otherwise the seller should sell the object to a high bidder for 100.

8. Implementation

A few remarks about the implementability of our optimal auctions should now be made. Once the f_i and e_i functions have been specified, the only computations necessary to implement our optimal auction are to compute the $\bar{c}_i$ functions and to evaluate (6.8). But these are all straightforward one-dimensional problems. The equilibrium strategies for the bidders are also easy to compute in our optimal auction, since each bidder's optimal strategy is to simply reveal his true value estimate.

In terms of sensitivity analysis, notice that (6.8) guarantees that our auction mechanism $(\bar{p},\bar{x})$ will be feasible, and yet the densities f_i do not appear in (6.8). So our optimal auction will satisfy the individual-rationality and incentive-compatibility constraints ((3.4) and (3.5)) even if the density functions are misspecified from the point of view of the bidders. However the revision-effect functions e_i do appear in (6.8) (through v_i), so if there are errors in specifying the e_i functions then bidders may have incentive to bid dishonestly in the auction we compute.

In general, we must recognize that an auction design problem must be treated like any problem of decision-making under uncertainty. No auction mechanism can guarantee to the seller the full realization of his object's value under all circumstances. Thus, the seller must make his best assessment of the probabilities and choose the auction design which offers him the highest expected utility, on average. The usual "garbage-in, garbage-out" warning must apply here, as in all operations research, but careful use of models and sensitivity analysis should enable a seller to improve his average revenues with optimally designed auctions.

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