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STOCHASTIC EQUILIBRIUM AND OPTIMALITY WITH ROLLING PLANS

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1. Introduction

There has been intensive discussion in the literature on macroplanning and growth about the appropriate time horizon for the economic decision-maker. The infinite planning horizon approach has served as a convenient standard of optimality, although the planner may be required to obtain and process too much information regarding future preferences, resources and technology. While the finite horizon approach has proven to be operational and provides a mathematically tractable framework, the length of the planning horizon and the terminal stock conditions are necessarily arbitrary. An interesting alternative is the "rolling plans" approach [2/] which is used in many countries by government planners (see Taylor [1975, p. 105], Johansen [1977, p. 108], and Zauberman [1967, p. 283]). A decision maker using the rolling plans approach faces a finite planning horizon of the same length in each period. The first period policy function, associating the optimal action with the state of the economy, is implemented in each period.

The rolling plan restricts the information requirements to those of the

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2/ In the literature on macroplanning, quite a number of terms have been used to describe this approach· "rolling" plans in Goldman [1968, p. 145], Radner [1975, p. 95] and Mirrlees and Stern [1972], p. 284], "roll-over" planning in Taylor [1975, p. 105], "sliding" plans in Johansen [1977, p. 209], "moving" plans in Frisch [1976, p. 135], in "rolling horizons" in Zauberman [1967, p. 283], and so on.

finite horizon problem, an important consideration when uncertainty is present. It is possible that the length of the rolling horizon may be determined by the planner based on computation requirements and other cost considerations. [3/] Assuming sufficient stationarity in the types of uncertainty faced by the decision maker (or equivalently a state space of sufficient size) the rolling plan shares the stationarity property of the infinite horizon plan. Thus, when the length of the rolling horizon is determined, there is no need to recalculate the optimal plan. The planner must only reassess the current state of the economy and use the optimal first period action of the finite-horizon rolling plan.

The stationarity property of rolling plans also has computational advantages over decision-making procedures which follow finite horizon plans to completion and then recalculate the next plan. Johansen [1977, p. 209] points out two additional advantages. First, the initial period decision rule of a medium -- or long-term plan will be more "reliable and relevant," while the decision rules of later periods become increasingly inaccurate as the effects of uncertainty accumulate. Secondly, the division of time into periods of four to five years "may tend to generate cycles in investment starting and completion which are not efficient from an overall point of view" (Johansen [1977, p. 209]). This paper will demonstrate that for a rolling horizon of sufficient length, following the stationary rolling plan in each period is approximately optimal [4/] in comparison with the stationary infinite horizon policy.

The uncertainty faced by economic planners with regard to future preferences,

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3/ A similar argument is made by Radner [1975, p. 113].

4/ Mirrlees and Stern [1972, p. 284] consider the qualities of a finite horizon plan followed to its completion but conjecture that "a 'rolling-plan' procedure is presumably superior, perhaps far superior, to the one we have assumed; and may well be satisfactory even for a short planning horizon."

available resources and technology emphasizes the need for a state-dependent, recursive decision procedure such as dynamic programming. The existence and optimality properties of rolling plans will be examined within the framework of stochastic dynamic programming in discrete time of Blackwell [1965], Maitra [1968], Strauch [1966] and Hinderer [1970]. The first period policy function and shadow prices will be shown to converge to the optimal infinite horizon policy function and shadow prices as the planning horizon is lengthened. Under additional assumptions, explicit forms may be obtained for the derivatives of the rolling plans.

Goldman [1968] has examined a deterministic, one sector growth model in continuous time where the planner is allowed continual revisions of a fixed horizon plan. Goldman finds that, for a fixed "target" capital-labor ratio, there is convergence to a quasi-stationary state which is sensitive to the value of the "target" capital-labor ratio. When the terminal capital-labor ratio is chosen to equal the existing capital-labor ratio, the resulting growth program is the same as for an infinite horizon planning period. The results presented here examine convergence in the space of policy functions and convergence of probability distributions on the state of the system when rolling plans are followed. Further work would be needed to explore the relationship between these convergence results and the optimum paths observed by Goldman.

The stationarity of rolling plans implies that the evolution of the system will obey a stationary Markov process. The transition equation for this process, formed through the interaction of environmental uncertainty and rolling plans can be used in analyzing the asymptotic behavior of the economic system. Using a discounted dynamic programming approach, Brock and Mirman [1972], Mirman [1972 and 1973] and Mirman and Zilcha [1975] have shown convergence to an invariant probability distribution on the size of the capital stock in a one-sector model of economic growth. These results were recently examined in an n-sector growth model by Brock and Majumdar [1978]. For the case of two controls, convergence to a stochastic equilibrium is shown here for a more general constraint correspondence and transition equation using the stronger properties of rolling plans (especially differentiability). The proof involves a detailed argument showing that the stochastic process on the states of the system can be restricted to a collection of disjoint invariant sets. The invariant distributions which are obtained induce "equilibrium" or steady-state distributions on the actions of the decision-maker, on the immediate returns to those actions and on shadow prices. Knowledge of these invariant probabilities is important for empirical work since they are the only observable results of actions taken by the decision maker.

In Section 2 the dynamic programming problem under uncertainty is defined and a general proof is given of the existence and optimality of a stationary plan for the infinite horizon problem. Then it is shown that under very general assumptions the finite horizon plan exists and converges to the infinite horizon plan. The continuity and convergence of rolling plans are then used to show that rolling plans with a horizon of appropriate length are ε-optimal. In Section 3, conditions are examined which guarantee differentiability of the finite and infinite horizon value functions. These results are seen to imply that the shadow price for the rolling plan converges to the stationary infinite horizon shadow price. In Section 4, the finite horizon problem is reconsidered and it is shown that for appropriate utility functions, the finite horizon plan and shadow prices are differentiable, and explicit forms of the derivatives may then be obtained. Noting that rolling plans are stationary and have the properties of the finite horizon plans, the existence of steady-state distributions on the state of the system, when rolling plans are used, is demonstrated in Section 5 for the case of two policy instruments. Section 5 shows that the evolution of the system follows a Markov process and indicates how the state space can be divided into invariant and transient sets. The question of convergence to steady-state distributions on the invariant sets is examined in detail.

2. Existence and Optimality of Rolling Plans

The optimality of rolling plans will be considered within the framework of discrete time stochastic dynamic programming when utility is discounted. A general theorem on the existence and continuity of a stationary optimal plan for the infinite horizon problem will be presented. Then it will be shown that finite horizon plans converge to the optimal infinite horizon plan. These results are of some independent interest. As a rolling plan involves the repeated use of the first period decision rule of a finite horizon plan, the convergence of finite horizon plans to the infinite horizon plan will be used to show the ε-optimality of rolling plans.

The Markovian decision model presented here is general enough to handle the one-sector growth models under uncertainty of Brock and Mirman [1972], Mirman [1972 and 1973] and Mirman and Zilcha [1975] as well as the multi-sector growth model of Brock and Majumdar [1978]. The results are not directly comparable with uncertainty models without discounting which employ alternative optimality criteria or which do not obtain representations of the optimal policy such as Radner [1973], Dana [1974] and Jeanjean [1974].

2.1 The Finite and Infinite Horizon Dynamic Programming Problem

We begin by introducing the discrete time dynamic programming problem with Markov disturbances.

Definition 1. A stationary Markov decision model is given by the tuple $((S,\mathcal{S}), (E,\mathcal{E},\phi), (A,\mathcal{A}), u,g,b)$ where:

(i) S describes the state space of the system, $s \in S$. S has σ-algebra $\mathcal{S}$.

(ii) E is the random events space, $\omega \in E$. E has σ-algebra $\mathcal{E}$. Let $\phi: E \times \mathcal{E} \to [0,1]$ define the Markov transition probability on $(E,\mathcal{E})$.

(iii) A is the action space for the decision maker, $a \in A$. A has σ-albebra $\mathcal{A}$.

(iv) The function $u: S \times E \times A \to \mathbb{R}_+$ is the immediate reward, i.e., $u(s,\omega,a)$ is the reward to the decision maker of taking action a when the system is in state (s,ω).

(v) The function $f: S \times A \to S$ is the transition equation.

(vi) The correspondence $b: S \times E \to A$ is the constraint correspondence, i.e., the set $b(s,\omega)$ describes the feasible actions available to the decision maker when the system is in state s after the random disturbance ω has occurred.

We will assume throughout that S,E and A are Borel subsets of complete separable metric spaces. It is important to note that (iv) implicitly assumes that utility is additively separable over time. We will assume that utility is discounted at rate α. From (iv) and (vi) it is evident that problems involving uncertainty with regards to preferences, endowments and technology can be handled.

We will now give a more precise definition of what is meant by "optimal plan" for the infinite horizon problem. Let $X = S \times E$ denote the state of the system where $x_n = (s_n, \omega_n)$ is an element of the state space at date n. For comparable definitions in the dynamic programming literature see Blackwell [1965], Maitra [1968], Strauch [1966] and Hinderer [1970].

Definition 2.

(a) A plan f is a sequence $f = (f_n)$ which selects an action on the n^{th} day as a function of the history $H_n = (x_1, a_1, \ldots, a_{n-1}, x_n)$ of the system by associating with each history H_n (Borel measurably) a probability distribution $f_n(\cdot \mid H_n)$ on the Borel subsets of A.

(b) A deterministic admissible plan is a sequence $f = (f_n)$ of measurable maps $f_n\colon X^n \to A$ with the property that

$$f_n(x_1, x_2, \ldots, x_n) \in b(x_n) \qquad x_n \in X.$$

(c) A stationary (deterministic admissible) plan will be denoted by $f = f^{(\infty)}$. The plan $f\colon S \times E \to A$ selects an action $a \in b(s,\omega)$ regardless of how the current state (s,ω) was arrived at.

Given the previous two definitions, when the plan f solves a discounted dynamic programming problem, then the plan f associates with each state (s,ω) an expected discounted total return.

Definition 3.

Given the plan f, an expected discounted total return is represented by

$$\text{(1)} \qquad I(f)(s,\omega) = E[u(s,\omega,f(s,\omega)) + \sum_{n=2}^{\infty} \alpha^{n-1} u(s_n,\omega_n, f(s_n,\omega_n))].$$

We are now ready to define an optimal plan.

Definition 4.

A deterministic admissible plan f^* will be called optimal if $I(f^*)(s,\omega) \geq I(f)(s,\omega)$ for all deterministic admissible plans f and $(s,\omega) \in S \times E$ (see Blackwell [1965] and Maitra [1968]).

Another definition which will be useful is that of the optimal value function.

Definition 5.

The maximal expected reward $V: S \times E \to \mathbb{R}_+$ will be defined by

$$(2) \quad V(s,\omega) = \sup \{ [u(s,\omega,a) + \alpha \int V(g(s,a),\tilde{\omega}) d\phi(\tilde{\omega} \mid \omega)] \ : \ a \in b(s,\omega) \}.$$

We will now state all of the assumptions which will be used in this section. 5/

Assumption 1. The action space A is compact.

Assumption 2. a. The immediate reward u is nonnegative and bounded above.

b. u is continuous.

c. u is strictly concave in a, concave in (s,ω), and increasing in (s,ω,a).

Assumption 3. $0 \leq \alpha < 1$.

Assumption 4. a. The transition function g is continuous.

b. g is strictly concave in a, concave in s, and increasing in s,a.

Assumption 5. The constraint correspondence b is continuous. Also, $b(s,\omega)$ is a convex set for any state (s,ω) and $b(\cdot,\omega)$ has a convex graph for each ω.

5/ For a discussion of state dependent preferences, see Hildenbrand [1970] and Bewley [1972].

Assumption 6. The measure $\emptyset(\cdot\,|\omega)$ is continuous in the weak topology.

To establish the existence of a continuous stationary optimal plan the following lemma will be useful.

Lemma 1. Given A1, A2a,b, A3, A4a, A5, A6:

(i) There exists an optimal value function $V: S \times E \rightarrow \mathbb{R}_+$ which is continuous and bounded on $S \times E$.

(ii) There exists a stationary optimal plan $f: S \times E \rightarrow A$ which is upper-semicontinuous on $S \times E$.

The proof will be given in Appendix A and draws upon the approach of Blackwell [1965], Strauch [1966], and Maitra [1968].

Given Lemma 1 it remains to show that f is single-valued, since a function which is upper-semicontinuous and single-valued is continuous. The proof of the following theorem is also given in Appendix A.

Theorem 1. Given A1 to A6:

(i) There exists an optimal return $V: S \times E \rightarrow \mathbb{R}_+$ which is continuous and bounded on $S \times E$ and concave on S.

(ii) There exists a stationary optimal plan $f: S \times E \rightarrow A$ which is continuous on $S \times E$.

The infinite horizon plan obtained in Theorem 1 may be used as a standard of optimality. The optimal policies for a finite horizon plan will be characterized for the discounted finite horizon problem with a general "scrap-value" on stocks remaining in the final period. Using arguments similar to the proof of Theorem 1 it will be shown that the finite horizon plan converges to the infinite horizon plan.

An important consideration for economic planners is the sensitivity of optimal investment plans to the length of the planning horizon or to the size of final stock requirements (see, for example, Chakravarty [1969] and Blitzer, et. al. [1975]). Our convergence result may be considered as a generalization to the policy space of the insensitivity results of Brock [1971], Brock and Mirman [1972], Nermuth [1978] and Nyberg and Viotti [1973]. 6/ Let $W: S \times E \times A \rightarrow \mathbb{R}_+$ denote the terminal condition on the actions of the decision maker. Clearly, if W were identical to the expectation of the value function for the infinite horizon plan, the solutions to the finite and infinite horizon problems would be identical. Any iterative search for this "correct" scrap-value would be equivalent to solving the infinite horizon plan. Thus, while the determination of W may depend on the computational and informational limitations of a particular economic planner, we restrict our attention to an arbitrary W satisfying:

Assumption 7. The terminal value $W: S \times E \times A \rightarrow \mathbb{R}_+$ is continuous and bounded on $S \times E \times A$ and concave on $S \times A$.

The value function for the finite horizon problem will now be introduced.

Definition 6. The maximal expected reward for a decision problem for T periods is given by

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6/ Modigliani and Hohn [1955, p. 65] suggest that for particular problems, "it may be useful to single out the factors that tend to limit the size of the relevant planning and expectation horizon." See also Chakravarty [1962 and 1966] and Manneschi [1966a and b] for a discussion of sensitivity. On the relationship between initial consumption and final capital requirements in models of economic growth see McFadden [1967] and Bliss [1971].

(3) $$V^T(s_1,\omega_1) \equiv \max_{(a^T,\ldots,a^1)} E[\sum_{t=1}^{T} \alpha^{t-1} u(s_t,\omega_t,a^{T+1-t}) + \alpha^{T-1} W(s_T,\omega_T,a^1)]$$

where $a^{T+1-t} \in b(s_t,\omega_t)$ and where $s_{t+1} = g(s_t,a^{T+1-t}), T \geq 1$.

Let $(a^T) = (a^T, a^{T-1}, \ldots, a^1)$ be the optimal plan for the T-period problem given by definition 6. This plan is characterized in the next theorem.

Theorem 2.[7/] Given A1 to A7:

(i) There exists a T period optimal plan (a^T) which is continuous on $S \times E$.

(ii) There exists a T period optimal return $V^T: S \times E \to \mathbb{R}_+$ which is continuous and bounded on $S \times E$ and concave and nondecreasing on S.

(iii) $\lim_{T\to\infty} a^T = a^\infty$ where a^∞ is the stationary optimal plan for the infinite horizon problem.

(iv) $\lim_{T\to\infty} V^T = V$ where V is the optimal return for the infinite horizon problem.

Proof. (The approach is similar to the proof of Theorem 1.) Let $C_S(S \times E)$ be the space of continuous and bounded functions on $S \times E$ which are also concave and nondecreasing on S. $C_S(S \times E)$ is a Banach space (as noted in (v) in the Appendix). As noted in Appendix (vi), the operator M defined by

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7/ A major step in the proof of Theorem 2 is the demonstration of the fact that a cluster point of the sequence of finite horizon plans is a plan which solves the infinite horizon problem. This result has been shown by Hammond and Kennan [1976] under their assumption of "valuation finiteness," which is satisfied here. Theorem 2 further demonstrates that under our assumptions (particularly discounting and strict concavity) the finite horizon plans actually converge to the infinite horizon plan.

$$(Mv)(s,\omega) = \max_{a \in b(s,\omega)} [u(s,\omega,a) + \alpha \int v(g(s,a),\tilde{\omega}) d\emptyset(\tilde{\omega} \mid \omega)]$$

for $v \in C_S(S \times E)$ takes $C_S(S \times E)$ into itself and is a contraction mapping. Let $C(S \times E)$ be the space of continuous and bounded functions on $S \times E$. The operator $L(f)$ on $C(S \times E)$ takes $v \in C(S \times E)$ into $L(f)v \in C(S \times E)$ where L is defined by

$$(4) \quad (L(f)v)(s,\omega) = u(s,\omega,f(s,\omega)) + \alpha \int v(g(s,f(s,\omega)),\tilde{\omega}) d\emptyset(\tilde{\omega} \mid \omega).$$

By remark (iv) of the Appendix, it follows that there exists a continuous and bounded map $a^T \in C(X)$ such that $V^T = MV^{T-1} = L(a^T)V^{T-1}$. This implies that a^T and V^T exist and are optimal.

By the Banach fixed point theorem, M has a unique fixed point. Let $V \in C_S(S \times E)$ be its unique fixed point, i.e., $MV = V$. From definition 6, we may write $V^T(s,\omega) = (M^{T-1}V^1)(s,\omega)$. Thus,

$$(5) \quad \lim_{T\to\infty} V^T(s,\omega) = \lim_{T\to\infty} (M^{T-1}V^1)(s,\omega) = V(s,\omega).$$

By Theorem 1 there exists a continuous function $a \in C(X)$ such that $MV = L(a)V$. Since V is the unique fixed point of M, $L(a)V = V$. So $V = I(a^\infty)$, and $MV = V$ can be rewritten as

$$(6) \quad I(a^\infty)(s,\omega) = \max_{a \in b(s,\omega)} [u(s,\omega,a) + \alpha \int I(a^\infty)(g(s,a),\tilde{\omega}) d\emptyset(\tilde{\omega} \mid \omega)].$$

Since $I(a^\infty)$ satisfies the optimality equation, a^∞ is an optimal plan. Note that $V^T = MV^{T-1} = L(a^T)V^{T-1}$. Hence $\lim_{T\to\infty} L(a^T)V^{T-1} = \lim_{T\to\infty} V^T = V = L(a^\infty)V$. Further $L(a)V^T \to L(a)V$, uniformly in a, from the definitions of L and the contraction M. This implies that $L(a^T)V \to L(a^\infty)V$. To see this remark more clearly, note that by the triangle inequality

$$(7)\quad \|L(a^T)V - L(a^\infty)V\| \leq \|L(a^T)V - L(a^T)V^{T-1}\| + \|L(a^T)V^{T-1} - L(a^\infty)V\| .$$

By the convergence results above it is possible, for any $\gamma > 0$, to select T^* such that both of the terms on the right side are less than $\gamma/2$. Hence, for any $\gamma > 0$ there is a T^* such that $\|L(a^T)V - L(a^\infty)V\| \leq \gamma$ for all $T \geq T^*$.

It will now be shown that the convergence of the sequence $L(a^T)V$ to $L(a^\infty)V$ implies that (a^T) is a convergent sequence **with limit** a^∞. Let $F = \{a:X \to A \mid a \text{ is measurable}\}$, the compactness of F follows from the Tychonoff theorem since A is compact. Hence (a^T) has a convergent subsequence $(a^{T_n}) \to \bar{a}$. Since $L(\cdot)V$ is continuous in a, $L(a^{T_n})V \to L(\bar{a})V$. But $L(\bar{a})V = L(a^\infty)V$ and the uniqueness of a^∞ implies that $\bar{a} = a^\infty$. Therefore $(a^T) \to a^\infty$.

2.2 Optimality of Rolling Plans

Rolling plans are made by a decision maker who has a fixed finite planning horizon at each date, say T periods in length. The decision maker determines his optimal action in the current period by solving the T-period dynamic programming problem and choosing action a^T in the current period. Due to our stationarity assumptions and the fact that the decision maker always looks ahead the same number of periods, the function $a^T: S \times E \to A$ is employed in each period without any need for recalculation. Note that the rolling plan obtained by employing the rule a^T in each period will have all the properties of the first period decision rule of the finite horizon problem.

As pointed out by Hammond [1975, p. 2], "a plan may be insensitive without being optimal." The optimality of rolling plans depends in an important way upon the continuity of the finite horizon policy function as well as on the convergence of the rolling plan to the infinite horizon plan. 8/

<u>Theorem 3</u>. Given A1 to A7, rolling plans for a horizon of length $T \geq T_\epsilon$ are ϵ-optimal; i.e., for any $\epsilon > 0$, $\exists\ T_\epsilon \in N$ finite, such that for all $T \geq T_\epsilon$,

$$E(\sum_{t=1}^{\infty} \alpha^{t-1} u(\cdot,\cdot,f(\cdot,\cdot))) - E(\sum_{t=1}^{\infty} \alpha^{t-1} u(\cdot,\cdot,f^T(\cdot,\cdot))) \leq \epsilon$$

where f is the optimal stationary plan for the infinite horizon problem and f^T is the rolling plan for the T-period horizon.

<u>Proof.</u> By Theorem 2, $f^T \to f$. Also by Theorems 1 and 2, note that f^T and f are continuous.

We wish to show that

$I(f^T) \to I(f)$ for $f^T \to f$, where $I(\cdot)$ is given by definition 3.

Applying definition 3 and rearranging terms yields:

8/ As the approach taken here examines the value of following the first period policy function, the result is not directly comparable with the "agreeable" or "strongly agreeable" plans criterion of Hammond [1975], Hammond and Mirrlees [1973], or Hammond and Kennan [1976]. According to Hammond [1975, p. 3]; "a plan is agreeable provided that the welfare loss from the wrong start becomes insignificant as the horizon H tends to infinity." This criterion is similar to the ϵ-horizon examined in Łoś [1967 and 1971] and Keeler [1974]. These papers focus on final stock requirements.

$$(9)\quad I(f^T)(s,\omega) - I(f)(s,\omega) = u(s,\omega,f^T(s,\omega)) - u(s,\omega,f(s,\omega))$$

$$+ E[\sum_{n=2}^{T} \alpha^{n-1}[u(s_n^T,\omega_n,f^T(s_n^T,\omega_n)) - u(s_n,\omega_n,f(s_n,\omega_n))]$$

$$+ \alpha^T I(f^T)(s_{T+1}^T,\omega_{T+1}) - \alpha^T I(f)(s_{T+1},\omega_{T+1})$$

where $s_n^T = g(s_{n-1}^T, f^T(s_{n-1}^T,\omega_{n-1}))$ and $s_2^T = g(s,f^T(s,\omega))$.
Since $I(f^T)$ and $I(f)$ are bounded and $\alpha^T \to 0$ as $T \to \infty$, the last two terms go to zero as $T \to \infty$. The first term $[u(s,\omega,f^T(s,\omega)) - u(s,\omega,f(s,\omega))] \to 0$ by the continuity of u. So, we may limit our attention to an arbitrary term in the sum $[u(s_n^T,\omega_n,f^T(s_n^T,\omega_n)) - u(s_n,\omega_n,f(s_n,\omega_n))]$. Note that $s_2^T \to s_2 = g(s,f(s,\omega))$ as $T \to \infty$ by the continuity of g. By induction (and using the continuity of f^T), $s_n^T \to s_n$. So, by the continuity of u, f^T and g,

$$(10)\quad \lim_{T\to\infty} [u(s_n^T,\omega_n,f^T(s_n^T,\omega_n)) - u(s_n,\omega_n,f(s_n,\omega_n)] = 0$$

Thus, $\lim_{T\to\infty} [I(f^T)(s,\omega) - I(f)(s,\omega)] = 0$ for any (s,ω).

The ε-optimality of rolling plans suggests that if the finite horizon plan converges rapidly to the stationary infinite horizon plan then there may be very small losses from the use of a short horizon in calculating the rolling plan. Further research may be worthwhile comparing the information costs of extending the planning horizon with the welfare loss of a shorter horizon.

3. Convergence of Shadow Prices

A system of shadow prices may be associated with the optimal actions of the planner in both the finite and infinite horizon problems considered in the last section. Given that the environmental disturbances form a stationary Markov process, the sequence of states of the system and shadow prices will also form a stationary Markov process. 9/ In this section shadow prices will be obtained by applying Lagrangean techniques to the recursive functional equation of dynamic programming. The shadow price for the first period rolling plan will then be shown to converge to the stationary shadow price for the infinite horizon plan given some additional assumptions which guarantee the differentiability of the finite and infinite horizon value functions. Thus the behavior of the shadow price over time for rolling plans provides a good indication of the movement of the shadow price for the optimal infinite horizon plan.

The stationarity of the infinite horizon price system was examined by Radner [1973] and Dana [1974] within a general Arrow-Debreu framework assuming a probability distribution on the set of sequences of states. 10/ The existence of prices supporting an optimal program has been established by Zilcha [1976a,b and 1978] where environmental disturbances are independent

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9/ As Radner [1976, p. 115] has pointed out, the sequence of prices alone is not a Markov process.

10/ Radner [1973], Dana [1974] and Jeanjean [1974] examine Lagrange multipliers in stochastic setting without an explicit consideration of first order necessary conditions for an optimal policy. For a functional analysis approach see Majumdar and Radner [1972].

and identically distributed. In a similar framework, Brock and Majumdar obtain a stochastic "turnpike" theorem. Föllmer and Majumdar [1978] examine prices supporting competitive programs in a general intertemporal allocation model under uncertainty. An interesting set of papers by Dynkin [1974], Evstigneev [1974], Kuznetsov [1974] and Tacsar [1974] have considered optimal planning under uncertainty by applying concave programming to the theory of controlled stochastic processes. In particular, Tacsar [1974] has considered prices for stochastic programs with discounting. The results derived here will focus only on the properties of the finite and infinite horizon shadow prices as functions of the state of the system and the last observed disturbance.

The following assumptions will be used to obtain differentiable value functions V^T and V.

Assumption 8. The state space and random events space are $S \subset \mathbb{R}_+^\ell$ and $E \subset \mathbb{R}^m$. At each date t, the optimal action a_t is given by $a_t = (\pi_t, \sigma_t)$ where $\pi_t \in \mathbb{R}_+^\ell$, $\sigma_t \in \mathbb{R}_+^\ell$.

Assumption 9. The immediate reward $u(s,\omega,\pi)$ is continuously differentiable in π.

Assumption 10. The transition equation is given by $g: \mathbb{R}_+^\ell \times \mathbb{R}_+^\ell \to \mathbb{R}_+^\ell$ where $s_{t+1} = g(s_t, \sigma_t)$ and $g \in C^1$.

Assumption 11. The action $(\pi,\sigma) = (\pi^i, \sigma^i)$, $i = 1, \ldots, \ell$ is constrained by

$$(11) \quad b^i(s,\omega) - h^i(\pi^i) - p^i(\sigma^i) = 0 \qquad i = 1, \ldots, \ell$$

where b^i, h^i, p^i are continuously differentiable for all i. Further, $h^{i\prime}(\pi) \neq 0$ for all $\pi \in \mathbb{R}_+^{\ell}$.

Mirman and Zilcha [1975] have shown differentiability of the value function for $\ell = 1$. Using the implicit function theorem, a generalization of this result is obtained.

<u>Lemma 2</u>. The value functions V^T and V are continuously differentiable in s, given assumptions 1 to 11.

Given Lemma 2, the Lagrangean approach may be used to obtain the first order necessary conditions for the infinite horizon problem.

$$(12)\quad \mathcal{L} = u(s,\omega,\pi) + \alpha \int V(g(s,\sigma),\tilde{\omega})d\phi(\tilde{\omega} \mid \omega) + \sum_{i=1}^{\ell} \lambda^i[b^i(s,\omega) - h^i(\pi^i) - p^i(\sigma^i)].$$

Differentiating $\mathcal{L}$ w.r.t. to π^i, σ^i and λ^i yields

$$(13)\quad u_{\pi^i}(s,\omega,\pi) - \lambda^i h^{i\prime}(\pi^i) = 0$$

$$(14)\quad \alpha \int V_{s^i}(g(s,\sigma),\tilde{\omega})g_{\sigma^i}(s,\sigma)d\phi(\tilde{\omega} \mid \omega) - \lambda p^{i\prime}(\sigma^i) = 0$$

$$(15)\quad b^i(s,\omega) - h^i(\pi^i) - p^i(\sigma^i) = 0$$

for $i = 1, \ldots, \ell$. A similar set of first order conditions may be obtained for the finite horizon problem. Then, rewriting (13) and suppressing the i superscript yields $\lambda = u_\pi(s,\omega,\pi)/h'(\pi)$. For the finite horizon problem we obtain $\lambda^T = u_\pi(s,\omega,\pi^T)/h'(\pi^T)$. By Theorems 1 and 2 there exist policies satisfying these conditions.

<u>Theorem 4</u>. The finite horizon shadow price converges to the infinite horizon shadow price.

<u>Proof.</u> Note from Theorem 3 that $\pi^T \to \pi$. Since u and h are continuously differentiable

(16) $u_\pi(s,\omega,\pi^T)/h'(\pi^T) \to u_\pi(s,\omega,\pi)/h'(\pi)$.

Therefore, $\lambda^T \to \lambda$.

4. Differentiability of Rolling Plans

The assumptions on the immediate reward, the transition equation and the constraint on the decision maker's actions must be strengthened to obtain sufficient conditions for the differentiability of rolling plans. For clarity of exposition, the case of two controls $(\pi,\sigma) \in \mathbb{R}_+^2$ and one state variable $s \in \mathbb{R}_+$ will be considered. Also let $\omega \in \mathbb{R}_+$. The following additional assumptions will be used.

<u>Assumption 12</u>. The immediate reward is a function of the action π only, i.e., the state of the system does not affect the immediate reward. So $u:\mathbb{R}_+ \to \mathbb{R}_+$. Also let $u \in C^\infty$ and $u'(\pi) \to \infty$ as $\pi \to 0$. The terminal value $W(s,\sigma)$ is bounded and concave on $\mathbb{R}_+^2$ and $W \in C^\infty$.

The assumption that u is infinitely continuously differentiable is made to assure that the optimal plan is differentiable for any large (but finite) planning horizon. This is not a very restrictive assumption since it permits, for example, any utility function which is a member of the exponential family such as $\exp(-\beta/\pi)$. Examples of these types of utility functions have appeared frequently in the literature on optimal growth and on portfolio theory. 11/

The transition equation and constraint correspondence will also have special forms:

Assumption 13. The transition equation is given by $g(s,\sigma)$, $g \in C^{\infty}$ where g is concave and strictly increasing.

Assumption 14. The action (π,σ) is constrained by

(17) $b(s,\omega) - h(\pi) - p(\sigma) = 0$, $b,h,p \in C^{\infty}$ where h and p are convex and strictly increasing and b is concave and strictly increasing.

The next assumption will allow us to exchange the differential and expectations operators.

Assumption 15. The disturbances (ω) are i.i.d.r.v. with distribution $\phi(\cdot)$.

Given these assumptions, a plan $(a^T) = ((\pi^T,\sigma^T),(\pi^{T-1},\sigma^{T-1}),\ldots,(\pi^1,\sigma^1))$ will be optimal if it satisfies

$$(18)\quad \max E\left[\sum_{t=1}^{T} \alpha^{t-1} u(\pi^{T+1-t}) + \alpha^{T-1} W(s_T,\sigma^1)\right] \quad \text{s.t. } b(s_t,\omega_t) - h(\pi^{T+1-t}) - p(\sigma^{T+1-t}) = 0 .$$

where $s_{t+1} = g(s_t,\sigma^{T+1-t})$. The optimal plan (π^T,σ^T) and the shadow price λ^T are differentiable and their partial derivatives w.r.t. s and ω have explicit forms.

11/ See Chakravarty [1969], Hahn [1970], Leland [1974], Levhari and Srinivasan [1969], Mirrlees [1974], Phelps [1962] and Tinbergen [1960].

<u>Theorem 5</u>. Given the strengthening of assumptions 1 to 7 implied by assumptions 12 to 15, for any $T < \infty$:

(i) $(\pi^T, \sigma^T) \in C^\infty$ and $\lambda^T \in C^\infty$.

(ii) $V^T \in C^\infty$ and V^T is increasing and concave in s.

<u>Proof</u>.[12/]

$$(19)\quad V^1(s,\omega) = \max_{(\pi^1,\sigma^1)} [u(\pi^1) + W(s,\sigma^1)] \text{ st } b(s,\omega) - h(\pi^1) - p(\sigma^1) = 0$$

implies that $\pi^1, \sigma^1 \in C^T$ where $\pi^* = \pi^1(s,\omega)$ and $\sigma^* = \sigma^1(s,\omega)$ since $u \in C^{T+1}$ and given assumptions 12 to 14. So $V^1(s,\omega) = u(\pi^1(s,\omega)) + W(s,\sigma^1(s,\omega))$ implies that $V^1 \in C^T$.

<u>Induction hypothesis</u>. Suppose $V^{t-1} \in C^{T+1-(t-1)}$ and $f^{t-1} \in C^{T+1-(t-1)}$. Then we will show that $\lambda^t \in C^{T+1-t}$, $f^t \in C^{T+1-t}$, $V^t \in C^{T+1-t}$. Consider

$$(20)\quad V^t(s,\omega) = \max_{(\pi,\sigma)} [u(\pi) + \alpha \int V^{t-1}(g(s,\sigma),\tilde{\omega})d\phi(\tilde{\omega})]$$

$$\text{s.t. } b(s,\omega) - h(\pi) - p(\sigma) = 0.$$

Form the Lagrangean:

$$(21)\quad \mathcal{L} = u(\pi) + \alpha \int V^{t-1}(g(s,\sigma),\tilde{\omega})d\phi(\tilde{\omega}) + \lambda(b(s,\omega) - h(\pi) - p(\sigma))$$

The first order necessary conditions are then:

$$(22)\quad u' - \lambda h' = 0$$

12/ The theorem is stated for $u \in C^\infty$ so that for any arbitrarily large (but finite) horizon, the optimal policy will be differentiable. The proof will show that for any T, $u \in C^{T+1}$ implies that

(i) λ^t, $(\pi^t, \sigma^t) \in C^{T+1-t}$ for all $t \le T$ and $(\pi^T, \sigma^T) \in C^1$

and

(ii) $V^t \in C^{T+1-t}$ for all $t \le T$ and $V^T \in C^1$.

(23) $\alpha \int V_s^{t-1} g_2 - \lambda p(\sigma) = 0.$

(24) $b(s,\omega) - h(\pi) - p(\sigma) = 0.$

The derivatives may be taken by the induction hypothesis.

The second order necessary conditions for a maximum are satisfied since

(25) $|H| = -(h')^2[\alpha \int V_{ss}^{t-1}(g_2)^2 + \alpha \int V_s^{t-1} g_{22} - \lambda p''] - (p')^2[u'' - \lambda h''] > 0.$

Since f^t is optimal and f^t solves the above problem, then $f^t \in C^{T+1-t}, \lambda^t \in C^{T+1-t}$. Substituting back into V^t we obtain

(26) $V^t(s,\omega) = u(\pi^t(s,\omega)) + \alpha \int V^{t-1}(g(s,\sigma^t(s,\omega)),\tilde{\omega})d\emptyset(\tilde{\omega}).$

This implies that $V^t \in C^{T+1-t}$. Then, by induction:

$\lambda^t \in C^{T+1-t},\ f^t \in C^{T+1-t}, V^t \in C^{T+1-t}$ for all $t \leq T$.

By differentiating the first-order necessary conditions (22-24) and applying Cramer's rule, we obtain explicit forms for the partial derivatives of the plans π and σ and the shadow price λ with respect to s and ω. Only $\frac{\partial\sigma}{\partial s}$ and $\frac{\partial\sigma}{\partial\omega}$ will be stated here since they will be used to characterize the asymptotic behavior of the system:

(27) $\frac{\partial\sigma}{\partial s} = = \frac{1}{|H|} \{- p'b_s[u''- \lambda h''] + (h')^2[\alpha \int V_{ss}^{t-1} g_1 g_2 + \alpha \int V_s^{t-1} g_{12}]\}$

(28) $\frac{\partial\sigma}{\partial\omega} = \frac{1}{|H|} \{ - p'b_\omega[u'' - \lambda h'']\}.$

Given the optimal policy $\sigma(s,\omega)$ the transition equation for the system is defined by $\theta(s,\omega) \equiv g(s,\sigma(s,\omega))$. The effect of the current state s and disturbance ω on the state of the system will depend on the properties of the transition function g and the optimal policy σ. Note that $\theta_s = g_1 + g_2\sigma_2$. From assumptions 12 and 14, and equation (28), $\sigma_\omega > 0$. Since $g_2 > 0, \theta_\omega > 0$, the derivative of θ with respect to s can be shown to be positive given the following assumption.

Assumption 16. The transition function g satisfies $g_{12} \geq 0$ and

$0 < g_2 \leq 1$.

The following lemma will be used in the next section to examine the long-run behavior of the system.

Lemma 3. Given the assumptions of Theorem 5 and Assumption 16,

$\theta_s > 0$.

Proof. Consider the derivatives of σ w.r.t. s. Since $u'' < 0$, $h'' \geq 0$,

$p' > 0$, $b_s > 0$, $\lambda > 0, g_{12} \geq 0$ and $h'' \leq 0$,

$$(29)\quad \frac{\partial \sigma}{\partial s} > (-g_1)\ [\frac{(h')^2}{|H|}\ (-\alpha \int V_{ss} g_2 d\phi)].$$

The term in brackets is less than one or equal to one so that

$$(30)\quad \frac{\partial \sigma}{\partial s} > -g_1$$

Then, since $0 < g_2 \leq 1$,

$$(31)\quad \theta_s = g_1 + g_2 \frac{\partial \sigma}{\partial s} > g_1 - g_2 g_1 \geq 0 .$$

5. Stochastic Equilibrium Generated by Rolling Plans

The stationarity of rolling plans implies that the sequence of states and actions for the economic system forms a stationary Markov process.

This process may be used to demonstrate convergence to an invariant distribution on the states of the system. Formally, the transition probability for the system, $P: S \times \mathscr{S} \to [0,1]$ is defined by $P(s,B) = \phi\{\omega \in E \mid \theta(s,\omega) \in B\}$ for any $s \in S, B \in \mathscr{S}$. P gives the probability measure on the next state, given the current state, that is generated by the interaction of the randomness and the optimal rolling plan. The transition probability P defines a Markov process on the state space S. 13/ The convergence result will be obtained by showing that P satisfies Doeblin's condition and that the stochastic process is limited to a collection of disjoint invariant sets. The state space for the Markov process will be "connected" by means of a link point in each ergodic set, where a link point is defined as follows:

<u>Definition 7</u>. $s_o \in S$ is a <u>link point</u> if for any integer $k \geq 1$, any point $s \in S$, and any neighborhood U of s_o there is an integer n such that $P^{nk}(s,U) > 0$.

The method used to divide the state-space is a generalization of a fixed point technique used by Brock and Mirman [1972]. Much of the analysis will draw upon the presentation of Futia [1976]. 14/ The differentiability of the transition equation θ distinguishes the approach taken here from the convergence results of Brock and Mirman [1972], Majumdar [1975] and Green and Majumdar [1975].

13/ The transition probability P defines a Markov process on state space S if; (1) $P(s,\cdot)$ is a probability measure on $(S,\mathscr{S})$ for all $s \in S$, (2) $P(\cdot,A)$ is a measurable function for all $A \in \mathscr{S}$. Both of these properties follow directly from the fact that θ is a measurable function. The measurability of θ results from the fact that the optimal rolling plan σ is stationary, deterministic, and single-valued.

14/ The analysis here differs from Futia [1976, section eight] in the analysis of the ergodic sets. Note that Futia assumes the differentiability of the policy function for the infinite horizon growth model. This condition will not hold in general.

Without much loss of generality, the event space is restricted to a compact interval.

Assumption 17. The event space is $E = [\gamma,\beta]$. Let η denote the Lebesgue measure on E and let $f(\omega)$ be any bounded, positive, measurable function on E such that $\mu(A) = \int_A f(\omega)\,\eta(d\omega)$ for all $A \in \mathcal{B}$ defines a probability on E. Assume that $\phi = \epsilon\mu + (1-\epsilon)\xi$ where ξ is any probability on E with $\xi(\gamma)$ strictly positive and $0 < \epsilon < 1$.

The set of all ϕ satisfying this assumption is an open and dense subset of $\mathcal{P}(E)$.

We will now place assumptions on the problem which will insure that the relevant state space for the optimal process is compact and that the process does not collapse.

Assumption 18. a. $g(0,0) = 0$, $b(0,\omega) = 0$ for all $\omega \in E$, $h(0) = 0$, and $p(0) = 0$.

b. There exists an $\bar{s}$ such that for all $s > \bar{s}$, $g(s,\sigma) < s$ for all feasible σ.

c. There exists an $\hat{s} > 0$ such that for all $s \in (0,\hat{s})$ and for all $w \in E$, $\theta(s,\omega) \geq s$.

Assumption 18.a simply normalizes the state space. Assumption 18.b insures that there is some maximum sustainable state and Assumption 18.c insures that the system does not collapse to zero. Parts b and c can be derived from Inada Conditions on the constraint equation. See Brock and Mirman [1972] for a discussion of these conditions in an optimal growth problem.

By Assumption 18 the state space S can be restricted to the interval $[0,\bar{s}]$. The importance of Assumption 18 is that it allows us to restrict our attention to a compact state space and not that any particular interval is selected.

The properties of the transition equation imply:

Theorem 6. Given assumptions 1, 2a, 3, and 12-18, the transition probability P satisfies Doeblin's condition, i.e., there is a probability η, an integer n, and an ϵ with $0 < \epsilon < 1$ such that if $A \in \mathscr{S}$ and $\eta(A) \leq \epsilon$, then $P^n(s,A) \leq 1 - \epsilon$ for all s.

Proof. The result $\theta_\omega(s,\omega) > 0$ for all s and ω and assumptions 17 and 18 are sufficient to insure that the conditions of Proposition 5.7 of Futia [8] are satisfied and hence the linear operator defined by P is quasi-compact. The linear operator defined by P is quasi-compact if and only if P satisfies Doeblin's condition (Futia [1976,Theorem 4.9]).

The results obtained up to this point are sufficient to insure that there exists an invariant probability or stochastic equilibrium and that the time averages of the sequence of probabilities generated by the optimal Markov process converges to an invariant probability. Let $\mathscr{P}(S)$ denote the space of all probability measures on $(S,\mathscr{S})$.

Definition 8. Define the linear operator $T:\mathscr{P}(S) \to \mathscr{P}(S)$ by, for any $\lambda \in \mathscr{P}(S)$, $T\lambda(A) = \int P(s,A)\lambda(ds)$ for all $A \in \mathscr{S}$.

Hence if the probability of A at date t is $\lambda(A)$, the probability of A at date $t+1$ is $T\lambda(A)$. A stochastic equilibrium is a fixed point for the mapping T.

Definition 9. A probability measure λ on $(S,\mathscr{S})$ is a stochastic equilibrium if $T\lambda = \lambda$.

Theorem 7. Given assumptions 1, 2a, 3 and 12-18 there exists at least one stochastic equilibrium and for any given initial probability λ on $(S,\mathscr{S})$, the sequence of time averages $\{ \frac{1}{n} \sum_{i=o}^{n-1} T^i \lambda \}$ converges at an arithmetic rate to a stochastic equilibrium.

Proof. The result follows directly from VIII.8 Corollary 4, Dunford and Schwartz, and the Corollary of Theorem 4, Yoshida and Kakutani, or see Futia [1976] Section 3.

Although the time averages of the probabilities converge to only one invariant probability for any given initial measure, there may be many invariant probabilities. With different initial measures, various invariant probabilities may be limit points. Theorem 7 demonstrates only convergence of the time averages and not necessarily convergence of the basic sequence of probabilities generated by the system. To obtain convergence of the actual sequence of probabilities to a unique stochastic equilibrium for a given initial state, the state space must be divided into invariant sets each of which has a linkpoint.

Definition 10. A subset C of S is called invariant if $P^n(s,C) = 1$ for all n and for all $s \in C$. A subset C of S is called transient if $\sum_{n=0}^{\infty} P^n(s,C) < \infty$ for all $s \in C$.

The expected number of visits to an invariant set from any point in that set is infinite, for a transient set the expected number of visits is finite. It will now be shown that the state space can be decomposed into invariant and transient sets and that there is an invariant probability relevant for each invariant set.

Since $\theta: S \times E \to S$ is continuous in s for each ω and S can be taken to be compact and convex, there are one or more fixed points $\bar{s}$, for each ω. Let $\theta(\cdot,\omega) \equiv \theta_\omega(\cdot)$. Let S^1_γ and S^1_β be the minimum strictly positive fixed points of θ_γ and θ_β respectively. There must be such minimum fixed points, greater than zero, since (a) for some $\epsilon > 0$, $\theta_s(\hat{s},\omega) > \hat{s}$ for all $\hat{s} \in (0,\epsilon)$ and for all ω; (b) the set of fixed points for θ_ω is closed [suppose $\theta_\omega(s_n) = s_n$ and $s_n \to \bar{s}$ then by continuity $\theta_\omega(\bar{s}) = \bar{s}$].

Lemma 4. $S_\gamma^1 < S_\beta^1$.

Proof. Clearly $S_\gamma^1 \neq S_\beta^1$, otherwise $\theta_\gamma(S_\gamma^1) = \theta_\beta(S_\beta^1)$ contradicting θ strictly increasing in ω. Suppose $S_\gamma^1 > S_\beta^1$. Then there is an $\hat{s} > 0$ such that $\theta_\beta(\hat{s}) > \theta_\gamma(\hat{s}) > \hat{s}$, since $\theta_s(s,\omega) > 1$ for all ω and for $s \in (0,\epsilon)$ given some $\epsilon > 0$, and $\theta_\gamma(S_\gamma^1) = S_\gamma^1 > \theta_\beta(S_\beta^1) = S_\beta^1$. This implies that there is an $\bar{s} \in [\hat{s},S_\beta^1]$ such that $\theta_\gamma(\bar{s}) = \theta_\beta(\bar{s})$ as θ_ω is continuous for all ω. This contradicts θ strictly increasing in ω. Hence, $S_\gamma^1 < S_\beta^1$.

The following outlines a technique for pairing up fixed points of θ_γ and θ_β to form minimal invariant and transient sets.

(1) Form the pair (S_γ^1,S_β^1).

(2) After forming a pair of fixed points, consider the next point at which the increasing sequence of fixed points of θ_γ and θ_β changes type, suppose it changes from γ type to β type. Let the next pair of fixed points consist of the maximum fixed point of type γ before the switch point and the mimimum fixed point of type β after the switch point, say (S_γ^n,S_β^n). If the change had been from β type to γ type then the next pair would have been (S_β^n,S_γ^n). For example, suppose the sequence of fixed points after (S_γ^i,S_β^i) can be written as $S_\gamma^{i+1},S_\gamma^{i+2}, \ldots, S_\gamma^{i+n},S_\beta^j$ then the next pair of fixed points is $(S_\gamma^{i+n},S_\beta^j)$.

(3) Continue this procedure, leaving all intermediate fixed points unpaired, until there is no further change of type among the remaining fixed points. Leave such remaining points unpaired.

These pairs of fixed points form closed intervals. Number these intervals in increasing order as $I^1, I^2, \ldots$ Let the end points of I^i have the superscript i, ignoring all fixed points in the interior of I^i. See Figure 1 for a simple case similar to those considered by Brock and Mirman [1972].

INSERT FIGURE 1

<u>Lemma 5</u>. If $S_\gamma^i < S_\beta^i$ then I^i is invariant. If $S_\beta^i < S_\gamma^i$ then I^i is transient.

<u>Proof</u>. Consider $I^i = [S_\gamma^i, S_\beta^i]$. Note that $S_\beta^i \geq \theta_\beta(s) \geq S_\gamma^i$ and $S_\beta^i \geq \theta_\gamma(s) \geq S_\gamma^i$ for all $s \in I^i$ since $S_\gamma^i = \theta_\gamma(S_\gamma^i) < \theta_\beta(S_\beta^i) = S_\beta^i$ and θ is increasing. Therefore, $\theta(s,\omega) \in I^i$ for any $s \in I^i$ and any ω.

Consider $I^i = [S_\beta^i, S_\gamma^i]$. Let $X = \{s \in S \mid s < S_\beta^i\}$. There is an $n > 0$ such that $P^n(s_o, X) \geq K^n > 0$, where $K = \phi(\gamma) > 0$, for all $s_o \in [S_\beta^i, S_\alpha^i)$ since $\theta_\gamma(s_o) < s_o$ for all such s_o. If $s_o = S_\alpha^i$ then for any $\omega > \alpha, \theta_\omega(s_o) > s_o$ and this event has probability $1 - K > 0$. Hence, there is an integer n such that the probability of leaving I^i in n steps is strictly positive. Since I^i cannot be entered from $(I^i)^c$ the expected number of visits to I^i is finite. Hence, I^i is transient.

The following is a method for expanding the invariant sets to form attracting sets· sets whose points eventually move into the contained invariant sets. Consider I^i.

(1) If the next paired fixed point above (or below) I^i is an end

point of a transient set, expand I^i to include the interval $(S_\beta^i, S_\beta^{i+1}]$ (or $[S_\gamma^{i-1}, S_\gamma^i)$).

(2) If an adjacent paired fixed point is on another invariant set, I^i cannot be expanded.

(3) Expand I^1 to include $(0, S_\gamma^1)$.

(4) If $I^i = [S_\gamma^i, S_\beta^i]$ is the last interval (in the positive direction on S) previously formed, expand it to include $(S_\beta^i, \bar{S}]$. If $I^i = [S_\beta^i, S_\gamma^i]$ is the last interval previously formed then let $I^{i+1} = [S_\gamma^i, \bar{S}]$ be an invariant set. Label these sets A^i, where $A^i \supset I^i$. This procedure decomposes the state space into attracting and transient sets. Examples are given in Figures 2 and 3.

<u>Lemma 6</u>. For any $A^i \supset I^i$, $\lim P^n(s, I^i) = 1$ for all $s \in A^i$.

<u>Proof</u>. Let $i \neq 1$, $I^i = [S_\gamma^i, S_\beta^i]$. Clearly if $A^i = I^i$, the result holds. At most $A^i = [S_\gamma^{i-1}, S_\beta^{i+1}] \supset [S_\gamma^i, S_\beta^i]$. Since the cases of $s \in [S_\gamma^{i-1}, S_\gamma^i)$ or $s \in (S_\beta^i, S_\beta^{i+1}]$ are symmetric, we will do the proof for $s \in [S_\gamma^{i-1}, S_\gamma^i)$. For $s \in (S_\gamma^{i-1}, S_\gamma^i)$, $\theta_\omega(s) > s$ for all ω, since θ is strictly increasing in both s and ω. For $s = S_\gamma^{i-1}$, $\theta_\omega(s) > s$ for all $\omega \neq \gamma$. Hence, $\lim P^n(s, I^i) = 1$ for all $s \in [S_\gamma^{i-1}, S_\gamma^i)$. If $i = 1$ we need to show that $\lim P^n(s, I^1) = 1$ for all $s \in (0, S_\gamma^1)$. Since $\theta_\omega(s) > s$ for all ω and for all $s \in (0, \varepsilon)$ and since S_γ^i is the first fixed point for θ_γ we have $\theta_\omega(s) > s$ for all $s \in (0, S_\gamma^1)$, for all ω. Hence, $\lim P^n(s, I^1) = 1$.

Note that $\theta_\omega(s) < s$ for any s above the last interval if it is of the form $I^i = [S_\gamma^i, S_\beta^i]$, for all ω, as otherwise $\theta(s, \omega) \notin S$ for some such s. Hence, if the last invariant set is $I^i = [S_\gamma^i, S_\beta^i]$ it can be expanded to include the interval to $\bar{S}$. If the last interval is of the form $I^i = [S^i_\beta, S^i_\gamma]$ there can be no further fixed points of type γ as otherwise $\theta(s,\omega) \notin S$ for some $s > S^i_\gamma$. Hence, $S^i \leq \theta_\gamma(s) < \bar{S}$ and $S^i_\gamma \leq \theta_\beta(s) \leq \bar{S}$ for all $s \in (S^i_\gamma, \bar{S}]$ as θ is increasing in s and ω. Therefore, $S^i_\gamma \leq \theta(s,\omega) \leq \bar{S}$ for all ω and all $s \in [S^i_\gamma, \bar{S}]$ and $[S^i_\gamma, \bar{S}]$ is an invariant set.

INSERT FIGURES 2 and 3

Convergence to a stochastic equilibrium will now be shown.

Theorem 8. If the initial state of the system s_o, is an element of A^i then the sequence of probabilities $(T^* \delta_{s_o})^n$ describing the evolution of the system, converges to a unique probability Ψ^i with support I^i.

Proof. Clearly the process P restricted to A^i is a Markov process on A^i as $P(s,A^i) = 1$ for all $s \in A^i$ and P was earlier shown to be Markov. By Theorem 6, P satisfies Doeblin's condition. Note that S^i_γ is a link point since $\theta^n_\gamma(s) \to S^i_\gamma$ and $\phi(\gamma) = K > 0$. These results are sufficient to insure that the conditions of Theorem 3.6 of Futia [1976] are satisfied. The result then follows from the Corollary to Theorem 4 of Yosida and Kakutani [1941].

If the initial state of the system is an element of a transient set then the resulting invariant measure will be one associated with one of the adjacent invariant sets. If the initial state of the system is $s_o = 0$ then the resulting invariant measure must be the point measure on 0, δ_o. Although the conditions presented here are not sufficient to insure uniqueness (across initial states) of the invariant measure they do not rule out such cases. For example, if I^1 is the only interval of the type formed earlier then $A^1 = (0,\bar{S}]$ and the unique invariant measure(for $s_o \neq 0$) is ψ^1.

Using an invariant probability, ψ, on states we can construct the corresponding invariant probabilities on actions ψ_A, shadow prices ψ_Δ, and immediate returns ψ_u. Define them by

(32) $\psi_A(B) = (\psi_o \phi)\{(s,\omega) \mid (\pi,\sigma)(s,\omega) \in B\}$ for all $B \in \mathcal{A}$

(33) $\psi_\Delta(B) = (\psi_o \phi)\{(s,\omega) \mid (s,\omega) \in B\}$ for all $B \in \mathcal{B}_\Delta$

(34) $\psi_u(B) = (\psi_o \phi)\{(s,\omega) \mid u(\pi(s,\omega)) \in B\}$ for all $B \in \mathcal{B}_{\mathbb{R}}$.

These equilibrium invariant probabilities have a number of interesting interpretations. The invariant probabilities on the action (π,σ) will be analogous to steady-state consumption and investment in models of economic growth. In models of renewable resource management, probabilities on (π,σ) offer an interesting parallel to the concept of sustained yield and optimal escapement. The corresponding equilibrium probability distribution on shadow prices may be useful in examining steady-state price distributions in market models. Finally, the invariant probability which is induced on immediate returns may be of interest to a firm manager concerned with "average" profits or to a government planner interested in the eventual steady-state utility of per-capital consumption. Empirical observation of the actions of the planner may be considered as random draws from these steady-state distributions. Thus, further specification of the form of these distributions is of particular interest. With explicit forms for the utility function and the constraint, explicit forms for the optimal finite horizon policy can be obtained (see, for example, Leland [1974]. If these policies were to be applied as rolling plans, then under certain restrictions on random disturbances, explicit parameterizations of the steady-state distributions could be derived. An interesting problem for future research, would be to investigate the effects of changes in the underlying parameters on the observed equilibriums probability distributions.

APPENDIX

The approach taken in the proofs of Lemma 1 and Theorem 1 follows the general outline of Maitra [1978]. The proof of Lemma 1 will employ the following well-known results:

(i) The function $m: X \to \mathbb{R}$ defined by

$$(1) \quad m(x) = \max_{a \in b(x)} v(x,a)$$

is continuous and the correspondence $f: X \to A$ defined by

$$(2) \quad f(x) = \{a \in b(x) \mid v(x,a) = m(x)\}$$

is upper semi-continuous given $b: X \to A$ continuous, A compact, and v continuous.

(ii) Given the metric $\|v - u\| = \sup_{x \in X} |v(x) - u(x)|$,

the class of all bounded and continuous functions on $X, C(X)$, is a Banach space.

(iii) Define $Tv \cdot X \to \mathbb{R}$ for any $v \in C(x)$ by

$$(3) \quad Tv(x) = \max_{a \in b(x)} [u(s,\omega,a) + \alpha \int v(g(s,a),\tilde{\omega})d\varphi(\tilde{\omega} \mid \omega)].$$

The fact that T maps $C(X)$ into $C(X)$ follows directly from assumptions 2b,4a,5,6 and result (i). Further T is a contraction map on $C(X)$ and consequently has a unique fixed point. See Maitra [1968], Lemma 4.3

Proof of Lemma 1: (Other than the result that the optimal return is continuous this proof directly parallels the proof of the main theorem of Maitra [1968].)

With any measurable f from X to A, associate the operator $L(f)$ from $C(X)$ to $C(X)$ defined by

$$(4) \quad L(f)V(X) = U(s,\omega,f(s,\omega)) + \alpha \int V(g,f(s,\omega)),\tilde{\omega})d\varphi(\tilde{\omega} \mid \omega).$$

By Theorem 5.1 of Strauch [1966], $L(f)$ is a contraction map on $C(X)$ and hence has a unique fixed point. Further this unique fixed point is $I(f^\infty)$ where f^∞ denotes the stationary plan $(f,f,f, \ldots)$.

Let V^* be the unique fixed point of T, see (iii). It then follows from (i) that there is an upper semi-continuous function f from X to A such that $TV^* = L(f^\infty)V^*$. Therefore $L(f^\infty)V^* = V^*$ and since the unique fixed point of $L(f^\infty)$ is $I(f^\infty)$, $V^* = I(f^\infty)$. Hence $TV^* = V^*$ can be rewritten as

$$(5) \quad I(f^\infty)(x) = \text{Max } [u(s,\omega,a) + \alpha \int I(f^\infty)(g(s,a),\tilde{\omega})d\psi(\tilde{\omega} \mid \omega)].$$

Thus $I(f^\infty)$ satisfies the optimality equation so that by Theorem 6 of Blackwell [1965] f^∞ is an optimal plan. Moreover, as $V^* = I(f^\infty)$ and $V^* \in C(X)$ the optimal return is continuous and bounded.

We now turn to Theorem 1. The proof will employ the following results.

(iv) The correspondence $f\colon X \to A$ defined by

$$(6) \quad f(x) = \{a \in A \cdot a \in b(x), v(x,a) = \underset{a' \in b(x,a')}{\text{Max}} v(x,a')\}$$

is single valued, given v strictly concave in a, and b continuous and convex valued. (Note that if f is upper semi-continuous it will then be continuous.)

(Note that if $a \in \mathbb{R}^n$, v must be strictly concave in all but one of $(a_1, \ldots, a_n)$, say a_i, and concave in a_i.)

(v) Given the metric $\|v - \mu\| = \sup_{x \in X} |v(x) - \mu(x)|$ the class of functions that are continuous and bounded on X and nondecreasing and concave on $S, C_S(X)$ is a Banach space.

(vi) Define $Tv\colon X \to \mathbb{R}$ for any $v \in C_S(X)$ by ,

(7) $$(Tv)(x) = \max_{a \in b(x)} [u(s,\omega,a) + \alpha \int v(g(s,a),\tilde{\omega})d\phi(\tilde{\omega} \mid \omega)]$$

It is easy to show that T maps $C_S(X)$ to $C_S(X)$. Then T has a unique fixed point in $C_S(X)$ by (iii).

<u>Proof of Theorem 1</u>:

Let $v^* \in C_S(X)$ be the unique fixed point of T in $C_S(X)$. Given Lemma 1, there exists an upper-semi-continuous function $f: X \to A$ such that $Tv^* = L(f)v^*$ where $(L(f)v^*)(x)$ is defined as before.

So $L(f)v^* = v^*$ and we can rewrite $Tv^* = v^*$ as

(8) $$I(f^{(\infty)})(x) = \max_{a \in b(x)} [u(s,\omega,a) + \alpha \int I(f^{(\infty)})(g(s,a),\tilde{\omega})d\phi(\tilde{\omega} \mid w)].$$

Thus, $I(f^{(\infty)})$ satisfies the optimality equation and $f^{(\infty)}$ is an optimal plan. Since $v^* = I(f^{(\infty)})$ and $v^* \in C_S(X)$, the optimal return is bounded and continuous on X and is nondecreasing and concave on S. Further, since f satisfies $L(f)v^* = Tv^*$ we may write

(9) $$u(s,\omega,f(x)) + \alpha \int v^*(g(s,f(x)),\tilde{\omega})d\phi(\tilde{\omega} \mid \omega)$$

$$= \max_{a \in b(x)} [u(s,\omega,a) + \alpha \int v^*(g(s,a),\tilde{\omega})d\phi(\tilde{\omega} \mid \omega)].$$

Since $v^* \in C_S(X)$, it is concave in s, so the term $\alpha \int v^*(g(s,a),\tilde{m})d\phi(\tilde{\omega} \mid \omega)$ is concave in s and a since g is concave. Since u is strictly concave in a, the optimal policy $f(x)$ will be single-valued and, therefore, continuous, by (iv).

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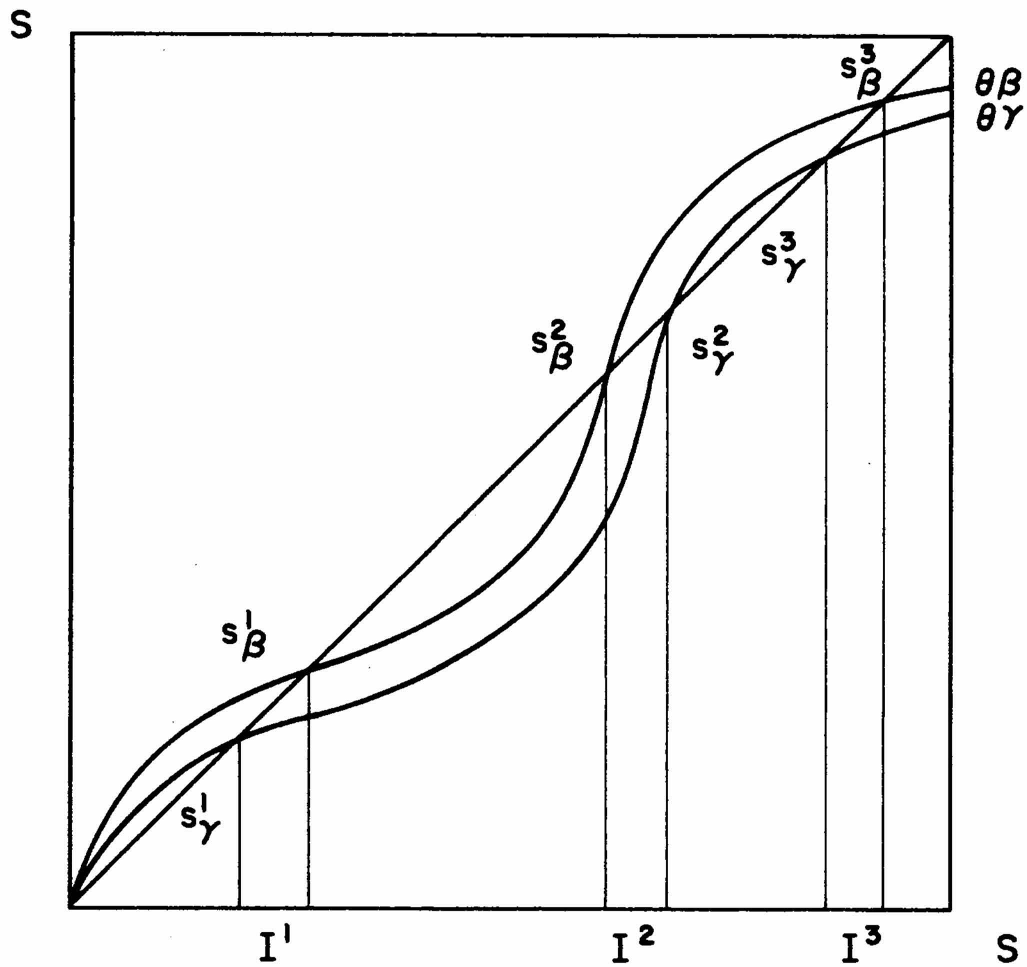
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Figure 1

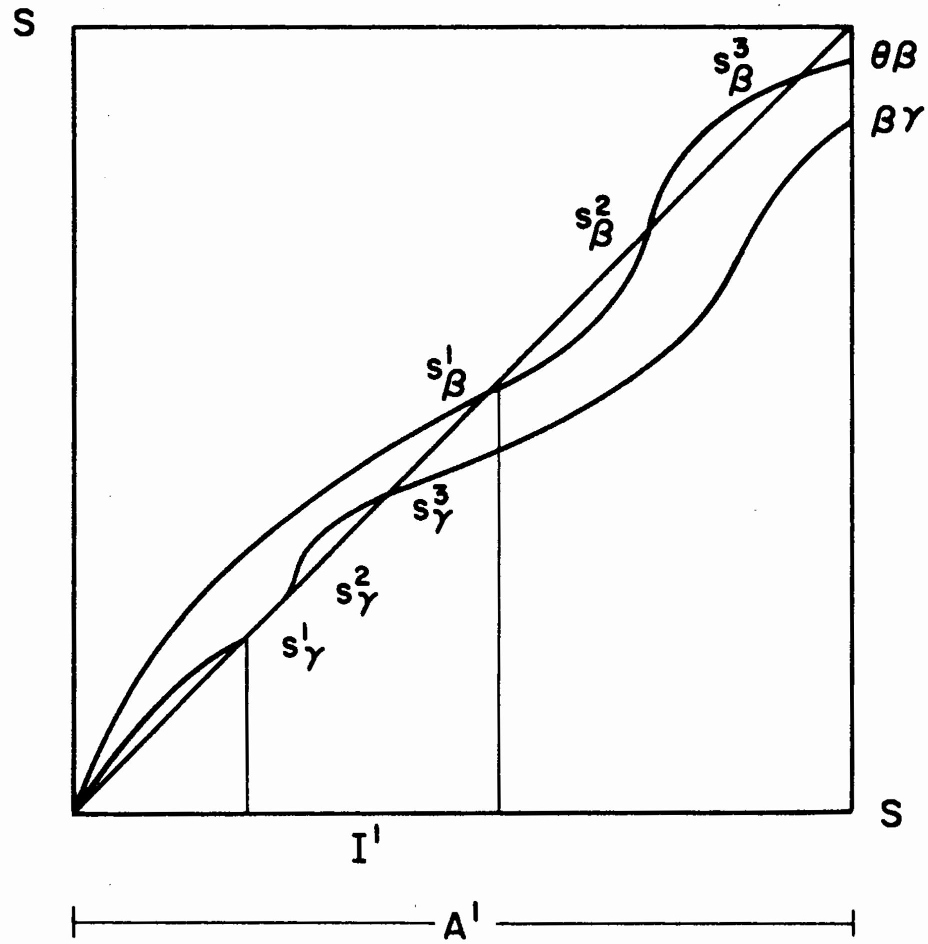
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Figure 2