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EXPECTATIONS AND EQUILIBRIUM WITH INCOMPLETE MARKETS

by

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Introduction

The object of this paper is to clarify the relationship between the existence (or non-existence) of an equilibrium with incomplete markets and two characteristics of an economy: the information that traders have about future prices and the structure of the securities markets in such an economy. Radner [8] has demonstrated the existence of a rational expectations equilibrium in a model with incomplete markets in which traders have consistent point expectations about future prices. However, because complete information, incomplete markets, the possibility of short selling and the existence of equilibrium prices may be incompatible (see Hart (6)), Radner imposed an arbitrary bound on futures trading.

The non-existence example that Hart discussed results from a collapsing of the individual's budget set at "equilibrium" prices. In effect, the price system has been requested to perform at least two functions in the economy. Relative prices determine the rate of exchange for the allocation of commodities available at any date and event pair and in conjunction with the set of allowable forward markets they may permit the transfer of income across states of the world. The latter role is performed by the contingent claims markets in the complete markets model, independently of the agents' price expectations. Therefore, because agents have point expectations on prices along with the specification of imcomplete markets, it is possible for agents to believe that some income transfers across states are possible for some price systems and not possible at other price systems.

In section 1 of this note we demonstrate that a rational expectations equilibrium exists for an economy with strictly monotone preferences if the futures markets are properly specified and that if the structure of securities markets does not satisfy this property then an equilibrium may not exist. The property of the market structure sufficient for existence is called positive price independence. A security structure is positive price independent if its matrix of expected security incomes is linearly independent for any matrix of strictly positive prices. The consequence of this property is that an agent can formulate portfolios of securities which will generate a given profile of uncorrelated incomes across some collection of states of the world and the ability to generate this profile of wealth is not dependent upon expected prices (as long as expected prices are positive). It will be immediate that if the economy has a price independent security structure then the economy consists of a partition of non-trading but possibly interdependent economies.

The proposed concept of price independence may be interpreted as relieving prices of the role of facilitating risk sharing and returning that role to the market. This property might occur in a world with endogenous security markets because only with positive price independent securities have all possibilities for arbitrage profit to issuers of securities been exhausted. That is, for any non-positive-price-independent security structure there is some environment in which individuals can not generate their desired profile of wealth and the potential for arbitrage profit exists. Therefore, if the exogenous security structure used here is viewed as the long run steady state of a process with endogenous markets the concept of positive price independence is natural. Positive price independence does not require that the security market be complete it only requires that the security market do its job among all candidates for equilibrium price systems.

The second section of the paper considers a relaxation of the informational requirements imposed upon the traders. The consequence of this relaxation is that if agents believe it is possible to alter incomes across the states of the world, they will always believe it is possible. Since agents no longer,have "rational" price expectations, this equilibrium is not a rational expectations equilibrium but rather a temporary equilibrium. The equilibrium is only in the current markets and the agents' plans and price expectations need not be consistent. This section also contains an example of the non-existence of a temporary equilibrium when the agents are permitted to hold point expectations. Therefore, if the market structure is not specified then uncertainty or imprecise information about future prices is an alternative assumption sufficient for the existence of a temporary equilibrium.

Several features of the formal structure of Radner-Hart deserve comment in the light of the concept of price independence. The set of allowable contracts is given a priori, there is no moral hazard, the information structures possessed by agents are identical in the sense that all agents possess the same partition of the states of the world and resources are not used to operate the futures markets. This interpretation suggests that the non-existence of certain markets, given the information structure, is to be explained by the costs of organizing the market for a new security and not by the costs of identifying when a given collection of events obtains. Further, it is clear that the agent's only motivation in trading securities is to transfer wealth across states of the world.

MARKET STRUCTURE

Consider an economy extending over two periods of time, dates one and two. It is assumed that there is a finite set $\mathscr{S}$ (cardinality S) of alternative states that can occur at date 2. There are ℓ_1 commodities at date 1 and ℓ_2 commodities at date 2. We assume that there are also ℓ_2 securities at date 1. Each security is associated with a date 2 good in which it pays off.[1/] The security yields some non-negative amount of its corresponding good where the amount may depend on the state that obtains. We also assume that $S \equiv \ell_2 \geq 2$; this assumption is made only for convenience and does not affect the nature of the results.

Commodities at date 1 are represented by points in R^{ℓ_1}, commodities at date 2 by points in $R^{\ell_2 S}$, and each of the ℓ_2 securities by points in R^{ℓ_2}. Let $A = [a_j(s)]$ be the ℓ_2 by S matrix of security payoffs. We assume that $a_j(s) \geq 0$ for all j and s. The interpretation is that $a_j(s)$ is the amount of good j that a holder of one unit of security j has contracted to receive in state s. It is possible that $a_j(s) = 0$ for all s, for some j; that is security j cannot be used for income transfer.

At date 1 we have $\ell_1 + \ell_2$ markets open. Let q^1 be the vector of first period goods prices and π the vector of security prices. The first period price vector is $p = (q^1, \pi)$ which is normalized such that $p \in \Delta^{\ell_1 + \ell_2}$ is the first period price simplex.

At date 2 the first period securities pay off and no new securities are issued (since date 2 is the last time period in the model). Hence, at date 2 there are only ℓ_2 markets. Let $q^2(s)$ be the vector of normalized second period prices in state s, $q^2(s) \in \Delta^{\ell_2}$ the set of normalized positive prices. In the rational expectation model, within each state, all agents have identical price expectations, although the agents need not agree on the probability of occurance for each state. In a temporary equilibrium model the agents must have some agreement in equilibrium about future prices, although they need not have identical price expectations and in general will have probability distributions over second period prices. The price expectations for date 2 can be written in matrix form as $q^2 = [q^2_j(s)]$.

Agents trade securities in order to transfer income both over time and across states. When an agent purchases a security he is interested in the monetary return from that security in each state. For a given matrix A of security payoffs and a given matrix q^2 of expected second period prices, let $\bar{M}$ represent the matrix of expected monetary returns, $\bar{M} = q^2 \times A = [a_j(s)q^2_j(s)]$. Row j of $\bar{M}$ gives the expected monetary return from purchasing security j in each possible state of the economy.

RATIONAL EXPECTATIONS MODEL

In this section we consider a rational expectations model (Radner, [8]). An equilibrium will be defined to be a set of prices (current and expected) and plans such that current markets clear and the plans of the agents are consistent.

There are I agents in the economy. Each agent i has a utility function $U^i \in \mathcal{U}^i =$

$\{U^i : \mathbb{R}^{\ell 1}_+ \times \mathbb{R}^{\ell 2 S}_+ \to \mathbb{R} \mid U^i$ is strictly quasi-concave, strictly monotone, and continuous.$\}$

Each agent has an endowment $w^{i1} \in R^{\ell 1}_+$ in period 1 and an endowment $w^{i_2}(s) \in R^{\ell 2}_+$ in period 2 for each state s. Let $w^{i_2} = (w^{i_2}(1),\ldots,w^{i_2}(S))$ and $w^i = (w^{i1}, w^{i2}) \in \mathbb{R}^{\ell 1}_+ \times \mathbb{R}^{\ell 2 S}_+$. It is assumed that the consumption set of agent i is $X^{i1} = R^{\ell 1}_+$ for period 1 and $X^{i2}(s)$ for period 2 for each state s. Let $X^i = X^{i1} \times X^{i2}(1) \times \ldots \times X^{i2}(s)$. We assume that w^i is in the interior of X^i for each agent. Let z^i_j be the number of units of security j purchased by i and z^i be the vector of such purchases.

In period 1 each agent will choose a consumption vector and a vector of security purchases. In period 2 once a state has obtained each agent will use his income to purchase some vector of period 2 commodities. The objective of the individual is to maximize $U^i(x^{i}1,x^{i}2)$ subject to his feasible set. An action $(x^{i}1,z^i,x^{i}2)$ is feasible for i, given $(q^1, \pi, q^2) = q$ if:

1) $(x^{i}1,x^{i}2) \in X^i$

2) $q^1 x^{i}1 + \pi Z^i \leq q^1 w^{i}1$

3) $q^2(s)x^{i}2(s) \leq q^2(s)w^{i}2(s) + \bar{M}(s)Z$ for all $s \in \mathcal{S}$

Let $\beta(q)$ denote this set of feasible actions, i.e.

$$\beta(q) = \{k \in \mathbb{R}^{\ell_2(2+S)}: k = (x^1,z,x^2), \text{ satisfying } 1),2),3)\}$$

The individual then solves, for any given q:

$$\underset{k \in \beta(q)}{\text{Max}}\ U^i(x^{i}1, x^{i}2)$$

Let $\xi^i(q)$ denote the set of actions optimal for q (note that $\xi^i(q)$ may be empty). Let $\beta_L(q)$ represent the budget set constructed by including the following condition in the definition of $\beta(q)$;

4) $|z^i| \leq L$, for some fixed $L \in R$.
Let $\xi^i_L(q)$ denote the set of optimal actions given q subjects to $\beta_L(q)$.

RATIONAL EXPECTATIONS EQUILIBRIUM

Let $\Delta = \Delta^{\ell_1 + \ell_2} \times (\Delta^{\ell_2})^S$.

A Rational Expectations Equilibrium (relative to L) is a set of prices $q^* \in \Delta$ and consumption and security trades (x^{i_1}, z^i, x^{i_2}) one for each agent, such that

$$(1) \quad (x^{i_1}, z^i, x^{i_2}) \in \xi_L^i(q^*) \text{ for all } i,$$

$$(2) \quad \sum_{i=1}^{I} z^i = 0, \quad \sum_{i=1}^{I} x^{i_1} = \sum_{i=1}^{I} w^{i_1} \text{ and}$$

$$(3) \quad \sum_{i=1}^{I} x^{i_2}(s) = \sum_{i=1}^{I} w^{i_2}(s) \text{ for all } s.$$

Theorem (Radner). If the assumptions presented in the preceeding section are satisfied then the bounded pure exchange economy has a rational expectations equilibrium.

Radner's proof uses the Debreu technique. He bounds the attainable plans, using L, and then applies a fixed point theorem.

The objection to this proof involves the use of an arbitrary bound, L, on security trades. There is no natural bound to security trading -- the supply of forward contracts is not limited. An agent can always offer any arbitrarily large number of offsetting forward contracts. In many examples of simple economies, the bound does not affect the equilibrium, but as Hart [6] has shown there are economies in which the bound will affect the equilibrium. Hart has provided an example of an economy which satisfies all of the conditions above, but in which a rational expectations equilibrium does not exist without some arbitrary bound on security trading.

The example demonstrates the possibility of nonexistence in a two person pure exchange economy, with $\ell_2 = 2$, $s = 2$, and $\ell_1 = 0$. In Hart's example there are two securities which pay off in a unit of the corresponding good in both states 1 and 2. Non-existence of equilibrium occurs because with this security structure and with endowments and utility functions chosen to give equal Arrow-Debreu equilibrium prices for all goods, the resulting collection of monetary returns are linearly dependent, permitting no income transfers across states. The reason that this leads to the non-existence of an equilibrium is, as Hart points out, due to demand correspondences which are not continuous at the Arrow-Debreu equilibrium prices for second period goods.

Each consumers' maximization problem determines an optimal constrained income level in each state, call it the profile of preferred or desired securities incomes. The consumers then buy and sell enough of the securities in period one to adjust their income appropriately. They are able to do so only as long as either buying or selling a security affects their income across future states. But when the monetary returns on all securities are independent of the state that results this is no longer possible. A sequence of prices approaching the Arrow-Debreu prices gives consumers the ability to transfer income across states except in the limit. Hence at the limit point each consumer is indifferent between any quantity of security transactions because those transactions do not affect his income. But if the consumers cannot attain their desired income levels in each state the Arrow-Debreu prices for period two will not result. Therefore, a rational expectations equilibrium is not possible in this example.

It is easy to see that the same problem can occur even with securities whose quantity returns are state dependent. Let E be a pure exchange economy of the type previously specified. The Arrow-Debreu contingent claims version of E is an economy with the agents and endowments of E, but in which trading in claims on all goods (both current and future) is allowed at date 1. Let E_s for $s \in \mathcal{S}$, be a single period economy with the agents of E and the endowments of state s.

In order that the discussion that follows be reasonably self-contained consider these two definitions of particular securities structures providing complete markets and the relationship between the equilibria of these economies and certain alternative market structures which are "incomplete."

Defn. 1 Arrow Securities

Let $A: J \times S \longrightarrow R^{\ell 2}$

Where $J = \{1, \ldots, \ell^2\}$ and

$S = \{1, \ldots, \ell^2\}$.

$a_j(s) \in \text{int } R_+^{\ell 2}$ whenever $j=s$

and

$a_j(s) = \{0\}$ whenever $j \neq s$.

Defn. 2 Contingent Claims

$A: J \times S \longrightarrow R_+^{\ell^2}$

$J = \{1, -----, \ell^2 s\}$

$= \{1_1, ----, \ell_1^2,\ 1_2, ----, \ell_2^2,\ ----, 1_{\ell^2}, ---, \ell_{\ell^2}^2\}$

where

$a_{j_s}(s') \geq 0$ and the j_s coordinate is strictly positive whenever $s=s'$ and zero otherwise and $j_s' \neq j_s$, j_s' coordinates are zero.

Remark 1. If $\overline{M}$ is of full rank in the rational expectations equilibrium, consumption allocations are Arrow or contingent claims equilibrium consumption allocations and there exist linear transformations (i.e. normalizations) of the equilibrium prices such that the transformed rational expectations equilibrium prices are Arrow-Debreu prices.

<u>Proposition 1</u>. Let E be a pure exchange economy of the type specified above, but with no endowments of first period goods. Suppose that the Arrow-Debreu version of E has a unique equilibrium price vector $q^* \in (\Delta^{\ell^2})^S$ and that E_s has a unique equilibrium price vector $\overline{q}(s) \in \Delta^{\ell^2}$, for each $s \in \mathscr{S}$. Let $q^{2*}(s) = (q_1^*(s), ----, q_{\ell^2}^*(s))$, the vector of Arrow-Debreu prices for second period goods in state $s \in \mathscr{S}$. If $\overline{q}(s) \neq q^{2^*}(s)$, $\forall\ s \in S$ then there is a security structure, A, such that no rational expectations equilibrium exists for E.

Proof: Choose A such that $q^{2^*} \times A$ is linearly dependent. This can be done by setting $a_j(s) = 1/q_j^{2^*}(s)\ \forall\ j,s$.

Suppose equilibrium rational price expectations q^2 are such that $q^2 \times A$ is linearly independent. Then the equilibrium consumption allocations are Arrow-Debreu, income transfer across all states is possible and the Arrow-Debreu second period price matrix, q^{2^*}, will result. But $q^2 \neq q^{2^*}$.

Suppose $q^2 = q^{2^*}$. Then $q^2 \times A$ is linearly dependent, has a rank of zero and income transfer across any collection of states is not possible. But $\bar{q}^2 \neq q^{2^*}$. Hence $q^2 \neq \bar{q}^2$.

Therefore no rational expectations equilibrium exists.

Hence, any model of an economy, with unbounded security trades, which incorporates potentially incomplete markets can not be assured of an equilibrium if the structure of the security market is left unspecified. In any model in which the security structure is derived endogenously, the mechanism which generates the securities must operate in a manner such that any security structure that gives nonexistence can not arise.

One means of insuring the existence of an equilibrium is to specify that the security structure is price independent.

Definition: A represents a price independent set of security markets if the dimension of the subspace spanned by $q^2 \times A$ is independent of $q^2 \in (\Delta^{\ell_2})^S$.

Definition: A represents a positive price independent set of security markets if the dimension of the subspace spanned by $q^2 \times A$ is independent of $q^2 \in \text{int } (\Delta^{\ell_2})^S$.

Arrow securities and a complete set of contingent claims markets are price independent. No other market arrangement is price independent except for Arrow or contingent claims markets which are incomplete in the sense that there are states for which no security pays off. However, a variety of market structures are positive price independent. If the economy is provided with a positive price independent security structure the ability of the agents to transfer income across states is constant. The markets need not be complete in the usual sense, they need only be such that if an agent is able to transfer income between states $s, s' \in \mathscr{S}$ for some q^2 then he is able to transfer income between s and s' for all strictly positive q^2. We will show that with this type of security structure, security demands will be finite and continuous at all expected prices and the usual equilibrium proofs apply directly.

It is easy to show that a security structure is positive price independent if and only if by doing simple row operations on it we can construct a matrix of the form

$$\begin{bmatrix} I_{nxn} & 0_{nxp} \\ & \\ 0_{pxn} & 0_{pxp} \end{bmatrix} \quad n + p = s;\ n,p \geq 0;\ n \text{ is constant for all } q^2 \in \text{int } (\Delta^{\ell_2})^S,$$

where I_{nxn} is the n by n identity matrix.[2/] That is, the security structure must be such that by purchasing various bundles of securities individuals can construct a portfolio of securities which are equivalent to Arrow securities in the sense that the same income transfers are possible for the states for which securities exist.[3/]

In effect, since agents are capable of distinguishing all states of the world costlessly, the securities structure should be such that those states are distinguished in terms of the ability to trade security incomes.

Proposition 2: For the pure exchange economy specified in the rational expectations model a rational expectations equilibrium exists if the security structure is price independent.

Proof: It is not possible to use a Theory of Value type of proof for this proposition since the set of attainable states for the economy can not be bounded independently of prices. We will instead follow the strucutre of proof presented in Debreu [2].

Let $D = \{q \in \Delta^{\ell_1 + \ell_2} \times (\Delta^{\ell_2})^S | q \gg 0$ and $\pi_j = \Sigma_{s=1}^{S} \alpha_s \overline{M}_j(s)\ \forall_j$ for some $(\alpha_1,\ldots,\alpha_S) \in \alpha_s > 0\ \forall\ s, \Sigma_{s=1}^{S} \alpha_s = 1\}$. Consider an arbitrary consumer i.

Let $\{\overline{M}(s)\}_{s\epsilon S}$ represent the finite set of columns of $\overline{M}$, hence $\overline{M}(s) \epsilon R^{\ell_2}$ and co $\{\overline{M}(s)\}_{s \epsilon s}$ represent the convex hull of this finite set.

Lemma (1). $\beta(q)$ is compact iff $q \epsilon D$ and $\overline{M}$ satisfied price independence (assume rank is ℓ^2 implying for all s there exists a j such that $a_j(s) > 0$).

Remark 2. Positive price independence implies that interior co $\{\overline{M}(s)\}_{s\epsilon S} \neq \emptyset$, $\forall\ q^2 \gg 0$.

Remark 3. If, for example, for all s there exists j_s such that $a_{j_s}(s) > 0$ and $j \neq j_s$ $a_j(s) = 0$ and $s \geq 2$, then price independence is satisfied and $\forall\ \pi' \gg 0$, $(q^1, \pi^1, q^2) \in D$. This conclusion holds for Arrow securities for example.

Remark 4. If, for example, there exists j such that $a_j(s) > 0$ and $a_j(s') > 0$ for some $s \neq s'$ and price independence holds, then there exists $\hat{\pi} \gg 0$ such that $(q^1, \hat{\pi}, q^2) \notin D$.

Proof of lemma 1. $\beta(q)$ is obviously closed.

Note that $\beta(q)$ is bounded if $\bar{A}B(q) = \{0\}$ where $\bar{A}$ represents the asymptotic cone.

It is clear that

$$\bar{A}B(q) = \{k : k = (x^1, x^2, z),\ q^1x^1 + \pi z \leq 0,\ q^2(s)x^2(s) - \bar{M}(s)z \leq 0,\ \forall_s \in S\}$$

Let $k \neq 0$, where $k \in \bar{A}B(q)$.

Case 1. If $x(s) > 0$ for some s then

$$\pi z \leq -q^1 x^1 \leq 0$$

$$-\bar{M}(s)\ z \leq -q^2(s)\ x\ (s) \leq 0 \text{ for all } s \in S$$

and the inequalities are strict for some of the linear inequalities or $x(s) = 0$ for all $s \in S$, but one of the inequalities must be strict which implies $z \neq 0$.

Since $q \in D$ and int co $\{M(s)\}_{s \in S} \neq \emptyset$, we have $\pi z = 0$ and $M(s)z \geq 0$, but $\pi \in$ int co $\{M(s)\} s \in S$.

Therefore $\pi z \leq 0$ and $\pi z \geq 0$ with one strict inequality.

Case 2. Let $x^2(s) = 0 = x^1\ \forall\ s \in S$, $\pi z = 0$, and $M(s)z = 0\ \forall s$, then by price independence this homogeneous system has a unique solution 0.

Assume $\beta(q)$ is bounded for all $\forall q$ but $\overline{M}$ is not price independent, then for some $\hat{q}$, $\overline{M}(\hat{q})$ is of rank strictly less than ℓ^2 call it $\hat{\ell}$, choose $\ell^2 - \hat{\ell}$ columns which are linearly dependent and let these columns be denoted, $\hat{\overline{M}}(\hat{q})$, then there exists $\hat{z} \in R^{\ell^2 - \hat{\ell}}$ with $\hat{z} \neq 0$ such that

$\hat{\overline{M}}(\hat{q}(s))\,\hat{z} = 0$ implying for all $\theta > 0$ that $\theta\,\hat{z}$ satisfies this relation.

If $\hat{\pi} = \hat{\overline{M}}(s)$ then $\hat{\pi}\,\hat{z} = 0$, hence $B(q)$ is not bounded.

If $\hat{\pi} \neq \hat{\overline{M}}(s)$ for $\hat{q}$, then a separation agrument establishes the result.

Assume that $\beta(q)$ is bounded but $\pi(0) \notin$ int co $\{M(s)\}_{s\in S} \neq \emptyset$. Then there exists $z \neq 0$ such that $\pi \cdot z \leq 0$ and $M\cdot z \geq 0 \;\forall\; M \in$ Co $\{M(s)\}_{s\in S}$ and $M\cdot z > 0$ for some $M \in$ int co $\{M(s)\} s\in S$. This implies that θz satisfies the system $\forall\theta > 0$ and is not bounded.

Lemma 2. Under the conditions of lemma 1, $\beta(q)$ is upper-hemi-continuous $\forall\; q \in D$.

Proof: By Lemma 1, $\beta(q)$ is compact valued. Therefore, it suffices to show that if $\{q^v\} \subset D$ such that $q^v \to q^o \in D$, $y^v \in \beta(q^v)$, then $\{y^v\}$ is bounded. Therefore, there exists a convergent subsequence $y^{v'} \to y^o$ and given the properties of $\beta(q)$, it is clear that $y^o \in \beta(q^o)$. A detailed proof may be constructed following the same lines as Green [lemma 2.2, lemma 3.3].

Remark: Again using arguments of Green it may be shown that the correspondence is lower hemi-continuous and therefore continuous. It follows by standard arguments that the demand correspondence is compact, convex-valued and upper hemi-continuous on D.

<u>Defn.</u> The Correspondence

$\eta^i\colon\; D \to R^{\ell_1 + \ell_2} \times R^{\ell_2 s}$

defines the set of excess demands for the i^{th} agent.

The Lemmata are adapted from Grandmont [4] and are included for the sake of completeness.

Lemma 3: Consider any sequence q^v in D which tends to some $q \in \partial D$ (boundary of D) and any sequence $\alpha^v \in \eta^i(q)$. Then $\bar{q}\,\alpha^v$ diverges to $+\infty$ for every $\bar{q} \in D$. (α^v represents a sequence of excess demands).

Proof: Suppose not. Then there exists a sequence q^v in D tending to some $q \in \partial D$, $\alpha^v \in \eta^i(q^v)$ and $\bar{q} \in D$ such that $\bar{q} \cdot \alpha^v$ is bounded above. Then it is easy to show, using an argument similar to the proof of lemma (1), that the sequence α^v is itself bounded. Hence (α^v) has a convergent subsequence, let α be its limit. By continuity this implies $\alpha \in \eta^i(q)$. But this contradicts lemma (1) since $q \in \partial D$.

Lemma 4: Let η: $D \rightarrow R^{\ell_1 + \ell_2} \times R^{\ell_2 S}$ be a correspondence which is compact and convex valued, upper hemi-continuous and satisfies Walras' Law, $q \cdot \alpha = 0$ for $\alpha \in \eta\,(q)$. If:

for every sequence q^v in D which tends to some $q \in \partial D$ and every sequence $\alpha^v \in \eta(q^v)$, there exists $\bar{q} \in D$ such that $\bar{q} \cdot \alpha^v > 0$ for infinitely many v then there exists $q^* \in D$ such that $0 \in \eta(q^*)$.

Proof: Grandmont [4].

Let $\eta(q) = \sum_{i=1}^{I} \eta^i(q)$. Using lemmas 2 and 3 it is easy to see that η satisfies the assumptions of lemma 4. Hence $(q^*, \eta^1(q^*), \ldots, \eta^I(q^*))$ is a rational expectations equilibrium. Therefore, proposition 2 is established.

The next proposition is not as strong as we would like. A better proposition would be the following:

Conjecture: For any security structure which is not positive price independent there is a pure exchange economy, with $U^i \in \mathcal{U}^i \;\forall\; i$ and strictly positive endowments, such that there does not exist a rational expectations equilibrium (recall the definition of $\mathcal{U}^i$).

If this conjecture can be shown to be true, then the essential although not necessary nature of positive price independence would be established. However, the proposition is true for a wider class of economies where the utilities are drawn from a larger set, the set $\hat{\mathcal{U}}^i$. $\mathcal{U}^i \subset \hat{\mathcal{U}}^i$ and these sets differ only with respect to the property of strict monotonicity. This property is no longer invoked.

Proposition 3. For any security structure which is not positive price independent there is a pure exchange economy, with $u^i \in \hat{\mathcal{U}}^i \;\forall\; i$ and strictly positive endowments, such that there does not exist a rational expectations equilibrium.

Proof: Since A is not positive price independent, there exists at least two states such that for some price system it is not possible to alter security incomes across these two states and for some other set of prices it is possible. Therefore, let the agents have preferences specified in the following manner.

Consider a two person economy in which there are no goods at date one, two goods at date two, and two equally likely states at date two. Consumer i's initial endowment will be denoted by $(\bar{x}^i(s), \bar{y}^i(s))$, $s = 1,2$, and his final portfolio by $(x^i(s),\ y^i(s))$, $s = 1,\ 2$. His state one preferences are represented by $\alpha^i \log x^i(1) + \beta^i \log y^i(1)$ and his state two preferences by $\gamma^i \log x^i(2) + \delta^i \log y^i(2)$; $\alpha^i, \beta^i, \gamma^i, \delta^i > 0$. Let q^{2*} be the Arrow-Debreu price matrix. Solving the equilibrium market balance equations for prices yields the following functions of the endowments and the parameters of utility functions;

$$q_1^{2*}(1) = \frac{\Sigma_i \frac{\alpha^i}{2} \bar{x}^i(1)}{\Sigma_i[\frac{\alpha^i}{2} \bar{x}^i(1) + \frac{\beta^i}{2} \bar{y}^i(1) + \frac{\gamma^i}{2} \bar{x}^i(2) + \frac{\delta^i}{2} \bar{y}^i(2)]}$$

$$q_2^{2*}(1) = \frac{\Sigma_i \frac{\beta^i}{2} \bar{y}^i(1)}{\Sigma_i[\frac{\alpha^i}{2} \bar{x}^i(1) + \frac{\beta^i}{2} \bar{y}^i(1) + \frac{\gamma^i}{2} \bar{x}^i(2) + \frac{\delta^i}{2} \bar{y}^i(2)]}$$

$$q_1^{2*}(2) = \frac{\Sigma_i \frac{\gamma^i}{2} \bar{x}^i(2)}{\Sigma_i[\frac{\alpha^i}{2} \bar{x}^i(1) + \frac{\beta^i}{2} \bar{y}^i(1) + \frac{\gamma^i}{2} \bar{x}^i(2) + \frac{\delta^i}{2} \bar{y}^i(2)]}$$

$$q_2^{2*}(2) = \frac{\Sigma_i \frac{\delta^i}{2} \bar{y}^i(2)}{\Sigma_i[\frac{\alpha^i}{2} \bar{x}^i(1) + \frac{\beta^i}{2} \bar{y}^i(1) + \frac{\gamma^i}{2} \bar{x}^i(2) + \frac{\delta^i}{2} \bar{y}^i(2)]}$$

Let $\bar{q}^2$ be the matrix of second period prices for the economies E_1 and E_2. Solving the market balance equations for E_1 and E_2 yields;

$$\bar{q}_1^2(1) = \frac{\Sigma_i \alpha^i \bar{x}^i(1)}{\Sigma_i[\alpha^i \bar{x}^i(1) + \beta^i \bar{y}^i(1)]}$$

$$\bar{q}_2^2\ (1) = \frac{\Sigma_i \beta^i \bar{y}^i(1)}{\Sigma_i[\alpha^i \bar{x}^i(1) + \beta^i \bar{y}^i(1)]}$$

$$\bar{q}_1^2(2) = \frac{\Sigma_i \gamma^i \bar{x}^i(2)}{\Sigma_i[\gamma^i \bar{x}^i(2) + \delta^i \bar{y}^i(2)]}$$

$$\bar{q}_2^2(2) = \frac{\Sigma_i \delta^i \bar{y}^i(2)}{\Sigma_i[\gamma^i \bar{x}^i(2) + \delta^i \bar{y}^i(2)]}$$

Let A be the non-price independent security structure. Using the above equations it is easy to see that one can choose utility parameters and endowments such that

$$q^{2^*} \times A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and } q^{2^*} \neq \bar{q}^2.$$

Then by the argument of Proposition 1, a rational expectations equilibrium does not exist for the economy chosen above when the security structure is A.

TEMPORARY EQUILIBRIUM

The temporary equilibrium model (Green [5]) is formally very similar to the rational expectations model. This type of model includes the case where agents have expectations which differ from "rational expectations" primarily in the confidence an agent places in his expectations. Here the agents are allowed to be uncertain about future prices and hence they need not all agree on which prices will result.

The examples and propositions in the previous section about non-existence of a rational expectations equilibrium do not apply to a temporary equilibrium. Those results demonstrated that if agents had rational expectations then equilibrium might not exist, but for a temporary equilibrium rational expectations are not required. However, a requirement having the force of price independence is essential for demonstrating the existence of a temporary equilibrium. An example of non-existence is provided when this requirement is not met. Only compatibility of first period actions for given price expectations is required for a temporary equilibrium.

We now search for an equilibrium in the first period only--for given price expectations.

A temporary equilibrium is a set of prices $p^* = (q^{1^*}, \pi^*) \in \Delta^{\ell_1+\ell_2}$ and consumptions and security trades $(x^{i_1 *}, z^{i^*})$ one for each agent i such that

(1) $(x^{i_1 *}, z^{i^*}) \in \xi^i (p^*)$ for all i

(2) $\sum_{i=1}^I z^{i^*} = 0, \sum_{i=1}^I x^{i_1 *} = \sum_{i=1} {}^i w^{i_1}$ Where $\xi^i(p)$ is agent i's demand correspondence at price p (formally defined later).

Now agent i's expectation on future prices is a collection of vectors

$q^{i}{}_2(s) \in \Delta^{\ell_2}$ for each $s \in \mathscr{S}$ and a collection of
$q^{i}{}_2 = (q^{i}{}_2(1), \ldots, q^{i}{}_2(S)) \in (\Delta^{\ell_2})^S$.

Assumption 1: For each $p \in \Delta^{\ell_1+\ell_2}$ agent i's expectations about q^2 are given by a random probability measure

$$\Psi^i : (\Delta^{\ell_1+\ell_2}, \mathcal{B}_{(\Delta^{\ell_2})^S}) \to [0,1]$$

such that

(i) $\Psi^i(p,\cdot)$ is a probability measure on $((\Delta^{\ell_2})^S, \mathcal{B}_{(\Delta^{\ell_2})^S})$ for each $p \in \Delta^{\ell_1+\ell_2}$.

(ii) $\Psi^i(\cdot, B)$ is a continuous function on $\Delta^{\ell_1+\ell_2}$ for each $B \in \mathcal{B}_{(\Delta^{\ell_2})^S}$.

The interpretation is that Ψ^i selects a probability measure on future prices for each vector of current prices.

In order to complete the analysis, the outline of an example of an economy with incomplete markets and point expectations on future monetary returns is offered for which there is no temporary equilibrium. Because Green's formulation is a complete markets model, point expectations are compatible with the existence of a temporary equilibrium. Again if market structures are price independent, then point expectations may be compatible with a temporary equilibrium. However, restrictions on the expectations function are still required to exclude the sort of phenomena that can occur with unit elastic price expectations.

In this example, there is a single consumer, two periods, two states of the world which become known in the second period, a single consumption good in period one and two consumption goods in period two for each state. Two securities are available in the first period; the first security pays off in one unit of the first consumption good in each of the two states in period two. First period prices are normalized with respect to the second security's price and in the second period prices in each state are normalized with respect to the second consumption good's price. The price expectations of the individual are such that the price of the first period's consumption good is expected to be realized as the price of the first commodity in period two state one. In state two the expected price of the first commodity is constant and equal to one.

Example

The preferences of the consumer are represented by:

(1) $U(x^1, x(1), x(2)) =$

$$\ln x^1 + \min \left[3/4\ x_1(1),\ 1/4\ x_2(1)\right] + \min\ [x_1(2),\ x_2(2)]$$

First period prices are:

(2) $P = (q^1, \pi) = (q^1, \pi^1, 1)$; and

expected prices are:

(3) $q^2 = (q^2(1), q^2(2))$

where

(4) $q^2(1) = (q^1, 1)$ and

(5) $q^2(2) = (1,1)$.

The endowments of the agent are:

(6a) $w^1 = 1/7$

(6b) $w^2(1) = (1/4, 3/4)$

(6c) $w^2(2) = (4,8)$ and

(6d) $z = (z^1, z^2) = (0,0)$.

The agent maximizes (1) subject to

(7a) $q^1x^1 + \pi^1z^1 + z^2 \leq q^1\,[1/7]$

(7b) $q^1x^1(1) + x^2(1) \leq q^1\,[1/4 + z^1] + 3/4 + z^2$

(7c) $x^1(2) + x^2(2) \leq 12 + z^1 + z^2$

(7d) $q^1\,[1/4 + z^1] + 3/4 + z^2 \geq 0$

(7e) $12 + z_1 + z_2 \geq 0$

(7f) $\pi^1 = 1/2\,(q^1 + 1)$.

(7f) insures that the maximization problem is well defined and the consumer's demands are well defined for all positive prices.

It can be shown that the budget correspondence for this individual is not upper hemi-continuous at the point, $p = (1,1,1)$. For all prices $p \neq (1, 1, 1,)$ at least one of the demands, z_1 and z_2, is different from zero. At the price system $p = (1,1,1)$, the first period demands are: $x^1 = 2/7$ and $z^1 + z^2 = -1/7$, i.e. z_1 or z_2 or both are different from zero. The choice of the utility function was solely for computational convenience and the lack of existence is attributable to the collapsing of the budget set at the prices, $p = (1,1,1)$.

In this case, the agent believes that at the equilibrium price system it is not possible to transfer income across states. However, at all other price systems, expectations are such that the agent believes that income transfer is possible. If instead of certain price expectations the agent has a probability distribution over prices which assigns other than unit mass to one price system, then the agent always believes it is possible to alter incomes across states.

The remainder of this section collects the relevant assumption and propositions which in the main are standard for temporary equilibrium. [see Green [6] or Grandmont [4]].

Assumption 2: $\text{supp } \Psi^i \subset \text{int } (\Delta^{\ell_2})^S$. Zero prices do not have positive probability.

Assumption 3: $\sigma^i : \Delta^{\ell_1 + \ell_2} \to (\Delta^{\ell_2})^S$ where $\sigma^i(p) = \text{supp } \Psi^i(p)$ is a continuous compact valued correspondence.

For any $q^{i2} \in \text{supp } \Psi^i$ define $\bar{M}^i = q^{i2} \times A$ as before. Let M^i be a matrix of normalized monetary returns expected by i, where the normalization is such that the sum of the elements of any column of M^i is one. Let Δ^M be the collection of all such matrices,

$\Delta^M = \{M: \sum_{j=1}^{\ell_2} a_j(s)\, q_j^{i2}(s) = 1 \text{ for all } s \in \mathcal{S}\}$. Let $\tilde{\pi} = \pi / \sum_{j=1}^{\ell_2} \pi_j$

be the vector of normalized security prices. These definitions will be used in stating necessary assumptions on the relationship between security prices and monetary returns.

The probability measure $\Psi^i(p)$ implies a probability measure $\eta^i(p)$ on Δ^M for each $p \in \Delta^{\ell_1 + \ell_2}$. For each vector of current prices, the agent has a probability measure on second period prices which implies a probability measure on monetary returns.

Assumption 4: Each agent has a von Neumann - Morgenstern utility function $U^i : R^{\ell_1 + \ell_2 S} \to R$ that is continuous, concave, strictly monotone, and bounded.

We now consider the individual's problem, supressing the superscript i. Suppose the consumer has taken the action (x^1, z) in period 1. Consider some $q^2 \in \text{supp } \Psi$. Then consumption in period 2 for each state s is the vector that solves $\underset{x^2}{\text{MAX}}\ U\ (x^1, x^2)$ subject to

(i) $q^2(s)\ x^2(s) \leq q^2(s)\ w^2(s) + \overline{M}(s)\ z$ for all $s \in \mathscr{S}$

(ii) $x^2 \in X^2$.

Let x^{2*} denote the optimum. Define $U\ (x^1, x^{2*}) = U^*\ (x^1, q^2(1) + \overline{M}(1)\ z, \ldots, q^2(S)\ w^2(S) + \overline{M}\ (S)z)$.

Using a result of Grandmont [3], we know that U^* is a well defined, continuous and bounded function of q^2 for (x^1, z) provided that

(i) $q^2\ (s) \gg 0$ for all $s \in \mathscr{S}$

(ii) $[q^2(s)\ w^2(s) + \overline{M}(s)\ z] \geqq 0$ for all $s \in \mathscr{S}$

(i) follows from assumption 2.

<u>Assumption 5</u>: (ii) above holds for all $q^2 \in \text{supp } \Psi$. That is, no consumer plans to be bankrupt with positive probability.

To find first period consumption and security demands the consumer solves

$$\underset{x^1,z}{\text{MAX}}\ V\ (p, x^1, z) = \underset{x^1,z}{\text{MAX}} \int_{(\Delta^{\ell_2})^S} U^*\ (x^1, \cdot)\ \Psi\ (p, dq^2)$$

subject to

$$(x^1, z) \in \beta(p) = \{\ (x^1, z) \in R^{\ell_1 + \ell_2}\ |\ x^1 \in X^1,$$
$$q^1x^1 + \pi\ Z \leqq q^1w^1, \text{ and } q^2(s)w^2 + \overline{M}(s)\ Z \geqq 0 \text{ for all}$$
$$q^2(s) \in \text{supp } \Psi(p)\ \}.$$

Where $\beta : \Delta^{\ell_1 + \ell_2} \to R_+^{\ell_1 + \ell_2}$ is the budget correspondence and $V(p,x^1,z)$ is the expected utility index.

<u>Proposition 4</u>. (Grandmont [3]) $V(p,x^1,z)$ is a continuous function of (x^1, z) on the set of feasible actions $\beta(p)$ for all $p \in \Delta^{\ell_1 + \ell_2}$.

Let $g(p) = \text{MAX}\{V(p,x^1,z) : (x^i,z) \in \beta\ (p)\}$, the maximal expected utility given p. Let $\xi(p) = \{(x^1, z) \in \beta(p) : V(p, x^1, z) = g(p)\}$, the set of optimal actions for p. Then if $\beta(p)$ is continuous and compact valued we can apply the maximum principle to find that $\xi(p)$ is a non-empty, upper hemi-continuous, and compact valued correspondence.

Let $A = \{ a \in R^{\ell_2} : a = M(s)$ for some $s \in \mathscr{S}$, for some $M \in \operatorname{supp} \eta\}$. A is the collection of columns of monetary return matrices which are assigned positive probability. Let N = convex hull A . N is the set of all linear combinations of columns of M for all $M \in \operatorname{supp} \eta$. Hence N is a subset of the ℓ_2 dimensional simplex Δ^{ℓ_2}.

Propositions 5. A solution to the consumers' problem as described above exists if and only if

1) $p \gg 0$
2) $\tilde{\pi} \in N$

$\tilde{\pi} \in \operatorname{Int} N$ (relative to Δ^{ℓ_2}) if int $N \neq \emptyset$.

Proof: The argument hinges on the continuity of the individual's budget correspondence. The proof is in all essential respects identical to the proof in the rational expectations section. One difference is that the system of inequalities must hold for every $p \in \operatorname{supp} \Psi(p)$. Therefore, the arbitrary intersection of compact sets is compact.

The remaining question is to provide some sufficient conditions for int N to be non-empty. Two possible cases are:

1) positive price independence of monetary returns and $s \geq 2$
2) genuine uncertainty about future prices in each state, i.e.

the interior of the convex hull of the support of future prices in each state be non-empty.

In either case, there will be a system of linearly independent monetary returns whose dimension is independent of equilibrium conditions. The equilibrium proofs all go through with the usual additional requirements of some agreement about future prices if agents have different price expectations and some form of a tightness condition in order to avoid the non-existence problems demonstrated by Grandmont [4] and Green [6].

Conclusion

It may now be of some interest to inquire whether or not an economy with endogenous futures market formation or perhaps "insurance firms" leads naturally to a price independent structure. Such an exercise, were it successful, might justify the use of the price independence assumption for environments characterized by Hart.

A second consideration of the price independence structure is for the case of an economy with production and a collection of stock markets (unlimited liability) substituted for the futures markets. If short selling is permitted then the same problems of non-existence may arise in the economy; however, two sources for the potential discontinuity in the agent's budget set now appear. The first instance would be due to the dependence of the dimension of the profile of the portfolio's return on equilibrium prices and the second instance could be due to the entry or exist of a firm. Again, in this situation, the production decision of a firm may provide a dual role:

1) the provision of goods and services and 2) facilitating the exchange of risks in the economy.

A condition analogous to price independence, given point expectations, may impose a restriction on the technologies that would be permitted and also a restriction on the behavioral rules firms are to follow since a conflict between the output decision of the firm affecting the possibilities of risk sharing and household's desires to share risks might arise. In fact, the question of existence in this case seems even more problematic than the case of an exchange economy. Perhaps, then, this suggests that rather than being treated as a fixed set of markets or contracts, that contracts or markets should be treated as decision variables in the problem.

Notes

[1]Hart used securities which payoff in a vector of date two goods rather than a single good. However, his nonexistence results hold even in this more restrictive setting.

[2]Under the assumptions in the rational expectations model $n = S$.

[3]Purchasing a bundle of securities is equivalent to preforming row operations on the security structure, A, since the rows of A represent the various securities available.

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