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COMPUTATIONAL DELAYS

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ABSTRACT

Decision-making processes usually involve time-consuming and costly operations of observation and communication of the state of the environment and generation and implementation of appropriate actions. Collectively these activities may be called computational and the procedure required to carry them out may be called an algorithm. If the environment is changing stochastically the fixed computational delay involved in operating the algorithm yields obsolete actions resulting in reduced expected return. The loss as a function of delay is a measure of efficiency of the algorithm which depends upon the stochastic properties of the environment. If it is possible to reduce the loss by employing a faster algorithm at a higher cost, the algorithm may be optimally designed and selected from a given family.

COMPUTATIONAL DELAYS

1. INTRODUCTION

Consider a decision-maker who has to take real-valued actions in face of a real-valued environment that is changing stochastically at discrete points in time, say every day. As a real-valued function of the environment and the action in effect on a particular day he receives a return. The main problem is that he can not instantaneously generate an action that is suitable for the environment on that day. Instead, he has to rely on a time-consuming algorithm (i.e. a procedure to generate actions) that requires a fixed number of days to compute an optimal action. We shall assume that he has enough computing equipment to carry on many computations simultaneously. However, due to the computational delays his responses are not perfect; every day the action available is optimal only with respect to the state of the world that existed a fixed number of days in the past. Knowing this, in each computation the decision-maker uses not the actual environment at the beginning of the computation but his expectation about the environment at the end of the computation, which the computed action is actually going to face. (Adjusting estimates for seasonal patterns and trends is common in forecasting techniques.) Nevertheless, since his forecasts are imperfect, everyday he suffers an opportunity loss due to a suboptimal action being in force. Our objective is to determine the loss due to this computational delay for different discrete as well as continuous time stochastic processes governing the envornment. In particular, in Section 3, we determine the loss as a function of delay and environment assuming quadratic payoff and different Markov processes.

Each computation involves observation and communication of the state of the environment and generation and implementation of an action. Each of these subactivities involves a time delay and a cost as a function of the delay. We may seek to optimally allocate the fixed total computational delay among these activities to minimize the total computational cost. Next, suppose the decision-maker has a family of algorithms available to him, each requiring a fixed total computational delay (optimally allocated among its subactivities) and a fixed total (minimum) computational cost. Then we may combine this operating cost with the loss function yielding an efficiency criterion for choosing an optimal algorithm from the family, as we shall see in Section 4.

Section 5 concludes the paper by some observations about the relationship of this paper with other related work. In Section 2 we establish some notation and give a general formulation.

2. NOTATION AND FORMULATION

In the discrete-time case suppose the environment changes every day according to a known Markov process $\{X_n : n = 1,2,\ldots\}$ taking its values in the (Borel) set $X \subset R$ and governed by a set of (regular) transition probabilities $\{P_{ij}(\cdot|\cdot): j \geq i,\ i,j = 1,2,\ldots\}$. Thus, given $X_i = x$, $P_{ij}(\cdot|\cdot)$ is probability distribution of X_j on (measurable) subsets of X and, given a (measurable) subset $B \subset X$, $P_{ij}(B|\cdot)$ is a (measurable) function on X. Suppose the corresponding transition densities $\{p_{ij}(\cdot|\cdot): j \geq i, i,j = 1,2,\ldots\}$ exist, so that $P_{ij}(B|x) = \int_B p_{ij}(y|x)dy$. Thus $p_{ij}(y|x)$ denotes the conditional density of X_j at y given $X_i = x$ for $i \leq j$. These conditional densities satisfy the following <u>Chapman-Kolmogorov</u> identity

$$p_{ik}(z|x) = \int p_{ij}(y|x)p_{jk}(z|y)dy, \; i \leq j \leq k.$$

Suppose $p_i(\cdot)$ denotes the density of X_i, with $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$ $i = 1,2,\ldots$ The Markovian property implies that for every finite collection $(X_{i_1}, X_{i_2}, \ldots X_{i_n})$ with $i_1 \leq i_2 \leq \cdots \leq i_n$ the conditional density of X_{i_n} given $(X_{i_1}, \ldots X_{i_{n-1}})$ is the same as p_{i_{n-1}, i_n}, the conditional density of X_{i_n} given in $X_{i_{n-1}}$.

The above probabilities and densities are called <u>time homogeneous</u> if $P_{ij}(\cdot|\cdot)$ and $p_{ij}(\cdot|\cdot)$ depend only on $(j-i)$; then it suffices to know only one-step transition probability $P(\cdot|\cdot)$ and density $p(\cdot|\cdot)$. The arbitrary d-step transition density $p^{(d)}(y|x)$ of X_{n+d} at y, given $X_n = x$ can be recursively calculated as $p^{(d)}(y|x) = \int p^{(d-1)}(z|x)p(y|z)dz$.

The process $\{X_n : n = 1,2,\ldots\}$ is said to be <u>stationary</u> if the joint distribution of any finite collection $(X_{i_1+k}, X_{i_2+k}, \ldots X_{i_n+k})$ is independent of any time shift k. This implies, in particular, that the density $p_i(\cdot)$ of X_i is the same for all i and then we let $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$.

Similar notation applies in the continuous-time case (for consistency we use the same letter n to denote discrete and continuous time instants) and we call the process $\{X_n : n \in [0,\infty)\}$ Markovian if for any finite collection $(X_{t_1}, X_{t_2}, \ldots X_{t_j})$ with $t_1 \leq t_2 \leq \cdots \leq t_j$ the conditional density of X_{t_j} given $(X_{t_1}, \ldots X_{t_{j-1}})$ is the same as the conditional density of X_{t_j} given $X_{t_{j-1}}$. The homogeneous transition probabilities are defined by

$$P^{(d)}(B|x) = \text{Prob }[X_{n+d} \in B|X_n = x] \text{ independent of } n$$

$$= \int_B P^{(d)}(dy|x) = \int_B p^{(d)}(y|x)dy \quad \text{and}$$

they satisfy the Chapman-Kolmogorov identity

$$P^{(n+d)}(B|x) = \int P^{(n)}(dy|x)P^{(d)}(B|y).$$

We have an algorithm $\alpha: X \to A \subset R$ which is a real-valued (measurable) function on X taking its values in the (Borel) set A of actions. (As we will see in Section 4, α is a composition of functions).

The return function $W: X \times A \to R$ is a real-valued (measurable) function, so that at any given time if the state of the world is $x \in X$ and the action in force is $a \in A$ we receive the payoff $W(x,a)$ at that time. For a given environment $x \in X$, let $a^*(x) \in A$ denote the optimal action (assumed to exist) maximizing $W(x,a)$ over A. Suppose the algorithm requires a fixed time period of length d to yield an optimal action. (As we will see in Section 4, d is the sum of individual delays involved in the activities required in operation of the algorithm.)

At time n we observe the environment X_n, the algorithm begins the computation and the action $\alpha(X_n)$ is generated at time $(n+d)$, to be put into force at time $n+d$ only. Thus, at each time we have a new action available and we switch to it instantaneously. We assume that the algorithm generates $\alpha(X_n)$ which maximizes $E[W(X_{n+d},a)|X_n]$ and not $a^*(X_n)$, and that the algorithm is not adaptive in the sense that it does not utilize the information about X_t, $n < t < n+d$, the environment passing by during the computation. We also assume that we have enough computing equipment to carry out several computations simultaneously. Finally the total payoff is assumed to be additive in time and there is no discounting.

If the computations were instantaneous, the optimal payoff we would expect to receive at time $(n+d)$ given the environment X_n is $E\{W[X_{n+d},a^*(X_{n+d})]|X_n\}$. Because the computation is time-consuming, we expect to receive at time $(n+d)$ $E\{W[X_{n+d},\alpha(X_n)]|X_n\} = \underset{a\epsilon A}{\mathrm{Max}}\, E[W(X_{n+d},a)|X_n]$. Thus, given X_n, the loss we would expect to suffer at $(n+d)$ due to the computational delay d is

$$(2.1) \qquad L_{n+d}(X_n,d) = E\{W[X_{n+d},a^*(X_{n+d})]|X_n\} - E\{W[X_{n+d},\alpha(X_n)]|X_n\}$$

$$(2.2) \qquad L_{n+d}(d) = E[L_{n+d}(X_n,d)]$$

We shall study the explicit form of this loss function with quadratic payoff and some specific Markov processes in the next section. For now we have the following general and intuitively plausible

Proposition 1

(a) The loss $L_{n+d}(d) \geq 0$

(b) If the environment is Markovian, the algorithm with a longer delay is inferior to one with a shorter delay in terms of loss i.e. $d_1 \geq d_2$ implies $L_n(d_1) \geq L_n(d_2)$ for all $n \geq d_1$.

<u>Proof:</u> (a) $E\{\max_{a\epsilon A} W[X_{n+d},a]|X_n\} \geq \max_a E\{W[X_{n+d},a]|X_n\}$

so that from (2.1) $L_{n+d}(X_n,d) \geq 0$ for almost all X_n and hence $L_{n+d}(d) \geq 0$. Thus a delayed algorithm is always inferior to the instantaneous one.

(b) $L_n(X_{n-d_1},d_1) - E[L_n(X_{n-d_2},d_2)|X_{n-d_1}]$

$$= E\{W[X_n,a^*(X_n)]|X_{n-d_1}\} - E\{E[W(X_n,a^*(X_n))|X_{n-d_2}]|X_{n-d_1}\}$$
$$+ E\{E[W(X_n,\alpha(X_{n-d_2}))|X_{n-d_2}]|X_{n-d_1}\} - E\{W(X_n,\alpha(X_{n-d_1}))|X_{n-d_1}\}$$

The first two terms are equal by Markovian assumption and the third term is

$$E\{\max_a E[W(X_n,a)|X_{n-d_2}]|X_{n-d_1}\}$$

$$\geq \max E\{E[W(X_n,a)|X_{n-d_2}]|X_{n-d_1}\}$$

$= \max_a E\{W(X_n,a)|X_{n-d_1}\}$, by Markovian assumption, which is the fourth term.

Thus $L_n(X_{n-d_1},d_1) - E[L_n(X_{n-d_2},d_2)|X_{n-d_1}] \geq 0$ for almost all X_{n-d_1}, so that $L_n(d_1) - L_n(d_2) \geq 0$. <u>Q.E.D.</u>

(Technically, the above result also holds if $\mathcal{J}(X_n) \subset \mathcal{J}(X_{n+1})$ for all n, where $\mathcal{J}(X_n)$ is the σ-field generated by X_n.)

The loss as a function of delay depends upon the return function and relevant properties of the stochastic environment. For example the greater the dependence of the return on the state of the environment the greater is the loss and similarly the greater the variability (in some sense) in the stochastic environment the greater will be the loss, or the greater the uncertainty, as measured by the informational entropy content in the future given the past, the greater will be the loss. In the next section we study the loss function assuming a fixed quadratic return function and different types of Markov processes.

3. THE LOSS FUNCTION

In this section we shall consider the quadratic payoff function (which has been extensively studied in the literature), $W(x,a) = 2\,x\,a - qa^2$ with $q > 0$ for concavity. In the discrete-time case $W(x,a)$ is the payoff at any time if x is the environment and a the action then. In the continuous-time case $W(x,a)$ is to be interpreted as the rate of payoff. Similar interpretations apply to the loss $L_n(d)$. The next proposition provides some general bounds.

Proposition 2

If the payoff function is quadratic, $W(x,a) = 2\,x\,a - qa^2$, with $q > 0$ then the bounds on the loss function are

$$(3.1) \qquad 0 \leq L_{n+d}(d) \leq \frac{\sigma^2_{n+d}}{q} \quad \text{for any } n$$

so that, in particular, if the environment is stationary the loss function is bounded for all n.

Proof: We have $a^*(X_{n+d}) = \frac{X_{n+d}}{q}$ and $W(X_{n+d}, a^*(X_{n+d})) = \frac{X_{n+d}^2}{q}$.

Also $\alpha(X_n) = \frac{E(X_{n+d}|X_n)}{q}$ and $E\{W[X_{n+d}, \alpha(X_n)]|X_n\} = \frac{[E(X_{n+d}|X_n)]^2}{q}$

Thus $L_{n+d}(X_n, d) = \frac{1}{q}\{E(X_{n+d}^2|X_n) - [E(X_{n+d}|X_n)]^2\}$, i.e.

$$(3.2) \qquad L_{n+d}(X_n, d) = \frac{1}{q} \mathrm{Var}(X_{n+d}|X_n), \text{ so that}$$

$$(3.3) \qquad L_{n+d}(d) = \frac{1}{q} E[\mathrm{Var}(X_{n+d}|X_n)] \geq 0$$

In case of deterministic environment $\mathrm{Var}(X_{n+d}|X_n) = 0$ so that $L_{n+d}(d) = 0$ as expected and it will be positive in case of uncertainty. Since

$$\mathrm{Var}(X_{n+d}) = E[\mathrm{Var}(X_{n+d}|X_n)] + \mathrm{Var}[E(X_{n+d}|X_n)]$$

and since variance is nonnegative we have

$$L_{n+d}(d) = \frac{1}{q}E[\mathrm{Var}(X_{n+d}|X_n)] \leq \frac{\mathrm{Var}(X_{n+d})}{q}$$

with equality holding iff $E(X_{n+d}|X_n)$ is a constant. (e.g. if the environments are independent and identically distributed. In that case, any delay is as bad as performing no computation at all and using the same fixed action maximizing $E[W(X,a)]$.)

With stationary environment $\mathrm{Var}(X_n) = \sigma^2$ for all n and the loss $L_{n+d}(d)$ will be nonnegative, independent of n and bounded from above by $\frac{\sigma^2}{q}$. Q.E.D.

Intuitively according to (3.2) the expected loss due to the computational delay is directly proportional to the amount of variability (or uncertainty, in some sense,) left in the future environment X_{n+d} at the end of the computation, given the present environment X_n at the beginning of the computation. Also the loss is inversely proportional to q, because the larger the q the smaller is the relative weight attached to the uncertain environment in the payoff function.

Suppose $d_1 \geq d_2$ and that $\{X_n\}$ is Markovian. Now we know

$$\mathrm{Var}\,(X_n|X_{n-d_1}) = E[\mathrm{Var}(X_n|X_{n-d_2}X_{n-d_1})|X_{n-d_1}]$$

$$+\mathrm{Var}[E(X_n|X_{n-d_2},X_{n-d_1})|X_{n-d_1}]$$

$$= E[\mathrm{Var}(X_n|X_{n-d_2})|X_{n-d_1}] + \mathrm{Var}[E(X_n|X_{n-d_2})|X_{n-d_1}] \geq E[\mathrm{Var}(X_n|X_{n-d_2})|X_{n-d_1}],$$

where the second equality follows by Markovian assumption.

Then we have

$$L_n(X_{n-d_1},d_1) = \frac{1}{q}\,\mathrm{Var}(X_n|X_{n-d_1}) \geq \frac{1}{q}\,E[\mathrm{Var}(X_n|X_{n-d_2})|X_{n-d_1}] = E[L_n(X_{n-d_2},d_2)|X_{n-d_1}]$$

for almost all X_{n-d_1} so that $L_n(d_1) \geq L_n(d_2)$ and, as already noted in proposition 1 (b), the slower algorithm is inferior in Markovian environment.

Next, in optimal selection of an algorithm it is important to know how sensitive the loss function is to the computational delay and the parameters of the stochastic process governing environments. To study this aspect we may classify the payoff functions and the stochastic processes according to the form of the resulting loss function in terms of the delay involved (e.g. increasing, concave, or convex etc.) or in terms of the parameters of the stochastic process (e.g. transition function, or mean or variance of X_n etc.) Here we will concentrate on the quadratic payoff function and Markov processes, classifying the latter according to the effect of their parameters on the loss function. Wherever possible we will indicate the applicability of a given process.

Since with the quadratic payoff the loss $L_{n+d}(X_n,d)$ is proportional to the conditional variance Var $(X_{n+d}|X_n)$, we may study Markov processes where the conditional variance Var $(X_{n+d}|X_n)$ is independent of X_n or when it is proportional to X_n (yielding $L_{n+d}(d)$ proportional to means) or when it is proportional to X_n^2 (yielding $L_{n+d}(d)$ proportional to second moments) etc. We do this in the remaining section, summarizing the results as Propositions.

An important class of Markov processes yielding $\mathrm{Var}(X_{n+d}|X_n)$ independent of X_n is the Markovian normal class. In the discrete-time case $\{X_n : n = 1,2,\ldots\}$ is called a Markovian normal process if X_n has normal density with mean 0 and variance σ_n^2 and if the conditional density of X_{n+d} given $X_n = x$, $p_{n,n+d}(\cdot|x)$ is the normal density with mean $\rho_{n,n+d}\dfrac{\sigma_{n+d}}{\sigma_n}x$ and variance $\sigma_{n+d}^2(1-\rho_{n,n+d}^2)$ for $d \geq 0$. (Feller [4]) Here $\rho_{ij} \in [-1,1]$ is the correlation coefficient between X_i and X_j so that $E(X_iX_j) = \rho_{ij}\sigma_i\sigma_j$. As d varies, the behavior of the loss function is entirely governed by the structure of the covariance matrices. However, for this process, and only for this process it can be shown (see Feller [4] p.94) that $\rho_{ik} = \rho_{ij} \cdot \rho_{jk}$ whenever $i \leq j \leq k$. Thus $d_1 \geq d_2$ implies $\rho_{n-d_1,n} = \rho_{n-d_1,n-d_2}\,\rho_{n-d_2,n}$, so that $\rho_{n-d_1,n}^2 \leq \rho_{n-d_2,n}^2$ again implying $L_n(d_1) \geq L_n|d_2$. Thus $\rho_{n,n+d}^2$ is non-increasing in d and thus more distant environment in the future is less predictable (in terms of the correlation coefficient which is what we intuitively mean by obsolescence due to delay). Alternative definition of this process may be given recursively as an <u>Autoregressive process</u> $\{X_n : n = 1,2,\ldots\}$ with $X_1 = \sigma_1 Z_1$

$$X_n = \rho_{n-1,n} \frac{\sigma_n}{\sigma_{n-1}} X_{n-1} + \sigma_n \sqrt{1-\rho^2_{n-1,n}} \quad Z_n \qquad n = 2,3,\ldots. \text{ where}$$

Z_n's are i.i.d. normal with expectation 0 and variance 1. In particular, we may define a <u>discrete Brownian motion</u> $\{X_n : n = 1,2,\ldots\}$ recursively by

$$X_1 = 0$$

$$X_n = X_{n-1} + Z_n \qquad n = 2,3,\ldots$$

where Z_n's are independent, each being normally distributed with mean 0 and variance σ^2. In that case $\mathrm{Var}(X_{n+d} | X_n) = \sigma^2 d$. Brownian Motion is a well-known model of certain economic phenomena (e.g. movement of stock market prices).

Another example of the Markovian normal process is the <u>Normal decomposable process</u> in which $E(X_n) = 0$ for all n and it has independent increments, each being normally distributed with zero expectation. Thus $(X_{n+d} - X_n)$ is independent of $(X_1 \ldots . X_n)$ and hence $E[X_n(X_{n+d} - X_n)] = 0$. This yields $\rho_{n,n+d} = \frac{\sigma_n}{\sigma_{n+d}}$ implying $\rho_{ik} = \rho_{ij}\rho_{jk}$ for all $i \leq j \leq k$, so that it is a Markovian normal process. Then $E(X_{n+d} | X_1, \ldots . X_n) = \rho_{n,n+d} \frac{\sigma_{n+d}}{\sigma_n} X_n = X_n$, i.e. $\{X_n : n = 1,2,\ldots.\}$ is a <u>Martingale</u>. (This means, for example, that the expected value of tomorrow's price is the same as today's price; this assumption is reasonable in many situations of economic interest.) Then $L_{n+d}(d) = \frac{1}{q} \sigma^2_{n+d} (1-\rho^2_{n,n+d}) = \frac{1}{q} (\sigma^2_{n+d} - \sigma^2_n)$. Since $\rho^2_{n,n+d} \leq 1$, the sequence $\{\sigma^2_n : n = 1,2,\ldots.\}$ is non decreasing. If there exists M such that $\sigma^2_n \leq M < \infty$ for all n, then a Martingale convergence theorem asserts that there exists a random variable Y such that $X_n \to Y$ with probability 1 (see, for example, Feller [4] p. 236) In that case $\lim_{n\to\infty} L_{n+d}(d) = 0$ independent of d, so that in the long run a computational delay does not adversely affect payoff.

With a stationary normal Markovian process $\rho_{ik} = \rho_{ij}\rho_{jk}$ for all $i \leq j \leq k$ requires $\rho_{ik} = \rho^{|i-k|}$ for all i and k, where $|\rho| \leq 1$. Thus $\mathrm{Var}\ (X_{n+d}|X_n) = \sigma^2(1 - \rho^{2d})$.

In the continuous - time case two outstanding examples of Markovian normal processes are the Brownian Motion and the Ornstein-Uhlenbeck process. The <u>Brownian Motion</u> $\{X_n : n \in [o,\infty)\}$ is specified by $X(0) = 0$, X_n normally distributed with mean 0 for all $n > 0$ and that the process has stationary, independent increments. It can be shown that $\mathrm{Var}\ (X_n) = \sigma^2 n$ and for $m \leq n$ $\rho_{m,n} = \sqrt{\frac{m}{n}}$, so that $\rho_{m,n} = \rho_{mp}\ \rho_{pn}$ for $m \leq p \leq n$ and the process $\{X_n : n \in [o,\infty)\}$ is Normal Markovian. For this process, given X_n conditional density of X_{n+d} is normal with mean X_n and variance $\sigma^2 d$.

The <u>Ornstein - Uhlenbeck process</u> is the most general stationary normal Markovian process with zero expectations. This process arises when the Brownian environment has built-in equilibrating controls, so that when $X_n = x$ the adaptive control exercises a linear force of magnitude $\mu \cdot x$ towards the origin (e.g. in an economic system, given a non-equilibrium price, institutional controls and the market mechanism tend to restore the equilibrium by means of a force proportional to the discrepancy). Increments are no longer independent and it turns out (see Breiman [1]) that, given X_n, X_{n+d} is normally distributed with mean

$e^{-\mu d} X_n$ and variance $\frac{\sigma^2[1 - e^{-2\mu d}]}{2\mu}$. Also $\rho_{m,n} = e^{-\mu(n-m)}$ if $m \leq n$,

so that the process can be seen to be Normal Markovian. Letting $\mu \to o$ yields the Brownian motion. We summarize next.

<u>Proposition 3</u> Suppose the payoff function is quadratic as above

(a) If the environment is Markovian normal, then

$$(3.4) \qquad L_{n+d}(X_n, d) = \frac{1}{q} \sigma^2_{n+d} (1 - \rho^2_{n,n+d})$$

is independent of X_n (hence equals $L_{n+d}(d)$). The loss is concave and non-increasing in $|\rho_{n,n+d}|$. In addition, if the process is stationary then the loss

$$(3.5) \qquad L_{n+d}(d) = \frac{1}{q} \sigma^2 (1-\rho^{2d})$$

which is concave in d and approaches $\frac{\sigma^2}{q}$ as $d \to \infty$.

(b) If the environment is normal decomposable, then

$$(3.6) \qquad L_{n+d}(d) = \frac{1}{q} (\sigma^2_{n+d} - \sigma^2_n)$$

(so that, if in addition the process is stationary then $L_{n+d}(d) = 0$)

If $\sigma^2_n \leq M < \infty$ for all n then $\lim_{n \to \infty} L_{n+d}(d) = 0$

(c) If the environment is Brownian, then the rate of loss

$$(3.7) \qquad L_{n+d}(d) = \frac{\sigma^2 d}{q}$$

which is linearly increasing in delay without a bound.

(d) In case of the Ornstein - Uhlenbeck process the loss

$$(3.8) \qquad L_{n+d}(d) = \frac{\sigma^2}{2\mu q} [1-e^{-2\mu d}]$$

which is concave and increasing in d to $\frac{\sigma^2}{2\mu q}$

Next we consider some Markovian environments yielding the loss proportional to means of the present environments. Two important continuous time examples are the special cases of the pure birth and the pure death processes (see Karlin [5]), while we illustrate the discrete-time process using the Branching process (see Karlin [5]), and the Morse-Elston process (see Morse and Elston [11]).

The Morse - Elston process has been considered in [11] as a probability model of obsolescence of library books; it represents an exponential decay in the average demand for a certain book as its age increases, the action to be computed might be the optimal inventory policy for that book, given a fixed lead time. If X_n denotes the average demand at time n, the stationary transition probability is assumed to be Poisson given by

$$P(j|i) = \text{Prob } [X_{n+1} = j | X_n = i] = e^{-(b+Ci)} \cdot \frac{(b+Ci)^j}{j!} \quad \text{for } i,j = 0,1,2....$$

where $b \geq o$ and $o \leq C < 1$ are constants.

$$\text{Then Var } (X_{n+d} | X_n) = \frac{(1-C^d)}{(1-C)} \left[\frac{b(1-C^{d+1})}{(1-C^2)} + C^d X_n\right]$$

which is linear in X_n. Also $E(X_n | X_1) = b \dfrac{(1 - C^{n-1})}{(1 - C)} + C^{n-1} X_1$,

where X_1, the initial demand is known and fixed.

In the (discrete - time) Branching process X_n denotes the number of offsprings at the beginning of the nth period, Y_i denotes the number of new offsprings the ith one gives rise to at the end of the nth period, where Y_i's are independent and identically distributed with mean m and variance S^2. (Here the action might be a design of the educational system or a child care center for the (n+d)th period, where such a design takes d periods starting at n.) Thus $X_{n+1} = \sum_{i=1}^{X_n} Y_i$ yielding

$E(X_{n+1}|X_n) = m X_n$ and $\mathrm{Var}(X_{n+1}|X_n) = s^2 X_n$. We may recursively compute

$$E(X_{n+d}|X_n) = m^d X_n \text{ and}$$

$$\mathrm{Var}(X_{n+d}|X_n) = \left[s^2 m^{d-1} \sum_{k=0}^{d-1} m^k \right] X_n,$$

which is linear in X_n. With respect to d it increases (decreases) geometrically if $m > 1$ $(m < 1)$ and linearly if $m = 1$.

The Yule process is a (continuous time) pure birth process in which X_n denotes the population size at time n and each member in the population has a probability $\lambda h + o(h)$ of giving birth to a new member in an interval of time length h $(\lambda > 0)$; there are no deaths. The action may refer to the industrial and agricultural production rates planned for the time n+d, such planning requiring the time duration d which may be short enough to assume no deaths. Then, assuming independence and no interaction among members of the population, it can be shown by induction on $j \geq i$ that

$$\mathrm{Prob}\,[X_{n+d} = j|X_n = i] = \binom{j-1}{j-i} e^{-i\lambda d} (1 - e^{-\lambda d})^{j-i} \quad \text{whenever}$$

$j \geq i$. Therefore we have $E(X_{n+d}|X_n) = e^{\lambda d} X_n$ and $\mathrm{Var}(X_{n+d}|X_n) = e^{\lambda d}(e^{\lambda d} - 1)X_n$ which is linear in X_n and increasing in d.

Consider a pure death process with a constant death rate $\bar{\mu} > o$ regardless of the population-size X_n, so that the probability of a death during an interval of length h is $\bar{\mu}h + 0(h)$. Then to find $\mathrm{Prob}\,[X_{n+d}=j|X_n = i]$ we note that a member alive at n will live beyond (n+d) with probability $e^{-\bar{\mu}d}$, so that the Binomial law gives the above probability as $\binom{i}{j} e^{-j\bar{\mu}d}(1 - e^{-\bar{\mu}d})^{(j-i)}$ Thus we have

$E(X_{n+d}|X_n) = e^{-\bar{\mu}d} X_n$ and $Var(X_{n+d}|X_n) = e^{-\bar{\mu}d}(1 - e^{-\bar{\mu}d}) X_n$, which is linear in X_n and decreasing in d to zero.

With the above notation we summarize the resulting loss function in

Proposition 4

Suppose the payoff function is quadratic as above.

(a) If the environment follows the Morse - Elston process, then

$$(3.9) \qquad L_{n+d}(d) = \frac{(1 - C^d)}{q(1 - C)} \left[\frac{b(1 - C^{d+1})}{(1 - C^2)} + C^d \mu_n \right]$$

which is linear in $\mu_n = E(X_n)$. Also the steady state loss

$$L(d) = \lim_{n\to\infty} L_{n+d}(d) = \frac{b(1 - C^{2d})}{q(1 - C^2)(1 - C)}$$

which is concave and nondecreasing in d

(b) if the environment follows the (discrete-time) Branching process, then

$$(3.10) \qquad L_{n+d}(d) = \begin{cases} \frac{S^2}{q} m^{d-1} \frac{m^d - 1}{m - 1} \mu_n & \text{if} \quad m \neq 1 \\ \frac{S^2}{q} d \mu_n & \text{if} \quad m = 1 \end{cases}$$

which is linear in μ_n. Also it is geometrically increasing (decreasing) in d if $m > 1$ $(m < 1)$ and linear in d if $m = 1$.

$$L(d) = \lim_{n\to\infty} L_{n+d}(d) = \begin{cases} 0 & \text{if } m < 1 \\ \frac{S^2 d}{q} & \text{if } m = 1 \\ \infty & \text{if } m > 1 \end{cases}$$

(c) If the environment follows the Yule process, then

$$L_{n+d}(d) = \frac{1}{q} e^{\lambda d} (e^{\lambda d} - 1) \mu_n \tag{3.11}$$

which is linear in μ_n and increasing in d. Also it increases without bound as $n \to \infty$.

(d) With the pure death process the loss rate is

$$L_{n+d}(d) = \frac{e^{-\bar{\mu} d}}{q} (1 - e^{-\bar{\mu} d}) \mu_n \tag{3.12}$$

which is linear in μ_n and decreasing in d. Also it decreases to 0 as $n \to \infty$.

We conclude this section by considering for illustration two discrete-time Markov processes which yield the loss function proportional to the second moments of the environment values. The first process arises naturally in the theory of <u>random splittings</u> (See Feller [4] p. 207,24). Here $\{X_n : n=1,2,\ldots\}$ is defined as $X_n = Y_1 Y_2 \ldots Y_n$ where Y_i's are independent and uniformly distributed on (0,1). We have $X_{n+1} = X_n \cdot Y_{n+1}$, so that, given X_n, $\frac{X_{n+1}}{X_n}$ is uniform on (0,1). Then using log transformation it can be shown that the d - step transition density of X_{n+d} at y given $X_n = x$, is given by $p^{(d)}(y|x) = \frac{1}{(d-1)!x} [\log (\frac{x}{y})]^{d-1}$ if $0 < y < x$ and 0 otherwise. From this we may calculate $E(X_{n+d}|X_n) = 2^{-d} X_n$ and $\mathrm{Var}(X_{n+d}|X_n) = (3^{-d} - 2^{-2d}) X_n^2$ which is increasing in X_n^2 and d.

As a second example, consider the <u>Ugaheri process</u> (see Feller[4])

$\{X_n : n = 1,2,\ldots\}$ on (0,1) such that, given $X_n = x$, X_{n+1} is uniformly distributed on (1-x,1). Then it can be shown that $E(X_{n+d} | X_n) = \frac{X_n}{2^d}$ and

$$(3.13) \qquad \mathrm{Var}(X_{n+d} | X_n) = \sum_{k=0}^{d-1} (1/3)^k \{ 1 + 1/12 [(-\tfrac{1}{2})^{d-k-1} X_n + \sum_{j=0}^{d-k-2} (-\tfrac{1}{2})^j]^2 \}$$

which is quadratic in X_n. Also it can be shown that the stationary density is triangular 2x on (0,1) with mean 2/3 and variance 1/18.

Proposition 5

Suppose the payoff function is quadratic as before.

(a) If the environment $X_n = Y_1 Y_2 \ldots Y_n$ with Y_i's i.i.d. uniform on (0,1) the loss

$$(3.14) \qquad L_{n+d}(d) = \frac{(3^{-d} - 2^{-2d})}{q} (\sigma_n^2 + \mu_n^2) \qquad \text{which is}$$

proportional to the second moment and increasing in d and

$$L(d) = \lim_{n\to\infty} L_{n+d}(d) = 0$$

(b) In case of the Ugaheri process described above

$$L_{n+d}(d) = \frac{1}{q} E\ [\mathrm{Var}(X_{n+d} | X_n)]$$

where Var $(X_{n+d} | X_n)$ is given by (3.13).

With this analysis of the loss function for a given algorithm under the assumptions of different fundamental processes governing the environment we now turn to the question of optimal design and choice of an algorithm.

4. OPTIMAL DESIGN

In the previous section we studied the undesirable effects of operating a given time and cost-consuming algorithm in a given stochastic environment and yielding a return according to a given payoff function. In this section we consider the problem of designing the decision-making process by choosing the algorithm that is optimal in some sense.

For any algorithm two related factors, the computational delay and the computational cost per unit time required for its application, are specified (at least in principle; empirical information about these factors is, however rare in most cases.) The complete decision-making algorithmic function α may be naturally decomposed into say four subalgorithms $\alpha_1, \alpha_2, \alpha_3$ and α_4, so that we may write $\alpha(X_n) = \alpha_4\{\alpha_3[\alpha_2(\alpha_1(X_n))]\}$. Here α_1 is the <u>observation (data collection) subalgorithm</u>. (e.g. lenses or cameras and observers) which observes the environment X_n, α_2 is the <u>communication subalgorithm</u> (e.g. pigeons or telephones) which transmits the information to the computing center, α_3 is the <u>computation subalgorithm</u> (e.g. computers or staff managers) which actually generates an action and α_4 is the <u>implementation subalgorithm</u> (e.g. clerks or line managers) which puts the generated action into effect. (Clearly, in practice, such a decomposition of α is difficult due to ambiguities in classification and possible overlaps between classes. Moreover, each subalgorithm may be further decomposed into its subcomponents and so on, yielding unmanageable sequences of subalgorithms from which to choose. For illustrational purposes we adhere to the simple classification stated above.)

Suppose the algorithm α_i requires a delay d_i time units, involves the operating cost of $C_i(d_i)$ per unit time and it can be purchased in the market place at a price $P_i(d_i)$ $i = 1,\ldots 4$. Suppose the functions $C_i(\cdot)$ and $P_i(\cdot)$ are non-increasing and convex in d_i with infinity as their limits as d_i tends to zero (so that it is prohibitively expensive to buy and operate instantaneous machines.) Thus, for example, having more expensive and powerful lenses results in quick observations but their use and care also requires a trained operator and the operating cost is subject to increasing marginal cost phenomenon. Similarly, using telephones instead of pigeons or using a large and fast computer instead of a small and slow one involves a smaller delay but higher purchasing and operating costs. Note that two algorithms α and α' requiring same total delay d may have it distributed differently among their subalgorithms,yielding different total costs.

For a given total delay d we may design an optimal algorithm requiring that delay by choosing the subalgorithms (i.e. system components) so as to minimize the total operating cost per unit time subject to the budget constraint on total purchasing cost.

<u>Proposition 6.</u> Suppose the algorithm α requiring delay d is decomposed into algorithms $\alpha_1,\ldots,\alpha_n$ where α_i can be chosen to require any delay $d_i \geq 0$. Suppose for the subalgorithm α_i the purchasing cost $P_i(d_i)$ and operating cost per unit time $C_i(d_i)$ are convex and nonincreasing differentiable functions of d_i with limits infinity as $d_i \to 0$. If M is the budget constraint on the total purchasing price and if costs are additive $C(d)$, the minimum cost of operating the algorith α, is a convex and non-increasing function of delay d. For a given d if there is no budget-restriction the optimal algorithm is obtained by choosing the subalgorithm delays in such a way that each subalgorithm has the same marginal operating cost.

Proof: The parametric convex program $\mathcal{P}(d)$ for given d, is

$$\text{Minimize } \sum_{i=1}^{n} C_i(d_i)$$

$$\text{(4.1)} \qquad \text{s.t. } \sum_{i=1}^{n} d_i = d$$

$$\sum_{i=1}^{n} P_i(d_i) \leq M$$

$$d_i \geq 0$$

Since $C_i(d_i) \to \infty$ as $d_i \to 0$ the optimal solution $d_i^* > 0$, $i = 1,2,\ldots n$. Also, since $C_i(\cdot)$ and $P_i(\cdot)$ are convex functions the following Kuhn-Tucker conditions are necessary and sufficient for $(d_1^*,\ldots d_n^*)$ to be optimal

(a) $(d_1^*,\ldots d_n^*)$ is feasible

(b) $\mu[M - \sum_{i=1}^{n} P_i(d_i^*)] = 0$

(c) $C_i'(d_i^*) + \mu P_i'(d_i^*) - \lambda = 0 \qquad i = 1,2,\ldots n$

where $\mu \geq 0$ and λ are the multipliers associated with the second and the first constraints respectively. If we denote the optimal value of (4.1) by $C(d)$, then $C'(d) = \lambda$, while from (c) $\lambda = C_i'(d_i^*) + \mu P_i'(d_i^*) \leq 0$, since $C_i(\cdot)$ and $P_i(\cdot)$ are nonincreasing and $\mu \geq 0$, so that $C'(d) \leq 0$, i.e. $C(\cdot)$ is nonincreasing.

To show convexity of $C(d)$ let $(d_1,\ldots d_n)$ be an optimal solution to $\mathcal{P}(d)$ and $(d_1',\ldots d_n')$ be an optimal solution to $\mathcal{P}(d')$. Then with $\beta \in [0,1]$

$$\beta C(d) + (1-\beta)C(d') = \beta \sum_{i=1}^{n} C_i(d_i) + (1-\beta) \sum_{i=1}^{n} C_i(d_i')$$

$$\geq \sum_{i=1}^{n} C_i(\beta d_i + (1-\beta)d_i') \quad \text{by convexity of } C_i(\cdot)$$

$$\geq C(\beta d + (1-\beta)d') \quad \text{since the feasible set is convex.}$$

With very large M, $\mu = 0$, so that (c) yields $\lambda = C_i'(d_i^*)$, the same marginal operating cost for all i. Q.E.D.

Without the budget constraint we have a simple algorithm for determining optimal $(d_1^*,\ldots d_n^*)$, namely: choose some $\lambda < 0$, find d_i^* by solving $C_i'(d_i^*) = \lambda$ for all i, if $\sum_{i=1}^{n} d_i^* > d$ decrease λ, if $\sum_{i=1}^{n} d_i^* < d$ increase λ and repeat until we find λ giving $\sum_{i=1}^{n} d_i^* = d$.

Thus, for any given delay d, we can choose optimal subalgorithms giving the best algorithm requiring that delay and the operating cost $C(d)$ per unit time. Now suppose we can choose the total delay $d \in D \subset \mathbb{R}^+$ by selecting among a class of algorithms each of which is optimally designed. Then assuming the utility function linear in payoff the natural selection criterion is to choose the algorithm requiring that d which minimizes the sum of the operating cost $C(d)$ and the long-run average loss $L(d) = (\lim_{n\to\infty} L_n(d))$ due to obsolescence.

If $L(d)$ and $C(d)$ are differentiable then the optimal d^*, if in the interior of D, satisfies

$$(4.2) \qquad L'(d^*) + C'(d^*) = 0 \quad \text{and } L''(d^*) + C''(d^*) > 0$$

i.e. when the marginal increment in the operating cost just equals the marginal loss due to delay. (Also note that $C'(d^*) = \lambda(d^*)$, the LaGrange multiplier in $\mathcal{P}(d^*)$.) If L(d) is also convex in d (we already know C(d) is) then (4.2) can be solved for optimal d^*. For example, if the environment is Brownian, $L(d) = \frac{\sigma^2 d}{q}$ by (3.7), so that optimal d^* is determined as a solution of $\frac{\sigma^2}{q} = -\lambda(d^*)$. Since $\lambda(d)$ is non-decreasing in d, it follows that the greater the σ^2 (representing greater variability and and thus resulting in greater obsolescence) or the greater the value of q, the smaller is the optimal d^*, as might be expected.

If L(d) + C(d) is unimodal on D, efficient search procedures such as the Fibonacci search technique may be employed to determine an optimal algorithm. More realistically the set D may be discrete (in fact finite) due to discontinuities in the (limited) technologies available. If $D = \{d_1, \ldots d_k\}$ and if k is small enough then the optimal delay $d^* \in D$ can be found simply by exhaustive search. Consider, for example, two algorithms α and α' respectively requiring d and (d-1) days at costs C and C' (with $C' \geq C$) to compute an optimal action with the quadratic payoff function and the stationary normal Markovian environment as in Section 3. Then, from (3.5), their respective losses will be

$$L_\alpha(d) = \frac{\sigma^2(1-\rho^{2d})}{q} \quad \text{and} \quad L_{\alpha'}(d-1) = \frac{\sigma^2(1-\rho^{2(d-1)})}{q}$$

Hence the relative advantage of using α' instead of α is

$\Delta L = L_\alpha(d) - L_{\alpha'}(d-1) = \frac{\sigma^2}{q}\rho^{2(d-1)}(1-\rho^2)$ so that the larger the $|\rho|$ (i.e. more predictable the environment) or the value of q the lesser important is the distinction between the algorithms. The simple strategy is then to choose α' if and only if $\Delta L \geq C' - C$.

In general with an arbitrary form of the function L(d) + C(d) and the set D we may have to resort to some ad hoc procedure to solve the one-dimensional minimization problem of finding an optimal algorithm.

5. CONCLUSION

Our purpose has been to explicitly take into account the repurcussions of the fact that decision processes involve activities that are both time-consuming and costly. The final objective is to choose an optimal decision procedure by balancing higher costs of obsolescence due to longer delays against higher computational costs due to employing procedures yielding shorter delays. Given the payoff function and the stochastic process governing the environment, obsolescence due to delay, as measured by loss in the expected payoff seems quantifiable without much difficulty, as in Sections 2 and 3. However, in Section 4 we have asumed that for any (sub)algorithm the information about delays and costs of purchasing and operating alternative procedures are known. The present state of knowledge of human and mechanical technologies involved falls considerably short of providing such information in required details. Economists, engineers and management scientists working together can contribute substantially to this important and difficult issue.

In optimal control theory literature some authors (e.g. see Oguztoreli [1]] Kharastishville [6], Mirza and Womack [10] and references therein) in the context of deterministic control problems have included delays in the state and control variables in the objective function and the law of motion of the system. Most of the work is deterministic and relates to either modification of the maximum principle for optimality or derivation of sufficient conditions for controllability of the system, rather than economic considerations of obsolescence due to delays in uncertain environments and choice of procedures for computing controls. Our main source of motivation for the latter has been Marschak [7], Marschak and Radner [8], and essays in McGuire and Radner [9].

In this paper our basic assumption has been that an algorithm is completely characterized by the computational delay and the costs it incurs, these being fixed for a given algorithm. In another paper (Deshmukh and Chikte [3]) we consider the problem of optimally utilizing a given algorithm by choosing, in each computation, the optimal delay for that computation. Typically, if the environment doesnot change stochastically, the effectiveness of generated actions improves with the amount of time spent in the computation. On the other hand, with stochastically changing environment the generated actions will become more obsolete with longer delays. The optimal stopping problem of choosing optimal delays by balancing improvement against obsolescence is treated in [2,3].

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