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"Approximate Convexity of Average Sums of Sets In

Normed Linear Spaces"

by

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Abstract

This note gives a sufficient condition under which the average sum of a large but finite number of sets in an arbitrary normed linear space is approximately convex. The result can be deduced as a simple corollary of an analogous result for a nonstandard universe. The latter has independent interest.

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In [3], building on the work of Brown, [1], and Starr, [10], it was shown that the average sum of a large but finite number of subsets of a finite space is approximately convex if the sets have uniformly bounded non-convexities. This theorem has found application in the development of a theory for economies with a large but finite number of agents; see, for example, [4] and [6]. In this note, I extend this result to sets in arbitrary normed linear spaces. The motivation underlying this extension is the development of a corresponding theory for economies with an arbitrarily large but finite number of commodities and agents. Section 4 below expands on this remark. Sections 1 and 2 develop the results for a nonstandard universe and Section 3 presents the standard asymptotic version.

1. Preliminaries

We work in a nonstandard universe. For a comprehensive introduction to nonstandard analysis, the reader is referred to Stroyan and Luxemburg, [11].

Let $^*\eta$ be the nonstandard extension of η, a normed linear space over the field of real numbers R. For any x in $^*\eta$, let $\|x\|$ denote the norm of x. Let *N be the nonstandard extension of N, the set of all positive natural numbers, and *R the nonstandard extension of R. Let T be an internal subset of *N where $|T|$, is ω, some infinite natural number. Without any loss of generality we assume T to be the set $(1, 2, \ldots, \omega)$, $\omega \in {}^*N - N$.

Let $G : T \to 2^{^*\eta}$ be an internal correspondence. Define $(1/\omega) \sum_{t \in T} G(t)$

as the set of points which are average sums of all internal selections from $\prod_{t \in T} G(t)$. For any set $\bar{G}$ in $^*\eta$, we shall take $(1/\omega) \sum_{t \in T} \bar{G}$ to mean the average sum of a constant correspondence.

A set B in $^*\eta$ is said to be *S-convex* if for all x, y ε B and any $\lambda \varepsilon {}^*(0,1)$, there exists a z ε B such that $\| z - (\lambda x + (1-\lambda)y) \| \simeq 0$. For any two elements a, b in *R, $a \simeq b$ means a differs from b by an infinitesimal. If the definition of S-convexity, we replace $\simeq$ by =, we shall say that B is *Q-convex*. For any internal set B, Q-con(B) will denote the Q-convex hull of B. By transfer, it is the set of all star finite convex combinations of points chosen from B. Note that B is S-convex if for all y ε Q-con(B), there exists z ε B such that $\| z - y \| \simeq 0$. The converse is, in general, false. We shall accordingly say that a set B in $^*\eta$ is *strongly S-convex* if for all y ε Q-con(B), there exists z such that $\| z - y \| \simeq 0$.

We shall continue to work with Starr's measure of non-convexity. Let x ε Q-con(S), $\mathcal{L}(x,S) = \{| A \subseteq S \mid x \varepsilon \text{Q-con}(A)\}$ and $\text{rad}(S) = \text{Inf}_{x \varepsilon S}\, r(x)$ where $r(x) = \{r \varepsilon {}^*N \mid B(x,r) \supseteq S\}$ and $B(x,r) = \{y \varepsilon {}^*\eta \mid \| x - y \| < r\}$. We can now define a measure of non-convexity of S, R(S), as $\text{Sup}_{x \varepsilon \text{Q-con}(S)}\ \text{Inf}_{A \varepsilon \mathcal{L}}\ \text{rad } A$.

We end this section with two remarks.

<u>Remark 1</u>: Let G be a correspondence from T into $^*R^\nu$ the ν-fold copy of *R, $\nu \varepsilon$ *N. Then irrespective of the magnitude of ν, $(1/\omega) \sum_{t \varepsilon T} G(t)$ is not, in general, S-convex. Consider, for example, $G : T \to 2^{*R}$ such

that for all $t \in T$, $t \neq \omega$, $G(t) = \{1\}$ and $G(\omega) = (\{1\}, \{\omega\})$. Then $(1/\omega) \sum_{t \in T} G(t)$ consists of the two points $\{1\}$ and $\{2 - 1/\omega\}$ and is obviously not S-convex.

Remark 2: Remark 1 is valid even for the case of a constant correspondence. Let $G(t) = (\{1\}, \{\omega\})$ for all $t \in T$. Then $1/\omega \sum_{t \in T} G(t)$ is not S-convex.

2. Principal Results

We begin by making precise the condition that there are "many sets of every type" in the range of the correspondence G.

Standing Hypothesis: Let $G(t) = G_i$ for all t in T_i and for all $i = 1, \ldots, \rho$, $\rho \in {}^*N$ where

(i) $T = \bigcup_{i=1}^{\rho} T_i$, $T_i \cap T_j = \phi$, T_i internal

(ii) For all i, $|T_i| = \omega(i) \in {}^*N - N$.

Note that we do not require $(\omega(i)/\omega) \neq 0$ for any i. We shall also need the following assumptions

Assumption 1: For all i, $(R(G_i)/\sqrt{\omega}) \simeq 0$

Assumption 2: For all i, there exist $n(i) \in N$ such that $G_i = \bigcup_{j=1}^{n(i)} A_{ij}$ where A_{ij} are convex subsets of ${}^*\eta$ and $n(i)/\sqrt{\omega}(i) \simeq 0$.

We can now state the principal result of this paper.

Theorem 1: Under Assumptions 1 and 2, $(1/\omega) \sum_{t \in T} G(t)$ is strongly S-convex.

The proof is in a series of lemmas. Let $\Delta_k = \{\lambda \in {}^*R^k \mid \lambda_j \geq 0, \sum_{j=1}^{k} \lambda_j = 1\}$.

Lemma 1: Let $G_i^c = \{x \in {}^*\eta \mid x = \sum_{j=1}^{n(i)} \lambda_j x^j, \lambda \in \Delta_{n(i)}, x^j \in A_{ij}\}$. Then $Q\text{-con}(G_i) = G_i^c$.

Proof: We first show that G_i^c is convex. Let x and y be elements of G_i^c. Then

$$x = \sum_{j=1}^{n(i)} \lambda_j x^j \qquad x^j \in A_{ij}, \quad \lambda \in \Delta_n(i)$$

$$y = \sum_{j=1}^{n(i)} \mu_j y^j \qquad y^j \in A_{ij}, \quad \mu \in \Delta_{n(i)}$$

Then for any $\beta \in {}^*[0,1]$

$$\beta x + (1-\beta)y = \sum_{j=1}^{n(i)} \beta\lambda_j x^j + (1-\beta)\mu_j y^j$$

Let $\gamma_j = \beta\lambda_j + (1-\beta)\mu_j$. Certainly $0 \leq \gamma_j \leq 1$ and $\sum_{j=1}^{n(i)} \gamma_j = 1$.

Thus, $\beta x + (1-\beta)y = \sum_{j=1}^{n(i)} (\gamma_j x^j + (1-\gamma_j)y^j)\gamma_j$. Given convexity of A_{ij}, we can see that $\beta x + (1-\beta)y$ is an element of G_i^c.

Since $Q\text{-con}(G_i)$ is the set of all star-finite convex combinations of points chosen from G_i, $G_i^c \subseteq Q\text{-con}(G_i)$. From above, $G_i^c \supseteq Q\text{-con}(G_i)$.

Q.E.D.

Lemma 2: Under Assumptions 1 and 2, $H_i \equiv (1/\omega(i)) \sum_{t=1}^{\omega(i)} G_i$ is strongly S-convex.

Proof: Certainly $H_i \subseteq$ Q-con(G_i). This implies that Q-con$(H_i) \subseteq$ Q-con(G_i). We thus need only show that for any $y \in$ Q-con(G_i), there exists $z \in H_i$ such that $\| z - y \| \simeq 0$. By Lemma 1, there exists $\lambda \in \Delta_{n(i)}$ and $y^j \in A_{ij}$ such that $y = \sum_{j=1}^{n(i)} \lambda_j y^j$. Without loss of generality, assume $\lambda_j > 0$ and $x^j = y^j - y^1$ for all $1 \leq j \leq \ell \leq n(i)$. Note that given Assumption 1, $\| x^j \| /\sqrt{\omega}(i) \simeq 0$.

Consider the set T_i of the natural numbers 1 through $\omega(i)$ and for any internal subset L of T_i define $\mu(L) = \sum_{t\in L} (1/\omega(i))$. It is easy to see by a transfer argument that for all $\lambda_j > 0$, there exist internal sets $S_j \subseteq T_i$ such that $| \lambda_j - \mu(S_j) | \leq 1/(\omega(i))$. One can alternatively show the existence of the sets S_j by successive application of a one-dimensional version of Loeb's Theorem [7]. Now define the function

$$h(t) = x^j \quad (\forall t \in S_j)(j=1,\ldots,\ell).$$

Certainly h is an internal function from T_i into $^*\eta$ such that for all t in T_i, $h(t) + y^1 \in G_i$. Now

$$\| (1/\omega(i)) \sum_{t=1}^{\omega(i)} (h(t)+y^1) - \sum_{j=1}^{\ell} \lambda_j (x^j+y^1) \| =$$

$$\| (1/\omega(i)) \sum_{t=1}^{\omega(i)} h(t) - \sum_{j=1}^{\ell} \lambda_j x^j \| =$$

$$|| (1/\omega(i)) \sum_{j=1}^{\ell} \sum_{t\varepsilon S_j} h(t) - \sum_{j=1}^{\ell} \lambda_j x^j || \leq$$

$$\sum_{j=1}^{\ell} | |S_j|/(\omega(i) - \lambda_j | \, || x^j || \leq \sum_{j=1}^{\ell} 1/(\omega(i)) \, || x_j || \leq$$

$$(\operatorname{Max}_{1\leq j\leq \ell} || x_j || (\ell/\omega(i)) \simeq 0 \qquad \text{Q.E.D.}$$

Remark 3: Note that the last step in the proof hinges crucially on the fact that $n(i)/\sqrt{\omega}(i)$ and $|| x^j || /\sqrt{\omega}(i)$ are infinitesimals.

The following lemma is essentially due to Robinson; see Brown, [1]. The proof produced in [1] for $^*R^n$, $n \varepsilon N$, can be straightforwardly extended.

Lemma 3: Let $\{A_t\}_{t\varepsilon K}$ be an internal family of nonempty subsets of $*\eta$ and $B = \prod_{t\varepsilon K} A_t$, the internal set of internal selections from the A_t. Suppose $\{\bar{y}_t\}_{t\varepsilon K}$ an internal function such that $(\forall t \varepsilon K)(\exists \bar{z}_t \varepsilon A_t) \; || \bar{z}_t - \bar{y}_t || \simeq 0$, then there exists $g \varepsilon B$ such that $\forall t \varepsilon K$, $|| g(t) - \bar{y}_t || \simeq 0$.

Proof: The following sentence is true in the standard universe U: for every positive $\delta \varepsilon R$, $(\forall K \subset N)(\forall \{A_t\}_{t\varepsilon K}, A_t \subset \eta)(\forall f \varepsilon (\eta)^K)(\forall t \varepsilon K)$
$(\exists b_t \varepsilon A_t)(||f(t) - b_t|| < \delta) \Rightarrow (\exists g \varepsilon \prod_{t\varepsilon K} A_t)(\forall t \varepsilon K)(||g(t) - f(t)|| < \delta)$.

Hence this sentence is true when translated into *U, the nonstandard universe: for every $m \varepsilon N$, $(\exists g_m \varepsilon (B)^N) (\forall t \varepsilon K)(||g_m(t) - \bar{y}_t|| < 1/m)$. Hence we have a sequence $\phi : N \to B$ such that $\phi_n = g_n$. Under the assumption of a η-saturated model, we can extend ϕ to $\rho : {}^*N \to B$ such that $\exists \nu \varepsilon {}^*N-N$ such that $g_\nu \varepsilon (B)^{*N}$ and $|| g_\nu(t) - \bar{y}_t || < 1/\nu \simeq 0$. Q.E.D.

Lemma 4: Let $\lambda \varepsilon \Delta_\rho$, $\rho \varepsilon {}^*N$ and A_t, $t \varepsilon K = (1,2,...,\rho)$ be an internal family of nonempty, strongly S-convex, subsets of $*\eta$. Then $\sum_{t\varepsilon K} \lambda_t A_t$ is strongly S-convex.

Proof: Pick x from Q-con $\sum_{t\varepsilon K} \lambda_t A_t$. Since the operators Q-con and $\sum_{t\varepsilon K}$ commute, $x = \sum_{t\varepsilon K} \lambda_t x_t$ where $x_t \varepsilon$ Q-con(A_t). Since A_t are strongly S-convex, there exist $z_t \varepsilon A_t$ such that $|| x_t - z_t || \simeq 0$. Now by Lemma 3, we can find an internal selection g from $(\prod_{t\varepsilon K} A_t)$ such that $|| g(t) - z_t || \simeq 0$. Let $h(t) = g(t) - x_t$. Since both g and x are internal, h is internal and thus $\bar{h} = \underset{t\varepsilon K}{\text{Max}} || h(t) ||$ is well-defined and equal to an infinitesimal. Thus $|| x - \sum_{t\varepsilon K} \lambda_t g(t) || \leq \sum_{t\varepsilon K} \lambda_t ||$ $x_t - g(t) || \simeq 0$. Q.E.D.

We can now furnish a

Proof of Theorem 1: Under the standing hypothesis, we can rewrite $(1/\omega) \sum_{t\varepsilon T} G(t)$ as $\sum_{i=1}^{\rho} (\omega(i)/\omega)(1/\omega(i)) \sum_{t=1}^{\omega(i)} G_i$. Since $\omega(i) \varepsilon$ *N-N, Lemma 2 allows us to deduce that $(1/\omega(i)) \sum_{t=1}^{\omega(i)} G_i$ is strongly S-convex. Application of Lemma 4 completes the proof. Q.E.D.

Remark 4: Let the internal family $\{A_t\}_{t\varepsilon K}$, $A_t \subseteq {}^*\eta$ be *essentially finitely spannable* if for all $t \varepsilon K$, for all $y \varepsilon$ Q-con(A_t), there exist $n(t) \varepsilon$ *N, $n(t)/\sqrt{|K|} \simeq 0$; $x^j \varepsilon A_t$, $j = 1, \ldots\ldots, \ell \leq n(i)$; and $\lambda \varepsilon \Delta_\ell$ such that $y = \sum_{j=1}^{\ell} \lambda_j x^j$. If n(t) is independent of t and an element of N, the famil- $\{A_t\}_{t\varepsilon K}$ is said to be *uniformly finitely spannable*. It is obvious that

Theorem 1 is valid with the replacement of Assumption 2 by the requirement that the family $\{G_i\}_{i=1}^{\rho}$ is essentially finitely spannable. Given Caratheodory's Theorem, it is also obvious that $A_t \subseteq {}^*R^n$, $n \in N$ implies that the family $\{A_t\}_{t \in K}$ is uniformly finitely spannable without any further assumptions.

Remark 5: Assumption 2 is not necessary for the validity of Theorem 1.

In the remaining part of this section, we expand on Remark 5. We shall now assume that the G_i, $\ell = 1, \ldots, \rho$, are subsets of ${}^*R^{\nu}$, $\nu \in {}^*N - N$. By transfer of Caratheodory's Theorem we know that for any i, for all $y \in Q\text{-con}(G_i)$, there exist $x^j \in G_i$, $j = 1, \ldots, \nu + 1$, such that $y = \sum_{j=1}^{\nu+1} \lambda_j x^j$. We shall say that the family $\{G_i\}_{i=1}^{\rho}$ is *basically spannable* if there exists $n \in N$, such that for all i and all y, for all coordinates $s = 1, \ldots, \nu$, the number of elements in the set $\{x^j\}_{j=1}^{\nu+1}$ with non-zero s^{th} coordinate is less than n and these elements have all other coordinates zero.

Theorem 2: Under Assumption 1 and the assumption of basic spannability, $(1/\omega) \sum_{t \in T} G(t)$ is strongly S-convex.

Proof: Note that we needed Assumption 2 only in the proof of Lemma 2. It can be easily checked that we can prove Lemma 2 under the substitution of the assumption of basic spannability for Assumption 2. Q.E.D.

3. An Asymptotic Interpretation

In this section we show how Theorems 1 and 2 can be used to derive standard results on the approximate convexity of the average sum of a large number of subsets of a normed linear space or a space with an arbitrarily large but finite dimension.

Let $\{G(t)\}_{t=1}^{\tau(k)}$ denote a family of $\gamma_t(k)$ sets identical to G(t) and thus the total number of sets in the family are $\sum_{t=1}^{\tau(k)} \nu_t(k) = \gamma(k)$ say. When G(t) are subsets of η, a normed linear space, we shall denote the family by $\mathscr{G}^k$ and when they are subsets of $R^{d(k)}$, we shall denote the family by $\mathscr{F}^k$. Let $\mathscr{A} = \{\mathscr{G}^k\}_{k\varepsilon N}$ and $\mathscr{B} = \{\mathscr{F}^k\}_{k\varepsilon N}$. We shall need the following condition of $\mathscr{A}$ and $\mathscr{B}$, with the concepts defined exactly as in the previous sections.

<u>Assumption 4</u>: For all k, for all G(t) in $\mathscr{G}^k$ or $\mathscr{F}^k$, $\text{Limit}_{n \to \infty} R(G(t))/n = 0$. In addition, $\mathscr{A}$ is uniformly finitely spannable and $\mathscr{B}$ is uniformly finitely *or* basically spannable.

We shall say that $A \subset R^k$ is *ε-convex* if $\forall x, y \in A$, $\forall \lambda \in (0,1)$, there exists $z \in A$ such that $\| z - (\lambda x + (1-\lambda)y) \| < \varepsilon$. We can now state

<u>Theorem 3</u>: Under Assumption 4, $(\forall\, \varepsilon > 0)(\exists m \in N)(\forall k \in N,\ k \geq m)$ the average sum of sets in $\mathscr{G}^k$ or $\mathscr{F}^k$, i.e., $(1/\gamma(k)) \sum_{t=1}^{\gamma(k)} G(t)$ is ε-convex.

Theorem 3 is a far-reaching generalization of the standard result presented in [3] or the one that can be derived from [1]. Note that it can be generalized slightly by relying on a standard version of essential finite spannability instead of uniform finite spannability.

<u>Proof of Theorem 3</u>: Follow the argument used in [3] to deduce this

result from Theorems 1 and 2. To see that $\lim_{n \to \infty} R(G(t))/n = 0$ implies Assumption 1, one has to use Robinson's Theorem [10, page 60, Theorem 3.3.7]. Q.E.D.

4. Remarks on Applications

In showing that core allocations can be sustained as price equilibria, we consider a correspondence whose range is a union of a point and a set of preferred elements; see for example, [2, page 133] or [4]. If we assume convexity of preferences, a natural assumption in this content (see Section 3.2 in [2]), our condition of uniform finite spannability is automatically fulfilled. The average sum of sets in the correspondence is thus approximately convex, allowing us to apply the separation theorems. In this way, the core equivalence theorem can be generalized to arbitrary normed linear spaces.

In proving the existence of a competitive equilibrium, it is a natural procedure to take the average sum of maximal elements in each trader's budget set; see for example [1]. If the sets of maximal elements is finitely spannable, uniformly or essentially, the average sum is approximately convex, opening the way for the use of fixed point theorems. However, we do not have any conditions on the preferences which are sufficient for the set of maximal elements to be finitely spannable. On the other hand, if the preferences are such that the "better-than set is given by $\{x \in R^n_+ \mid \sqrt{\sum_{i=1}^{n} x_i^2} > \alpha\}$, $\alpha \in R_+$, then the set of each agent's maximal elements will be basically spannable.

Both of these applications will allow the development of a point of view that is novel. It is assumed in the literature that for the above two results to extend to spaces of arbitrary dimension, one needs to restrict the commodity space relative to the number of agents in the economy; see, for example, [6] and [8, page 265]. Using the results of this paper in the manner indicated above, one can show that all one requires is that *there be many agents of the same type, and that the number of agents of any given type need not be a non-negligible fraction of the number of all agents in the economy.*

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