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PATH INDEPENDENT CHOICES

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Abstract. The choice functions considered here are rules for selecting a probability distribution (or other types of convex combinations) for every finite non-empty set of alternatives. Path-independence means $c(S \cup T) = c(\{c(S), c(T)\})$ for all disjoint pairs of sets of alternatives. It is proved that path-independence choice-functions are so degenerate that the choice for any set S must coincide with the choice for some pair of elements of S. Hence, in the lottery chosen for S, at most two elements may appear with positive probability. It is also proved that path-independent choice-functions cannot be continuous, except for the one-dimensional case.

1. Introduction and discussion of definitions and results

In this paper we examine situations in which an individual or society has to choose one element out of a given finite set of elements. For example, given a list of candidates for presidency, the society has to choose a president. Another example: given a set of commodity bundles in the budget set, a consumer has to choose one bundle.

Very frequently, a consumer may find two bundles which he likes equally, and in order to choose one out of the two he needs a random device that will make the decision for him. Similarly, a society may find itself in situations in which there are several candidates with identical amounts of support. Under such conditions, choosing one of the candidates would have to rely on some randomization. Thus, we shall be dealing with situations in which a decision maker (either an individual or a society) has to choose a lottery (i.e., a probability distribution or any convex combination) over a finite set, when that set is given to him to choose from.

As a matter of fact, the set of alternatives, from which the decision maker has to choose, may itself involve probability distributions. Consider the following example. A committee has to decide where some convention will take place. Two cities have suggested to host the convention. One of the cities has a 50% probability for rain on the convention day, so that all outdoors activities would have to be cancelled, if that city is chosen and it rains. Thus, the committee has to make a decision without really knowing what the actual outcome will be.

In view of the above discussion, we would like to incorporate lotteries both in the given alternatives, and in the choices based on them. Thus, we formalize choices in the following way. Let our universe of <u>alternatives</u> X be a subset of some topological real linear space (for example, a Euclidean space). The structure of a real linear space enables us to interpret a point x, in the convex hull of a set S, as a lottery over the elements of S. Let X^* denote the set of all finite non-empty subsets of X.

<u>Choice-functions:</u> A mapping $c: X^* \to X$ is called a <u>choice-function</u>, if for every $S \in X^*$, $c(S) \in$ convex-hull(S).

Thus, a choice-function is a rule, that selects for every finite non-empty subset S of the universe of alternatives, a unique lottery c(S) over the elements of S. This lottery may of course be equivalent to one of the elements of S, if that element has probability one in the lottery.

<u>Path - indepencence</u>: A choice-function $C: X^* \to X$ is said to be <u>path-independent</u> if for every $S \in X^*$, the elements of S may be considered in any order $p = (x_1, \ldots, x_s)$ and the pairwise choices $y_i = c(\{y_{i-1}, x_i\})$ $(i=2,3,\ldots,s; y_1=x_1)$ always lead to $y_s = c(S)$. Equivalently, c is path-independent if for all $S, T \in X^*$, such that $S \cap T = \emptyset$,

$$c(S \cup T) = c(\{c(S), c(T)\}).$$

Path-independent choice-functions are easier to implement since they are determined by the choices over pairs of alternatives. Thus, the decision maker does not have to consider the entire set of alternatives all at once, and may rather confine himself to pairwise comparisons. Another benefit of path-independent choice-functions is that they eliminate the possibility that a chairman may manipulate the resolution made by some legislative body, merely by putting the different motions to vote in a suitable order.

It should be mentioned here that the notion of path-independence has been extensively studied by Plott [1] (see also [2] for further references and discussion). However, Plott's definition differs from ours in the following way. The choice-function in Plott's definition is a rule, that assigns to every set S of alternatives, a <u>subset</u> C(S) of S. No lotteries are allowed. In that setup, path-independence is defined by $C(S \cup T) = C(S) \cup C(T)$ for all S, T. Thus, Plott's definition is somewhat more restrictive than ours, since not only disjoint sets, but also overlapping ones have to satisfy the condition.

The goal of this paper is to demonstrate that the path-independence condition is very restrictive. The main theorem is as follows.

Theorem 1. *If* $c: X^* \to X$ *is a path-independent choice-function, then for every* $S \in X^*$ *there exist* $x, y \in S$ (not necessarily distinct) *such that* $c(S) = c(\{x,y\})$.

Notice that no additional conditions are assumed in this theorem. A consequence of Theorem 1 is displayed in the following example.

Consider a situation where a society has to choose one out of a set of m condidates. We define the universe of alternatives to be $X = \{x = (x_1, \ldots, x_m) : \Sigma x_i = 1, x_i \geqq 0\}$. The points of X naturally correspond to probability distributions over the set of candidates, and particularly, the extreme points of X correspond to the "sure" lotteries, where a certain candidate is chosen with probability one. Theorem 1 implies that for every finite subset S of X, the choice over S is a lottery in which at most two elements of S participate with positive probability. In particular, the choice over the entire set of candidates is a lottery over two candidates at most. Thus, even if there are three or more candidates that are symmetric with respect to the social profile of preferences, still the choice (if it satisfies path-independence) has to be some lottery in which no more than two candidates participate. That does not seem to be a reasonable way of breaking such ties, since it involves an arbitrary discrimination against some of the candidates.

We will also prove that path-independence does not comply with continuity, even in the weak sense defined below. A choice function is said to be continuous if for every $x,y \in X$ and every sequence $\{y_k\}_{k=1}^{\infty} \subset X$, if $\lim_{k\to\infty} y_k = y$ then $\lim_{k\to\infty} c(\{x,y_k\}) = c(\{x,y\})$.

Theorem 2*. *If X contains the convex hull of three non-colinear points, then a choice-function over X cannot be both path-independent and continuous.*

2. Auxiliary lemmas and proofs

We denote in short xy for $c(\{x,y\})$, for every $x,y \in X$. Thus, $xy = yx$ and $xx = x$.

Lemma 1. *If for every set* $\{x,y,z\} \subset X$ *of three pairwise distinct elements* $(xy)z = x(yz)$, *then for every* $x,y \in X$, *either* $x(xy) = xy$ *or* $x(xy) = x$.

Proof. Suppose, to the contrary, that there are $x,y \in X$ such that both $x(xy) \neq xy$ and $x(xy) \neq x$.

This implies $x \neq y$. Also, since $xy = x \Rightarrow x(xy) = xx = x$, if follows that $xy \neq x$, and since $xy = y \Rightarrow x(xy) = xy$, it follows that $xy \neq y$.

*We wish to thank Hugo Sonnenschein for suggesting the investigation of the relationship between path-independence and continuity.

Denote $z = xy$, $w = xz$, and $t = zy$. Thus, x,y,z are pairwise distinct and x,w,z are pairwise distinct. It follows that

(1) $xt = x(zy) = x(yz) = (xy)z = zz = z$, and

(2) $wy = (xz)y = (zx)y = z(xy) = zz = z$.

Note that since z is in the open line segment (x,y) and w is in (x,z), then $w \neq y$. Anologously, $z \in (x,y)$ and $t \in [z,y]$ imply $t \neq x$. Also, $w \in (x,z)$ and $t \in [z,y)$ imply $t \neq w$. Some more equalities follow:

(3) $wt = w(zy) = w(yz) = (wy)z = zz = z$,

(4) $wz = w(xt) = w(tx) = (wt)x = zx = xz = w$,

(5) $xw = x(wz) = x(zw) = (xz)w = ww = w$,

(6) $wz = w(xt) = (wx)t = (xw)t = wt = z$.

Equalities (4) and (6) are contradciting and that completes the proof of this lemma.

Remark: Surprisingly, the conditions of Lemma 1 do not imply $x(xy) = xy$ as may have seemed natural to expect. This is shown by the following example.

Let $X = [0,1]$. Define $c(\{0,1\}) = .5$, $c(\{0,.5\}) = 0$, $c(\{.5,1\}) = 1$. Also, for every $\lambda \in [0,1] \setminus \{0,.5,1\}$ define $0\lambda = 1\lambda = .5\lambda = \lambda$ and for $\mu \in [0,1] \setminus \{0,.5,1\}$ define $\lambda\mu = \mathrm{Max}\{\lambda,\mu\}$. All the conditions of Lemma 1 are met, however, $0(01) = 0 \neq 01$.

Lemma 2. *If* $(xy)z = x(yz)$ *for every set* $\{x,y,z\} \subset X$ *of three pairwise distinct elements, then for all* $x,y,z \in X$, $(xy)z \in \{xy,xz,yz\}$.

Proof. In view of the condition assumed in the lemma, we may use the symbol xyz whenever x,y,z are pairwise distinct. The assertion of the lemma is obvious if x,y,z, are not pairwise distinct. Assume, to the contrary, that x,y,z are pairwise distinct and $xyz \notin \{xy,xz,yz\}$. It follows that $\{x,y,z\} \cap \{xy,xz,yz\} = \emptyset$. (If, for example, $xy = x$ then $xyz = xz$, and if $xy = z$ then $(xyz) = (xy)(xy) = xy$, and in both cases our assumptions are contradicted.)

Note that $(xy)(xz) = (xy)(zx) = (xyz)x = x(xyz)$, and by Lemma 1 (with yz here playing the role of y in the lemma) either $(xy)(xz) = xyz$ or $(xy)(xz) = x$. We now distinguish two cases.

Case I. x,y,z are colinear. We may assume, without loss of generality, that y belongs to the open line segment (x,z) and xyz belongs to the closed line segment $[x,y]$ (otherwise, the names x,y,z may be changed so as to conform with these assumptions).

Since $xyz \in (xy,z)$ it follows that $xy \in (x,xyz)$. Analogously, since $xyz \in (xz,y]$ it follows that $xz \in (x,xyz)$. Thus,

$(xy)(xz) \in [xy,xz] \subset (x,xyz)$. In other words, $(xy)(xz) \neq x$ and $(xy)(xz) \neq xyz$ and that contradicts what we have found before.

Case II. x,y,z are affinely independent. In this case x,yz,xz are affinely independent and hence $(xy)(xz) \neq x$. Also, since $xyz \in (xy,z]$, and since xy,xz,z are affinely independent, it follows that $(xy)(xz) \neq xyz$. Again, we arrive at a contradiction.

The proof of Theorem 1.

The proof is by induction on $|S|$. The assertion is trivial for $|S| \leqq 2$. If $S = \{x,y,z\}$ and $|S| = 3$ then the conditions of Lemma 2 are met:

$$(xy)z = c(\{c\{x,y\},z\}) = c(\{x,y,z\}) =$$

$$= c(\{x,c(\{y,z\})\}) = x(yz)$$

and hence

$$c(S) = xyz \in \{xy,xz,yz\} = \{c(\{x,y\}),\ c(\{x,z\}),\ c(\{y,z\})\}.$$

Suppose, by induction, that the claim is true for sets of cardinality not greater than n. Let $S = \{x_1,\ldots,x_{n+1}\}$. By the induction hypothesis, there are i,j $(1 \leq i,\ j \leq n)$ such that $c(\{x_1,\ldots,x_n\}) = c(\{x_i,x_j\})$. Thus,

$$\begin{aligned} c(S) &= c(\{c(\{x_1,\ldots,x_n\}),\ x_{n+1}\}) \\ &= c(\{c(\{x_i,x_j\}),\ x_{n+1}\}) \\ &= c(\{x_i,x_j,x_{n+1}\}) \end{aligned}$$

and by what we have already proved,

$$c(S) \in \{c(\{x_i,x_j\}, c(\{x_i,x_{n+1}\}), c(\{x_j,x_{n+1}\})\}.$$

This completes the proof.

The proof of Theorem 2.

Assume that $c:X^* \to X$ is path-independent and we will prove that c is not continuous.

Without loss of generality, assume that $X = \text{convex-hull } \{x,y,z\}$ where x,y,z are affinely independent. We distinguish two cases.

Case I: There exist $a,b\in X$ such that $ab\notin\{a,b\}$. First, note that for every $d\in X$ which is not colinear with a and b, $ad\in\{a,d\}$; this is because $(ab)d = (ad)b$ and in order for $c(\{a,b,d\})$ to belong to $[a,b] \cup [a,d] \cup [b,d]$ (Theorem 1) and to $[ab,d] \cap [ad,b]$, ad must be a vertex of the triangle a,b,d. Now let $\{d_k\}_{k=1}^{\infty}$ be sequence of points that are not co-linear with a and b, and such that $\lim_{k\to\infty} d_k = b$. Obviously, if $\lim_{k\to\infty} c(\{a,d_k\})$ exists then it must equal either a or b, and hence cannot equal ab. That contradicts continuity.

Case II. For all $a,b\in X$, $ab\in\{a,b\}$. In this case a complete linear order R is induced on X by $aRb \iff ab = a$. [Anti-symmetry: aRb & $bRa \Rightarrow ab = a$ & $ba = b \Rightarrow a = b$. Transitivity: aRb & $bRd \Rightarrow ab = a$ & $bd = b \Rightarrow ad = (ab)d = a(bd) = ab = a \Rightarrow aRd$].
Without loss of generality, assume xRyRz. Suppose, per absurdum,

that c is continuous. Let $A = \{w \in X : wRy,\ w \neq y\}$ and $B = \{w \in X : yRw,\ w \neq y\}$. It is easy to verify that A and B are both open relative to X and non-empty. Obviously, $A \cap B = \emptyset$ and $A \cup B = X \backslash \{y\}$. That implies that $X \backslash \{y\}$ is not connected, and hence, a contradiction. Thus, c cannot be continuous.

References

1. C. R. Plott (1973): "Path Independence, Rationality and Social Choice," Econometrica, 41 (1973), 1075-1091

2. A. Sen (1977): "Social Choice Theory: A Re-examination," Econometrica, 45 (1977), 53-89.

<u>Lemma 2</u>. <u>If</u> $(xy)z = x(yz)$ <u>for every set</u> $\{x,y,z\} \subset X$ <u>of three pairwise distinct elements, then for all</u> $x,y,z \in X$, $(xy)z \in \{xy,xz,yz\}$.

<u>Proof</u>. In view of the condition assumed in the lemma, we may use the symbol xyz whenever x,y,z are pairwise distinct. The assertion of the lemma is obvious if x,y,z, are not pairwise distinct. Assume, to the contrary, that x,y,z are pairwise distinct and $xyz \notin \{xy,xz,yz\}$. It follows that $\{x,y,z\} \cap \{xy,xz,yz\} = \emptyset$. (If, for example, $xy = x$ then $xyz = xz$, and if $xy = z$ then $(xyz) = (xy)(xy) = xy$, and in both cases our assumptions are contradicted.)

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Since $xyz \in (xy,z)$ it follows that $xy \in (x,xyz)$. Analogously, since $xyz \in (xz,y]$ it follows that $xz \in (x,xyz)$. Thus, and by what we have already proved,

$$c(S) \in \{c(\{x_i,x_j\}, c(\{x_i,x_{n+1}\}), c(\{x_j,x_{n+1}\})\}.$$

This completes the proof.

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