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A Dynamic Process of Exchange *

by

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A Dynamic Process of Exchange

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1. The model presented in this paper deals with dynamic behavior of a market not in equilibrium. It is a characteristic feature of markets out of equilibrium that opportunities to trade on different terms exist simultaneously. One explanation of how a uniform price comes to prevail in a market is in terms of arbitrage. In broad terms, the model studied here formalizes a dynamic arbitraging process and explores the extent to which that process can bring about results of the kind usually envisaged for it.

The literature on markets out of equilibrium contains models which deal with search behavior** and models in which a market "process" is formalized as a game***. The model studied in this paper differs from these in many details but mainly in two respects. First, the behavior of agents is modelled from a "bounded rationality" point of view, rather than, say, in terms of optimal searching. Second, the main focus is on the dynamic process and its long run tendency rather than on a solution concept such as Nash equilibrium or market clearing. In these respects the viewpoint of [5] is retained here.****

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*I have benefitted from discussions with Ehud Kalai, Glenn C. Loury, and Michael Magill, and from comments by Truman Bewley, Hugo Sonnenschein and an anonymous referee.

**A bibliography of the job search literature may be found in [3]. For a reference on market search see [6].

***See [7] and [8].

****This paper is a revision of [5]. The basic objective of [5] was to investigate the equilibria of a process of exchange in unorganized markets. The ideas of bounded rationality, of stochastic behavior as a way of coping with insufficient information and the stochastic nature of equilibrium are all taken from [5], although the present formulation is different.

A dynamic market process can not satisfactorily be based on static excess demand behavior of economic agents (traders, in a pure exchange setting) because it is unsatisfactory to assume that agents will maximize utility treating the budget constraint as if it were a certainty in a situation in which it is necessarily uncertain. Away from the static equilibrium no agent can be assured of his opportunities. If the behavior of agents away from equilibrium is qualitatively different from their behavior at equilibrium, then they can, in effect, recognize whether the market is or is not in equilibrium. While it is possible that behavior in equilibrium should be in some sense a limit of disequilibrium behavior, we take the view that behavior of agents should in some sense be qualitatively the same throughout the process.

The role of information and its effect on the appropriate behavior of agents then becomes important. We try to take account of two aspects of the role of information. First, the institutional structure of the market determines the information agents get from the market process, the "structural" aspect. Second, the restricted capacity of economic agents to handle information restricts behavior, the "bounded rationality" aspect. The structural aspect of information is modelled by looking at an extreme case, one in which each agent acquires information only from his own direct experience, and at that only information indispensable to making trades, namely bids or offers tendered to or by himself. Moreover, trading is assumed to be unorganized and anonymous. Traders meet one another at random, exchange bids, do or do not agree to trade and then separate, each ignorant of the identity of the other.

The "bounded rationality" aspect enters in two ways. First, in economic life each individual takes part in many different economic activities, including trading in markets for many different commodities. Each person has limited time and resources, and typically devotes a substantial portion of them to some specialized productive activity, his job, from which he earns his living. Such a person cannot devote a large amount of his time and resources to searching out trading opportunities. If the number of commodities which an individual trades is large relative to his capacity, then the individual agent can do relatively little in the way of searching or acquiring information in "most" markets. Thus, in any market we may expect to find many agents whose capacities for search are relatively small. In the kind of situation that would typically result, opportunities for arbitrage profit may be expected to exist. Therefore, some agent would have incentives to "specialize" in such a market, i.e. to devote a large portion of his resources to trading in that market and in effect make trading his "job". Such considerations led to the formulation of a pure exchange model in which there are two types of agents, those with relatively large capacity and those with low capacity, and correspondingly with different appropriate behavior.

Second, the behavior of each type of agent is itself restricted by restricted capacity to process information in the face of complex and changing circumstances. To derive and characterize explicitly rules for fully rational behavior in such circumstances is a difficult problem. Instead the model specifies certain plausible modes of behavior supported only by heuristic arguments. This leads, as will be seen below, to behavior involving stochastic elements .

More specifically, the process is as follows. A market consists of a finite collection of agents.

The market process goes on in time, which is discrete, consisting of a sequence of periods. Contact among agents takes place at random during each period.

There are two types of agents. One is characterized by low capacity for contact per period, (reflecting a low allocation of information-processing capacity to this market). The other is characterized by a high capacity for contact per period. There are many low capacity agents and a few high capacity ones.*/ I shall call the agents consumers and traders respectively.

The situation envisaged was one in which the various agents having very little information about opportunities might be prepared to trade on very different terms. This is a situation in which the possibility of arbitrage profit exists.

When a trader and a consumer meet some form of bargaining takes place. Because of the trader's high capacity for contact he is likely

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*/ In [5] the low capacity agents were called "flounders" and the high capacity ones "sharks". The process was visualized as taking place in an ocean in which the flounders form a numerous collection of widely separated and slow-moving individuals while the sharks circulate among them at high speed. It was assumed that each flounder can make at most one contact per period while each shark is restricted to some number greater than one.

to be better off devoting his capacity to seeking arbitrage profits rather than to spend scarce time in hard bargaining with a single consumer, from whom he is likely to make a relatively small gain at best. This is modelled as follows. When trader and consumer meet, the consumer makes a bid. The trader either accepts it or rejects it and moves on to another consumer. Thus, each participant makes and receives a bid as a result of each contact he makes. These bids are the sole basis of the information acquired from the market.

I turn next to the behavior of agents. First, the consumers. The specification of rational behavior for a consumer presents an interesting and largely unsolved problem of behavior under uncertainty. In the nature of the case a consumer can have at best an estimate of the alternatives confronting him. The only point that he can be certain is available to him is the point involving no trade. His problem is complicated by three factors. First, the set of alternatives facing him may change from time to time. Second, his ability to acquire information and hence the rate at which he acquires information may be slow relative to the rate at which the set of alternatives is changing. This means that aging information becomes increasingly irrelevant. Third, each "observation" involves both information and payoff. The means of acquiring information is the making and receiving of offers. The process involves the risk of making deals. In this respect the consumer's problem is like problems of the two-armed bandit class.

In view of these difficulties we shall prescribe intuitively appealing modes of behavior for consumers, the rationality of which is supported only by heuristic arguments.

Roughly speaking, each consumer selects his current bid probabilistically on the basis of the current state of his information about his opportunities.

Now, the traders. Each trader, in pursuit of arbitrage profit, accepts or rejects the bids received by him, using a decision rule chosen at the beginning of the current period and maintained unchanged through that period. Thus a trader is assumed not to be able to respond to variations in the pattern of bids received which take place within a period. That is, he cannot condition his response to a bid received on the others received before it in the same period. Such a restriction on behavior reflects considerations of "bounded rationality". The specification of rational behavior for a trader presents difficulties similar to those already mentioned in connection with the same problem for consumers. Our approach is the same in this case as in that; we prescribe intuitively appealing modes of behavior supported by heuristic arguments. The prescribed behaviors represent conjectured solutions to a problem of statistical decision theory, namely this particular form of the "two-armed bandit" problem.

The difference between traders and consumers is the higher capacity for contact enjoyed by traders. This makes it plausible for traders to attempt to take advantage of the possibility for profit inherent in the simultaneous existence of offers on different terms. In such circumstances it is possible for a trader to find several transactions the net effect of which is preferred by him to no trade, while no single one of them is so preferred. Therefore, the behavior of traders , being directed toward finding advantage in the sum of many transactions, does not depend on an evaluation of each offer individually with immediate reference to his own preferences.

In addition, the asymmetry between traders and consumers provides a basis for strategic behavior. A trader, knowing that he is a trader, may consider that he has a degree of monopoly power, namely the power to control the information received by individual consumers, and thereby to influence the perception on the part of consumers of the opportunities present in the market. One type of monopoly power, or perhaps it is better to say one source of monopoly power, is the ability of an agent to distort the "true" opportunities confronting the others. In this view the stick-up man who holds a pistol to his victim's head and says, "Your money or your life!" exercises a kind of monopoly power not so different from that, say, of a product monopolist, who present consumers with marginal rates of transformation different from those determined by the technology. A consumer, knowing that he is a consumer in a world in which there are traders, also has a strategic problem. These will be explored below.*/

In what follows the behavior of agents is stated more precisely. A stochastic process resulting from the behavior of agents and their

*/ The monopolistic aspects of this process were not explored in [5].

interactions represents the market process. The long run behavior of this process is studied. The main results are stated in Propositions 1 and 2 .

Generally, we expect arbitraging to eliminate price differences within a market, and for the resulting allocations to be "optimal". For instance, a process based on recontracting <u>à la</u> Edgeworth has been shown to lead to Core allocations. In the present model the unorganized structure of the market and the restricted capacity of agents to make contacts limits the allocations achievable via the process. Specifically, the process cannot be guaranteed to achieve Core allocations. Rather the process tends toward allocations in the K-core. (The K-core is the set of allocations which cannot be blocked by coalitions involving fewer than $K+1$ agents, where K , one of the parameters of the model, is a positive integer.*) If a core allocation results it is essentially accidental. The results contained in Propositions 1 and 2 are established in a model in which each agent acquires very little information about what is happening in the market generally. However, these results would remain even if agents were provided with information about all transactions. They depend on the restricted capacity of traders to make contacts. As long as individual arbitrageurs are small relative to the market, a process of unorganized arbitrage, such as the one in this model, cannot guarantee Core allocations, nor competitive ones. Thus, it appears that some organized market institution is needed. "Natural" arbitrage is not enough. Furthermore, increasing the number of traders will not save the situation. Increasing the number of competing arbitrageurs seems to lead to a sort of monopolistically competitive solution rather than one in which full equalization of opportunities prevails.

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*The K-core is a concept closely related to ψ - stability [4] . Here the function ψ allows all changes resulting in a coalition of no more than K players. The connection with ψ-stability was pointed out to me by L. Hurwicz.

2. Agents are of two kinds, <u>consumers</u> and professional traders or intermediaries, called <u>traders</u>. Let

$$\mathcal{C} = \{1, \ldots n\}$$

be the set of consumers and let

$$\mathcal{T} = \{n + 1, \ldots, m\}$$

be the set of traders. Then the set of agents I is given by

$$I = \mathcal{C} \cup \mathcal{T} .$$

Let $\bar{X}$ be the commodity space. I shall suppose that $\bar{X}$ is a (countable) discrete subset of $\mathbb{R}^{\ell}$. The role of this assumption is to avoid technical complexities connected with probability and measure, while retaining the main ideas.

Consumer i is characterized, as usual, by his consumption set $Y^i \subseteq \bar{X}$, his initial endowment $\omega^i \in Y^i$ and his preferences, $\underset{i}{\succsim}$, assumed to be represented by a utility function. Correspondingly, the trade space of consumer i is

$$X^i = Y^i - \{\omega^i\} = \{x \in \mathbb{R}^{\ell} \,|\, \omega^i + x \in Y^i\} .$$

A preference relation $\underset{i}{\succsim}$ on Y^i induces a preference relation $\underset{i}{\succsim}$ on X^i in the usual way, i.e. $\bar{x}^i \underset{i}{\succsim} \bar{\bar{x}}^i \Leftrightarrow \bar{x}^i + \omega^i = \bar{y}^i$, $\bar{\bar{x}}^i + \omega^i = \bar{\bar{y}}^i$ and $\bar{y}^i \underset{i}{\succsim} \bar{\bar{y}}^i$.

I assume that $\underset{i}{\succsim}$ is represented by a utility function $U^i : X^i \to \mathbb{R}$, where $\underset{i}{\succsim}$ is strictly monotone for each $i \in \mathcal{C}$, and $U^i(0) = 0$;

$\underset{i}{\succsim}$ is automatically continuous since X^i is discrete.

Bids.

Each agent makes bids. The bid of consumer i is a point of X^i.

Denote by $b^i(t)$ the bid of consumer i at time t. Let $b(t) = (b^1(t), \ldots, b^n(t))$ be the vector of bids of consumers, where $b^i(t) \in X^i$.

Traders respond to bids received. The response of trader j is a function

$$\beta^j : \underline{\overline{X}}^n \to \underline{\overline{X}}^n$$

defined as follows.

First, let

$$\rho^j : \underline{\overline{X}}^n \to \underline{\overline{X}}^n$$

where $\rho^j(b^1, \ldots, b^n) = (b^{1j}, \ldots, b^{nj}) = b^{\cdot j} \qquad j \in \mathcal{T}$

and $b^{ij} = \begin{cases} b^i & \text{if } i \in \mathcal{C}_j \text{ */} \\ 0 & \text{otherwise} \end{cases}$.

Thus, $\rho^j(b) = b^{\cdot j}$ is the vector of bids received by trader j when b is the vector of bids made by the consumers. The response β^j of trader j to each bid received is either that bid, indicating that he accepts the bid, or the zero vector in $\underline{\overline{X}}$, indicating that he rejects that bid. Thus, when the consumers bid b,

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*/ The symbol $\mathcal{C}_j$, defined in the Meeting Process below, denotes the subset of consumers who meet trader j.

$\beta^j(b^{\cdot j}) = a^j$, where $a^{j\cdot} = (a^{j1}, \ldots, a^{jn})$ and

$$a^{ji} = \begin{cases} b^{ij} \text{ if } b^{ij} \neq 0 \text{ and } b^{ij} \text{ is accepted, i.e., if } j \text{ receives } i\text{'s bid } b^i \text{ and accepts it.} \\ 0 \quad \text{otherwise} \end{cases}$$

The functions β^j must also satisfy the following condition. For trader j there is a scalar c^j, which may be interpreted either as representing j's costs of being a trader, or alternatively, as representing trader j's aspirations to monopoly profit. It is required that trader j accept only sufficiently profitable trades, i.e., if

$$\beta^j(b^{\cdot j}) = (a^{1\cdot}, \ldots, a^{n\cdot}) = a \ .$$

then $\sum_{i=1}^{n} a \leqq - c^j \cdot \underline{1} < 0$, where $\underline{1} = (1,1, \ldots, 1) \in \overline{X} \subset \mathbb{R}^\ell$.

Anonymity of trading imposes a further restriction on the responses β^j, as follows.

If $b^{\cdot j} = (b^{1j}, \ldots, b^{nj})$ is an array of bids received by j, and if

$$\bar{b}^{\cdot j} = (b^{i_1 j} \ldots, b^{i_n j})$$

where $i_1 \ldots i_n$ is the permutation of $(1, \ldots, n)$, given by

$$\begin{pmatrix} 1 & \ldots & n \\ i_1 & \ldots & i_n \end{pmatrix}$$

then

$$\beta^j(\bar{b}^{\cdot j}) = \beta^j(b^{\cdot j})$$

Thus, trader j's responses to a given array of bids received must be independent of who made those bids. Note that j's response to a bid depends on the array of bids received, not just on the individual bid.

In order to express the idea that agents have limited capacity to contact other agents, I shall assume that consumers can make at most one contact with another agent in each period and that traders can make some given finite number, $k^j \geq 1$, for trader j. I assume further that the nature of the (random) meeting process is such that consumers meet only traders and vice versa. Thus there is a positive integer k^j such that trader j can receive no more than k^j bids in any period. Then if k^j is the number of non-zero bids in $b^{\cdot j}$, then 2^{k_j} is the number of different functions β^j which satisfy the anonymity condition; only some of these will meet the profitability condition. I shall suppose that trader j chooses his response to bids received, $b^{\cdot j}$, probabilistically from the set of possible responses satisfying the requirements of anonymity and profitability given $b^{\cdot j}$. We may identify a function β^i with its graph in $X^n \times X^n$, i.e., with the (discrete) set

$$\{(a^{j\cdot}, b^{\cdot j} \in X^n \times X^n \mid a^{j\cdot} = \beta^j(b^{\cdot j})\}$$

and the response of trader j may be represented as a (discrete) conditional probability measure

$$p^j(\cdot|x) = p^j(x)$$

where $p^j(y|x) = \text{Prob}\,\{\beta^j(b^{\cdot j}) = y | b^{\cdot j} = x\}$.

Thus, $p^j(x)$ is a discrete measure non-zero on the set of vectors $y = (y^1, \ldots, y^n)$ such that $y^i = x^i$ or $y^i = 0$.

Let $\mathcal{P}^j$ be the set of all these conditional probabilities

$$p^j : X^n \times X^n \to [0,1]$$

Thus, an <u>action</u> of trader j at time t is a function $p^j_t \in \mathcal{P}^j$.

The information trader j has about his trading opportunities may be described by a function which assigns a subjective probability to each bid he might conceivably receive.

Let $\hat{Q}^j$ denote the class of functions

$$\hat{q}^j : X^j \rightarrow [0,1] , \qquad j \in \mathcal{J},$$

where X^j denotes the trade set of trader j. Thus, if $a \in X^j, \hat{q}^j_t(a)$ is the subjective probability that trader j has in mind at time t to receive the bid a. */

Thus, if $a \in X^j$, $\hat{q}^j_t(a)$ is the subjective probability that trader j has in mind at time t to receive the bid a.

The behavior, or strategy, of trader j is given by a function

$$\varphi^j : \hat{Q}^j \rightarrow \mathcal{P}^j .$$

These functions will be specified further below.

Bidding of Consumers.

Bids of a consumer are chosen probabilistically. Let Q^i denote the set of probability measures on X^i. Thus, if $q^i \in Q^i$, then

$$q^i: X^i \rightarrow [0,1] \qquad \text{for } i \in \mathcal{C}$$

is a probability measure; $q^i_t(x^i)$ is the probability that $b^i_t = x^i \in X^i$.

The <u>action</u> of consumer i in period t is q^i_t.

This is interpreted and justified as follows. If consumer i knew his present and future trading opportunities for sure, he would select a bid which maximized his utility given his opportunities. However, in the present context he cannot in general know his opportunities. He is uncertain about

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*/ If trader j knows the meeting process, and if he assumes individual bids to be independent, then he can calculate the probability of receiving a given vect of bids from knowledge of $\hat{q}^j$. An alternative formulation would express traders' information in the latter form to begin with i.e., by a subjective probability measure over the vector of bids received.

them and must use his meetings with traders to explore those opportunities, i.e., to search. Consumers use randomized bidding to try to avoid being trapped in a mistakenly perceived set of opportunities. However, a consumer also uses the information so acquired to guide his further exploration in directions of advantage to him.

I shall suppose that consumer i's state of knowledge about his trading opportunities is represented by a subjective conditional probability function, as follows.

Let $\hat{\mathcal{P}}^i$ denote the class of functions

$$\hat{p}^i : X^i \to [0,1] \qquad i \in \mathcal{C}$$

then $\hat{p}^i_t \in \hat{\mathcal{P}}^i$ for $i \in \mathcal{C}$ and $t = 0,1,\ldots$. The interpretation of this function is ; $\hat{p}^i_t(x)$ = Prob. {x is accepted by the trader i meets at t given that b^i_t equals x} .

A behavior rule or strategy for consumer i is a way of choosing his action, i.e. bidding distribution, q^i on the basis of his knowledge of his opportunities $\hat{p}^i$. Thus, a behavior rule of agent i is a function

$$\varphi^i : \hat{\mathcal{P}}^i \to Q^i \qquad i \in \mathcal{C}$$

where

$$q^i = \varphi^i(\hat{p}^i) \qquad i \in \mathcal{C}$$

is the bidding distribution of i chosen when $\hat{p}^i$ summarizes his current knowledge. I shall specify the function φ^i more particularly below.

The Meeting Process

At each time t agents meet one another according to a random

Let E be the (discrete) space of possible observations, (M,b,a), and denote by $\underset{\sim}{E}$ the set of subsets of E.

Given the actions of each agent, and the probabilities of meetings, the meetings which take place and the bids made by consumers are probabilistically determined, hence so are the bids received by traders, and so are their responses. Thus, given π, the actions (q,p) together determine a probability measure on E . Thus,

$$\Pi: Q^{(n)} \times \mathcal{P}^{(m-n)} \times E \rightarrow [0,1],$$

where $\Pi(q,p,e)$ is the probability of the point $e \in E$ given q,p where $q = (q^1,\ldots,q^n)$ and $p = (p^{n+1},\ldots p^m)$.

Because

$$q^i = \varphi^i(\hat{p}^i) \quad \text{for} \quad i \in \mathcal{C}$$

and

$$p^j = \varphi^j(\hat{q}^j) \quad \text{for} \quad j \in \mathcal{T},$$

we may write

$$\Psi((\hat{p}^i)(\hat{q}^j),e) = \Pi((\varphi^i(\hat{p}^i)),(\varphi^j(\hat{q}^j)),e).\ ^{*/}$$

I next define a stochastic process representing the market process, using the stochastic kernel Ψ as its transition kernel.

The states of the process are

$$s = ((\hat{p}^i),(\hat{q}^j)), \quad i \in \mathcal{C} \quad j \in \mathcal{T} .$$

Thus, $\mathcal{J} = \hat{\mathcal{P}}^n \times \hat{Q}^{m-n}$, is the set of states, and their projections are

*/ The notation (x^i) denotes the vector whose components are x^i, where it is understood that $i = 1,\ldots,n$. Similarly for (x^j), where it is understood that $j = n+1,\ldots,m$. No confusion need result from this abuse of notation.

$$\mathscr{S}^i = \begin{cases} \hat{\mathscr{P}}^i & \text{if } i \in \mathscr{C} \\ Q^i & \text{if } i \in \mathscr{T} \end{cases}$$

To complete the specification of the process, define the function

$$\lambda^i : \mathscr{S}^i \times E \rightarrow \mathscr{S}^i \qquad \text{for } i \in I \quad ,$$

as follows. Let

$$\eta^i : E \rightarrow D^i \qquad i \in I$$

denote the function which associates with each point in E the datum observed by i when that event occurs. Thus, if e obtains, then i observes $\eta^i(e) = d^i \in D^i$.

On the basis of this "new" information, i can revise his state of knowledge. Let

$$\bar{\lambda}^i(\hat{p}^i_t \, , \, d^i_t) = \hat{p}^i_{t+1} \qquad i \in \mathscr{C}$$

and

$$\bar{\lambda}^i(\hat{q}^j_t \, , \, d^i_t) = \hat{q}^j_{t+1} \qquad j \in \mathscr{T}$$

be the functions which represent this "learning" process. Then

$$\lambda^i(s,e) = \bar{\lambda}^i(s^i, \eta^i(e)), \qquad \text{for } i \in I \, ,$$

where

$$s = (s^1, \ldots, s^m)$$

and

$$s^i = \begin{cases} \hat{p}^i & \text{for } i \in \mathscr{C} \\ \hat{q}^i & \text{for } i \in \mathscr{T} \, . \end{cases}$$

The Markov process just defined is given by the mappings

1) $\Psi : \mathscr{S} \times \underset{\sim}{E} \to [0,1]$

and

2) $\lambda : \mathscr{S} \times E \to \mathscr{S}$

where $\lambda = \lambda^1 \times \ldots \times \lambda^m$.

I am interested in several variants, each of which is in the class of Markov processes given by 1) and 2).

By imposing some additional properties on the mappings 1) and 2), certain general theorems could be applied to establish the existence of stochastic equilibria for these processes (For an exposition of these theorems see [1]). However, those theorems do not provide the kind of information we would like to have about the set of states to which the system tends in the long run. Therefore my approach is to specify more particularly the behavior of the various agents in the process and to study the long run behavior of the process more directly.

The following example helps make clear some of the motivations for the specifications made below.

An Example.

Suppose the number of commodities and the number of consumers is 2, i.e., $\ell = n = 2$, and that there is just one trader, i.e., $m = 3$. Given their characteristics, we can represent the economy consisting of the two consumers in an Edgeworth Box as in Figure I.

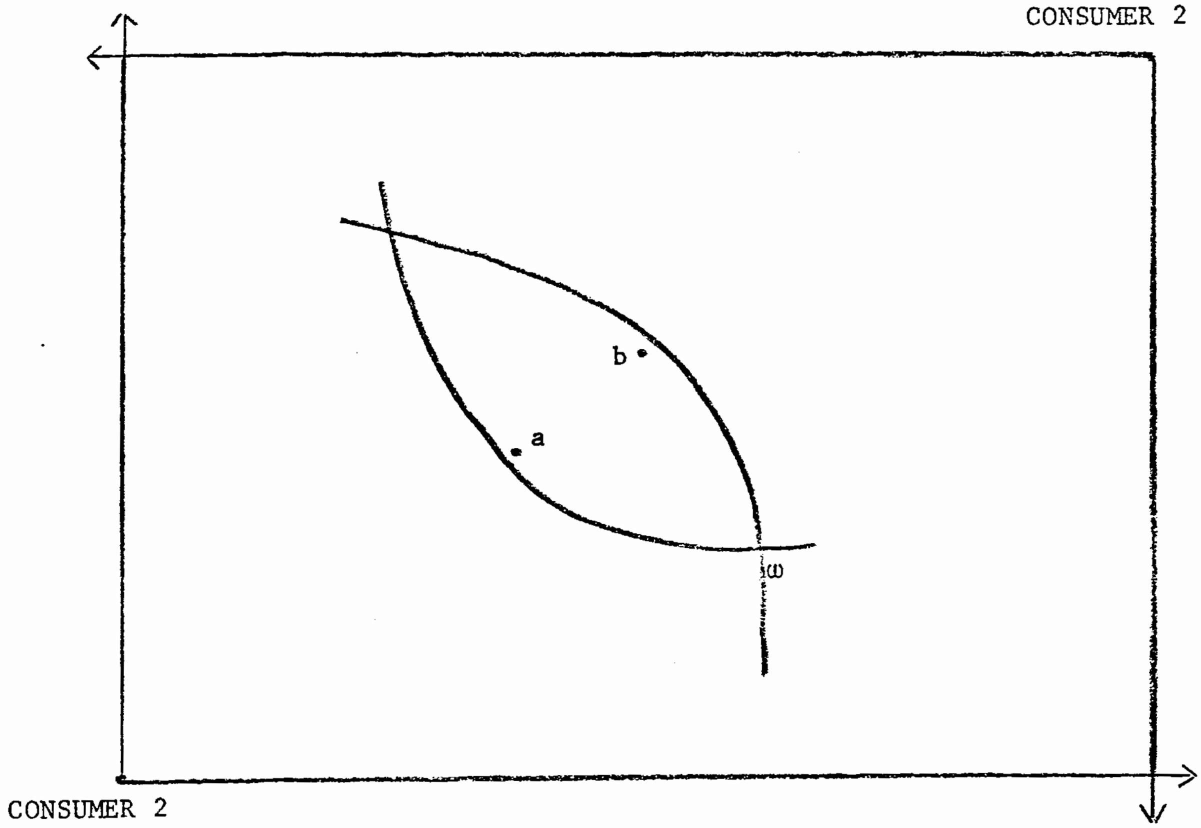

FIGURE 1

As the trader views this situation, he, seeking arbitrage profits, can assure himself of a permanent flow of profit, e.g., by inducing the consumers to make bids corresponding to a and b respectively. This the trader can do by adopting an action which gives positive probability only to accepting the bids corresponding to a and b. If the consumers give positive probability to bids a and b, respectively, then they will acquire data which reinforce the belief that their opportunities include a and b, respectively, and nothing else but the initial endowment. In terms of trades as shown in Figure 2., the trader

Figure 2

can ensure a profit of $(a - \omega^1 + b - \omega^2) < 0$*/ by allowing the consumers to "learn" that their opportunities are confined to a, b, and the initial endowment.

Consider next a two commodity economy consisting of four consumers. Represented in the Edgeworth Box in Figure 3 are the four consumers. It is assumed that there are two pairs each with the same initial endowment but different preferences. Thus the indifference curves labelled 1, 2, 3 and 4 refer to different agents.

*/ Profit to the trader is measured here from the viewpoint of the consumer, i.e., a negative quantity indicates a net flow from consumers to trader.

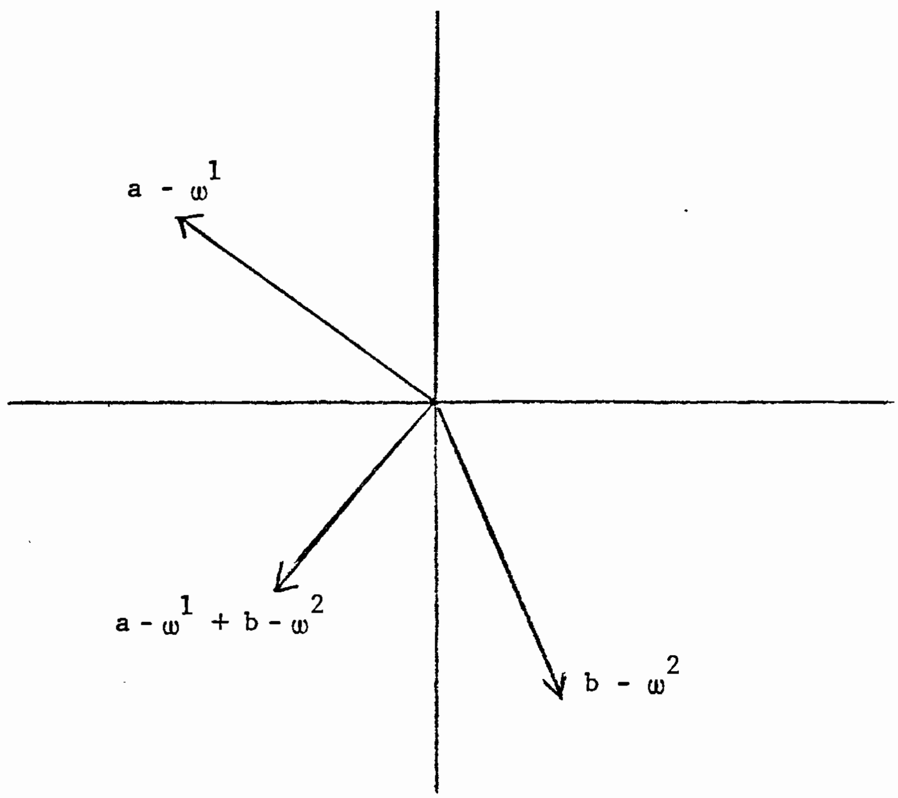

FIGURE 2

Figure 3

The single trader facing this situation could make a profit of $a + b + c + d$ if he could induce the agents to make those bids persistently. However, if he accepts only the bids a and b, he must forego the profit $c + d$. If he always accepts $c + d$, since he cannot identify the agents bidding, he will allow agents 1 and 3 to learn that c and d are acceptable bids. They will therefore tend not to bid a and b, and rather bid c and d. In that case, the trader's arbitrage profit may be expected to tend to $2(c + d)$.

If, however, by sometimes rejecting bids c and d, he can "teach" the consumers 1 and 2 that the probability of acceptance of c and d respectively is so small that it is worthwhile for them to continue bidding a and b, at least sometimes, then the trader can make a long run profit between $a + b + c + d$ and $2(c + d)$. Thus, by falsifying the "true" terms of trade in the economy, a trader can make persistent monopoly profit. This I am suggesting he might do by randomizing his acceptance of bids received.

The question is whether there is any behavior open to the consumers which protects them against the trader(s)? There is indeed such a course of behavior and it, in part, motivates the behavior rules prescribed for consumers. We turn now to these specifications.

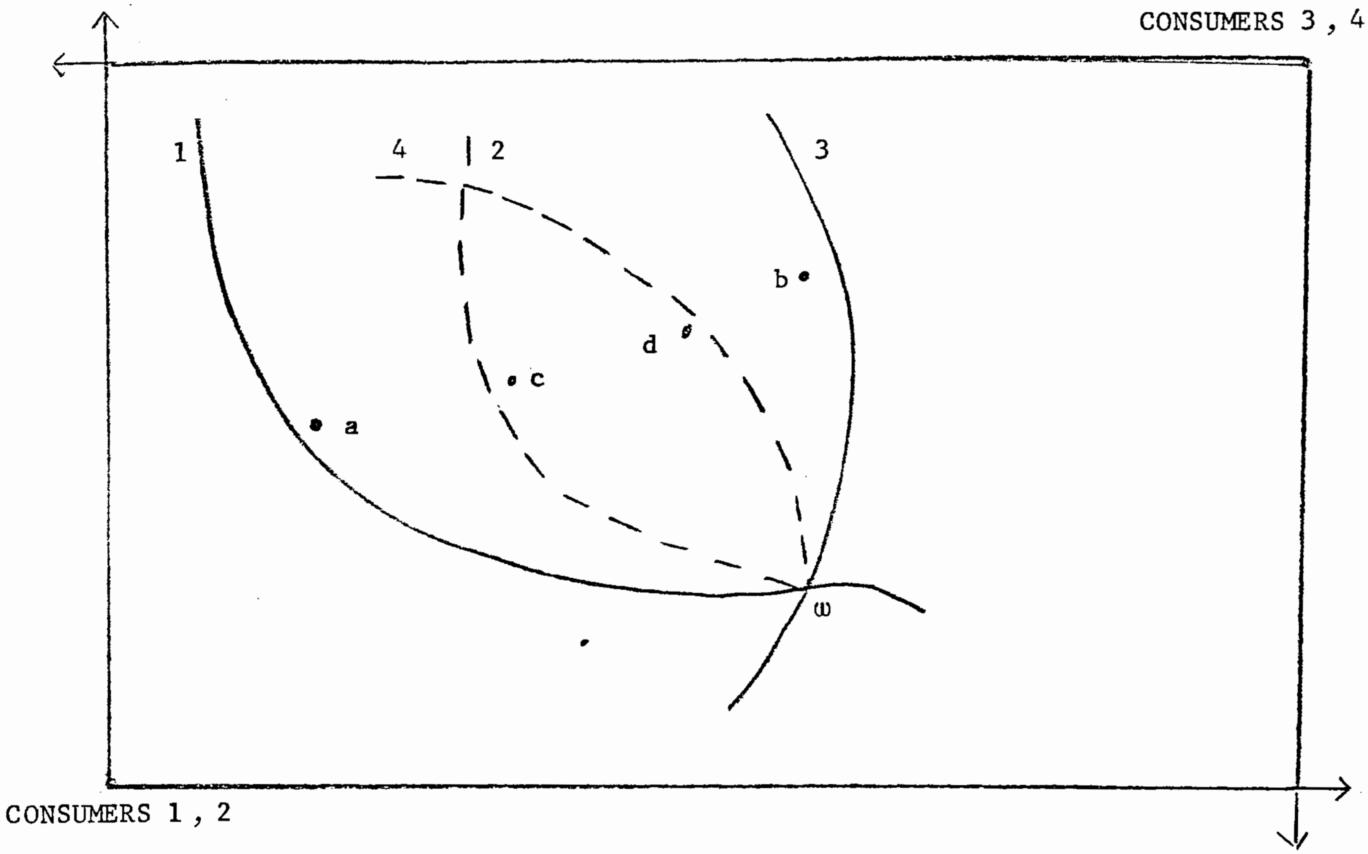

FIGURE 3

The functions φ^i and λ^i must be specified more closely. I consider consumers first.

Let

$$\mathcal{G}^i : \mathbb{R}_+ \to X^i \qquad\qquad i \in \mathcal{C}$$

be the correspondence that associates to each non-negative utility level of consumer i the (upper contour) set of trades individually feasible for i which afford him at least that level of utility, i.e., for $r \in \mathbb{R}_+$,

$$\mathcal{G}^i(r) \equiv \{x^i \in X^i | U^i(x^i) \geqq r\}.$$

Also define

$$C^i : X^i \to X^i \qquad\qquad i \in \mathcal{C}$$

by

$$C^i(y^i) = \mathcal{G}^i(U^i(y^i)) .$$

Next, define

$$\mathcal{V}^i(\hat{p}^i) \equiv \max_{x^i \in X^i} \hat{p}^i(x^i) U^i(x^i)$$

and denote the value of $\mathcal{V}^i(\hat{p}^i_t)$ by

$$v^i_t \equiv \mathcal{V}^i(\hat{p}^i_t) \qquad \text{for } t \geqq 1 \text{ and } v^i_0 = 0.$$

Then v^i_t is the maximum expected utility level that consumer i considers available to him at time t . That $v^i_0 = 0$ is implied by setting

$$\hat{p}^i_0(x^i) = \begin{cases} 1 & \text{if } x^i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

<u>Assumption I</u>. (1) For all $\hat{p}^i \in \hat{\mathcal{P}}^i$ and $i \in \mathcal{C}$, if $q^i = \varphi^i(\hat{p}^i)$, then the support of q^i is $\mathcal{S}^i(v^i)$ where $v^i = \mathcal{V}^i(\hat{p}^i)$. I.e., for any subset $A \subset X^i$, $\varphi^i(\hat{p}^i)\ (A \cap \mathcal{S}^i(v^i)) > 0$ and $\varphi^i(\hat{p}^i)\ (A \cap \text{comp.}\ \mathcal{S}^i(v^i)) = 0$.

(2) For each $i \in \mathcal{C}$, $\hat{p}^i_0(x^i) = \begin{cases} 1 & \text{if } x^i = 0 \\ 0 & \text{otherwise} \end{cases}$.

Thus, initially q^i_0 is positive on $\mathcal{S}^i(0)$, and subsequently bids are selected from a set which i regards as affording him a sure utility at least as great as the highest level of expected utility he thinks available to him. */

Figure 4 shows this assumption for the case in which X^i is one dimensional (a case not naturally interpreted in terms of trades, but easy to see).

<u>Figure 4</u>

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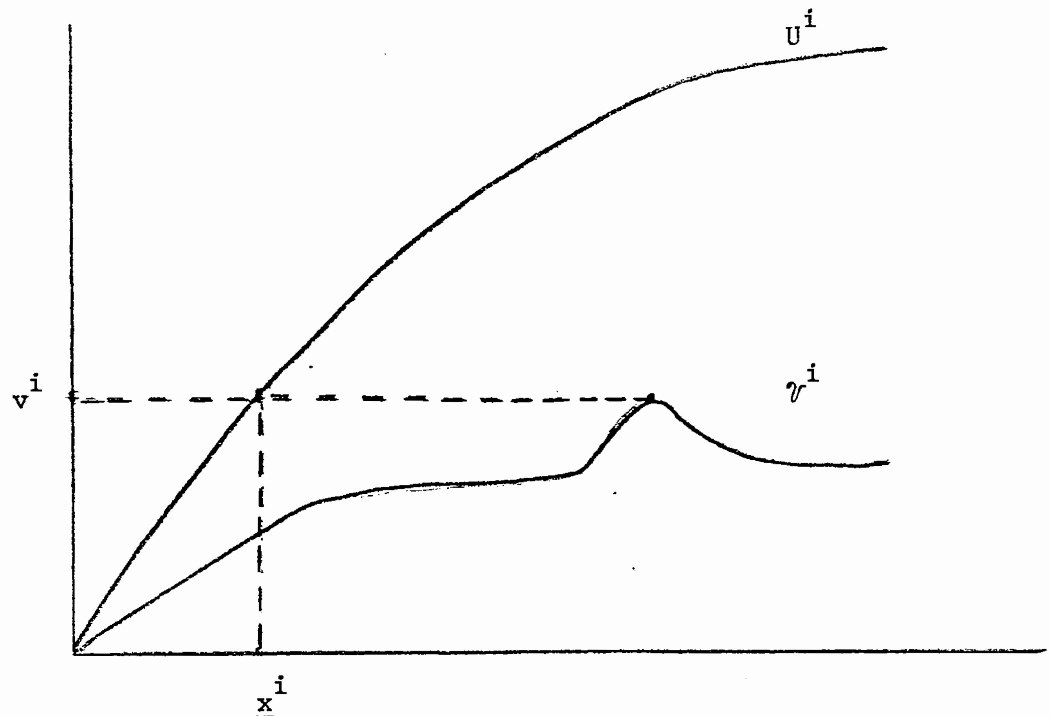
*/ Notice that if consumer i is certain of his alternatives then his bidding distribution is positive on the set of trades at least as preferred by him as the best trade available to him in a form of the static demand responses.



$$\mathcal{S}^i(v^i) = \{x^i \in X^i \mid x^i \geq \underline{x}^i\}$$

FIGURE 4

Def. We say that $x^i \in X^i$ is a basis for q^i, or that q^i is based on x^i if supp. $q^i = C^i(x^i)$, i.e., if supp. $q^i = \mathscr{B}^i(v^i)$ and $U^i(x^i) = v^i$

We say that the joint bidding distribution $q = (q^1, \ldots, q^n)$ is based on $x = (x^1, \ldots, x^n)$ if q^i is based on x^i for each $i \in \{1, \ldots, n\}$.

Let q_t^i be based on x_t^i with $U^i(x_t^i) = v_t^i$. If $v_{t+1}^i > v_t^i$ and x_{t+1}^i is such that $U^i(x_{t+1}^i) = v_{t+1}^i$ then $\mathscr{B}^i(v_{t+1}^i) \subseteq \mathscr{B}^i(v_t^i)$. Let q_{t+1}^i be the conditional measure on $\mathscr{B}^i(v_{t+1}^i)$ corresponding to q_t^i .

Thus, if $A \subseteq \mathscr{B}^i(v_{t+1}^i)$ then

$$q_{t+1}^i(A) = \frac{q_t^i(A)}{q_t^i[\mathscr{B}^i(v_{t+1}^i)]} .$$

Learning.

Let B_t^i denote the response received by consumer i to his bid at t . Then,

$$B_t^i = \begin{cases} b_t^i \\ 0 \end{cases}$$

depending on whether i's bid b_t^i is or is not accepted by the trader whom he meets at t . The function η^i which represents the structure of observation in the market determines what agent i observes as a result of his participation in the process. In the present case, for each $i \in \mathcal{C}$

$$\eta^i(M, b, \dot{a}) = (b^i, B^i)$$

where

$$B^i = \beta^j(\rho^j((b^i))) = \begin{cases} x^i \text{ if } m_{ij} = 1 \text{ and } \beta^j(\rho^j((b^i))) = x^i \\ 0 \text{ otherwise} \end{cases}$$

Thus,

$$d_t^i = \begin{cases} (b_t^i, 0) \text{ if } b_t^i \text{ was bid by } i \text{ at } t \text{ and rejected} \\ (b_t^i, x^i) \text{ if } b_t^i \text{ was bid by } i \text{ at } t \text{ and } x^i = b_t^i \text{ was accepted.} \end{cases}$$

I consider two types of learning. The first, which may be called <u>probability matching</u>, has two properties.

(1) The revised estimate $\hat{p}^i$ is monotone with respect to d^i in the following sense:

If $d_t^i = (b_t^i, x^i)$ then $\lambda^i(\hat{p}_t^i, d_t^i) = \hat{p}_{t+1}^i$ is such that $\hat{p}_{t+1}^i(x^i) > \hat{p}_t^i(x^i)$ $(0 < \hat{p}_t^i < 1)$, and if $d_t^i = (b_t^i, 0)$ then $\hat{p}_{t+1}^i(x^i) < \hat{p}_t^i(x^i)$ for $x^i = b_t^i$, and all $t \geq 0$.

I.e., an accepted bid leads to increased perceived probability of acceptance of that bid, while a rejected bid leads to a decrease in that probability.

(2) Let $\bar{p}_t(x^i)$ denote the "objective" probability that a bid $x^i = b_t^i$ will be accepted at t, i.e. that i will meet some trader j who will accept x^i in combination with some array of bids received from others. This is just a complicated combination of the true acceptance probabilities (p^j) of traders.

If

$$\bar{p}_t^i(x^i) = \bar{p}^i(x^i) \quad \text{for all} \quad t > \bar{t} \quad \text{and} \quad x^i \in X^i$$

then

$$\hat{p}_t^i(x^i) \underset{t}{\to} \bar{p}^i(x^i) \quad .$$

Thus, the probability matching type of learning is one in which the consumer responds to positive and negative reinforcement in such a way that his estimate of the objective probability of acceptance he faces would converge to the true probability, if it were constant, given sufficient time.

The second type of learning, which may be called defensive (or stragetic) ignores rejection of offers and responds only to bids accepted.

In this case,

$$\lambda^i(\hat{p}_t^i, d_t^i) = \hat{p}_{t+1}^i$$

where $\hat{p}_{t+1}^i\,(x^i) > \hat{p}_t^i\,(x^i)$ if and only if $d_t^i = (b_t^i, x^i)$ where $x^i = b_t^i$, and ,

$$\hat{p}_{t+1}^i\,(x^i) = \hat{p}_t^i(x^i) \quad \text{if} \quad d_t^i = (b_t^i, x^i) \quad \text{and} \quad x^i \neq b_t^i.$$

A particular function with this property is

$$\lambda^i(\hat{p}^i, d)(x^i) = \begin{cases} \hat{p}^i(x^i) + \theta^i(1 - \hat{p}^i(x^i))\; 0 < \theta^i < 1 & \text{if } d^i = (b^i, x^i) \text{ and } x^i = b^i \\ \hat{p}^i(x^i) \text{ otherwise.} & \end{cases}$$

According to this learning rule, consumer i increases his estimate of the probability that a bid of his will be accepted only when such a bid is accepted, and never decreases his estimate of such a probability.

Such learning behavior may be partly justified as follows. If the consumer is aware that he is dealing with traders who may be attempting to take advantage of him by misrepresenting trading opportunities, as in the example, he can realize that in order to profit from transactions with him, a trader will have to accept his bid sometimes. He might then ignore all rejections as attempts to confuse him and give consideration only to acceptances. He does not take "No" for an answer. */

Behavior of Traders.

Bidding.

The behavior of traders is directed toward arbitrage profit. As the examples above suggest, a trader may attempt to exploit his position by sometimes refusing profitable bids so as to mislead consumers.

Assumption II. The action p_t^j of trader j at time t satisfies the following condition.

(1) Let $x_t = (x_t^1, \ldots, x_t^n)$ be the bid received by trader j at t; if y_t is a possible response of j to x_t, i.e., $y_t^i \in \{x_t^i, 0\}$, and

$$\sum_{i=1}^{n} y_t^i \leq -c^j \cdot \underline{1},^{**/} \text{ then } p_t^j(y_t | x_t) \geq \delta > 0$$

I consider two cases.

1. $c^j = 0$
2. $c^j > 0$.

Regarding the learning behavior of traders, I assume that it is of the probability matching type. Thus, for $j \in \mathcal{J}$

$$\lambda^i(\hat{q}_t^j, d_t^j) = \hat{q}_{t+1}^j$$

*/ A class of learning behaviors "between" defensive and probability matching are also possible. It would be interesting to investigate such cases.

**/ $\underline{1}$ denotes the vector all of whose components are unity.

satisfies the monotonicity and consistency conditions above. Here $d_t^j = (\beta_t^j(\rho^j(b_t), \rho^j(b_t)))$, i.e., trader j observes the bids he receives and his response to them.

(Since β_t^j is not permitted to depend on the identity of the consumers making bids, we may regard $\rho^j(b_t)$ as a representative element of the equivalence class consisting of all bids received which are obtained from $\rho^j(b_t)$ by permutation of the names of consumers.)

The learning behavior of traders is not significant in the models considered here, because only the restriction of φ^j by Assumption II will play a role. This restriction is to the effect that any sufficiently profitable array of bids received has a positive probability of being accepted.

What is the long-run behavior of this process? First, if the consumers use the probability matching type of learning, depending on the specific properties of λ^i, $i \in \mathcal{C}$ (and λ^j for $j \in \mathcal{J}$) the process may or may not converge. However, it is clearly possible that the process have recurring states in which the trader(s) earn monopoly profit. This is suggested by Figure 3. Given the four utility functions U^i $i = 1, 2, 3, 4$ of consumers, and the points a, b, c, d, it is possible to find probabilities $\bar{p}(a)$, $\bar{p}(b)$, $\bar{p}(c)$, $\bar{p}(d)$ such that

$$\gamma^1(a) = \gamma^1(c) \qquad \gamma^2(c) \geqq 0 \qquad \gamma^3(b) = \gamma^3(d) \qquad \gamma^4(d) \geqq 0.\ ^{*/}$$

Suppose the (sole) trader uses these values $\bar{p}(\cdot)$ as his acceptance probabilities. Suppose further that all consumers meet the trader in each period. Then each

*/ $\gamma^1(a)$ denotes $\gamma^1(p_a)$ where $p_a(a) = 1$, $p_a(x) = 0$ if $x \neq a$.

consumer will eventually learn the objective probabilities $\bar{p}$, i.e., $\hat{p}^1(a) = \bar{p}(a)$ $\hat{p}^1(c) = \bar{p}(c)$, etc. Consequently $q^1(a) > 0$ $q^1(c) > 0$ $q^2(c) > 0$ $q^3(b) > 0$ $q^3(d) > 0$ and $q^5(d) > 0$.

It is not difficult to construct a numerical example in which the average profit of the trader is strictly greater than $2(c + d)$. If there is more than one trader, the acceptance probabilities facing each consumer are a mixture of the acceptance probabilities of traders, the same for each consumer . From their point of view it is as if there is one trader.

Consider next the case of defensive learning by consumers.

Given the current array of bids of consumers, $b_t = (b_t^1,\ldots,b_t^n)$ and the current matrix of meetings M_t, the behavior of traders determines an array of bids accepted. Call this array the current trades and denote it by $y_t = (y_t^1,\ldots,y_t^n)$,

where
$$y_t^i = \begin{cases} b_t^i & \text{if there exists } j \text{ such that } i \in \mathcal{C}_j(t) \text{ and } b_t^i \text{ is accepted by } j. \\ 0 & \text{otherwise} \end{cases}$$

<u>Definition</u>: Let z and x and w be allocations. We say that <u>z K-dominates x from w</u> if there exists a subset $\{i_1, \ldots i_r\} \in \mathcal{C}$ where $r \leqq K$, such that

$$\sum_{\nu=1}^{r} z^{i_\nu} \leqq \sum_{\nu=1}^{r} w^{i_\nu}$$

and

$$U^{i_\nu}(z^{i_\nu}) \geqq U^{i_\nu}(x^{i_\nu}) \quad \text{for each} \quad \nu \in \{i, \ldots, r\}$$

with strict inequality for at least one value of ν. Let

$y_1 = x - w$ and $y_2 = z - w$ we say that y_2 K-dominates y_1 from w if z K-dominates x from w.

<u>Proposition 1.</u> Under Assumptions I (defensive learning) and II, with $c^j = 0$ for $j \in \mathcal{J}$, let $\{s_t : t \geqq 0\}$ be a sequence of states and let

$$q_t = \varphi(s_t) \quad t \geqq 0$$

be the corresponding bidding distributions of consumers. Suppose that for some $t \geqq 0$ there are two distinct allocations x and z such that (i) the bids $y_1 = x - w$ and $y_2 = z - w$ are each in the support of q_t and (ii) y_2 K-dominates y_1 from w, <u>then</u> s_t is a transient state.

<u>Proof of Proposition 1.</u> To show that s_t is a transient state it suffices to show that for some consumer i and for some trade y^i and time $t' > t$, $\hat{p}^i_{t'}(y^i) \neq \hat{p}^i_t(y^i)$. This suffices because $\hat{p}^i$ is monotone.

Since y_2 K-dominates y_1, there exists a subset $\{i_1, \ldots, i_r\}$ of consumers such that

$$\sum_{\nu=1}^{r} y_2^{i_\nu} \leqq 0 \quad \text{and} \quad U^{i_\nu}(y_2^{i_\nu}) \geqq U^{i_\nu}(y_1^{i_\nu}) \quad . \quad \forall\, \nu \in \{1,\ldots,r\}$$

and $y_2^{i_\nu}$ is individually feasible for i_ν from w.

Since $r \leqq K$, the probability that

$\{i_1, \ldots, i_r\} \subset \mathcal{C}_j(\tau)$ is a positive constant (independent of τ) for each j.

If $\hat{p}^i_\tau \neq \hat{p}^i_t$ for some i and $\tau > t$, then s_t is transient. Hence consider the case $\hat{p}^i_\tau = \hat{p}^i_t$ for all i and for all $\tau \geqq t$. But that implies that $\hat{p}^i_\tau$ is constant for $\tau \geqq t$. It follows that $q^i_\tau = \varphi^i(\hat{p}^i_\tau) = \varphi^i(\hat{p}^i_t) = q^i_t$ for all $\tau \geqq t$, and for all i. In particular $q^{i_\nu}_\tau = q^{i_\nu}_t$ for $\nu \in \{1, \ldots, r\}$ and all $\tau \geqq t$.

By hypothesis $y_2^{i_\nu} \in \operatorname{supp} q^{i_\nu}_t = \operatorname{supp} q^{i_\nu}_\tau$ for all $\tau \geqq t$. Hence the probability that $b^{i_\nu}(\tau)$ equals $y_2^{i_\nu}$ is a positive number, the same for all $\tau \geqq t$. Hence, the probability that both $b^{i_\nu}(\tau)$ equals $y_2^{i_\nu}$ and that $\{i_1, \ldots, i_r\} \subset \mathcal{C}_j(\tau)$ for a given $j \in \mathcal{J}$, is the product of two positive constants (these events are independent) and hence is also a positive number the same for all $\tau \geqq t$.

Under Assumption II, the probability that the combination of bids $(y_2^{i_1}, \ldots, y_2^{i_r})$ will be accepted by j given that it is received by him as a positive number bounded away from 0 . It follows that the waiting time until the bid containing $(y_2^{i_1}, \ldots, y_2^{i_r})$ is made to and accepted by the given trader j is finite with probability 1 , since the probability of that event at any one time is bounded from below by a positive constant.

Therefore with probability 1 there exists some $t' > t$ such that $\hat{p}_{t'}^{i_\nu} \neq \hat{p}_t^{i_\nu}$ for $\nu \in \{1, \ldots, r\}$ and hence $s_{t'} \neq s_t$.

Under Assumption I , because of monotonicity, supp $q_\tau^{i_\nu} \subsetneq$ supp $q_t^{i_\nu}$ for all $\tau \geqq t' > t$. Hence, the state s_t never recurs.

<u>Definition</u>. The K-core from w is the set of feasible allocations which are not K-dominated from w .

<u>Proposition 2.</u> The set of states s such that the corresponding bidding distributions $q = \varphi(s)$ are based on allocations in the K-core from ω is an absorbing set.

<u>Proof of Proposition 2.</u> Suppose s is a state such that $q = \varphi(s)$ is based on an allocation $x \in$ K-core from ω . In order to leave s, there must exist a set of consumers $\{i_1, \ldots, i_r\}$ with $r \leqq K$, and a set

of trades $y^{i_1}, \ldots, y^{i_r}$ such that $\sum_{\nu=1}^{r} y^{i_\nu} \leqq 0$ and such that $y^{i_\nu} \in \operatorname{supp} q^{i_\nu}$. But since $\operatorname{supp} q^{i_\nu} = C^{i_\nu}(x^{i_\nu})$, $U^{i}(y^{i_\nu}) \geq U^{i}(x^{i_\nu})$ for $\nu = 1,\ldots,r$. Hence the allocation z, where

$$z^i = \begin{cases} \omega^i + y^i & \text{if } i \in \{i_1,\ldots, i_r\} \\ \omega^i & \text{if } i \notin \{i_1,\ldots, i_r\} \end{cases},$$

K-dominates x from ω, which contradicts the hypothesis that $x \in$ K-core from ω.

Furthermore, under the assumptions on preferences of consumers used in [2], we conjecture that the reasoning used in [2] can be employed to show that this process actually converges to its set of absorbing states; namely, those based on the K-core from ω.

When the cost of trading is positive, i.e. $c^j > 0$ for $j \in \mathcal{J}$, trader j will accept only such combinations of bids that yield at least $c^j \cdot \underline{1}$ profit. This is based on costs being costs per period rather than costs per transaction. [If the costs are related to the number of transactions, say by, $c^j \cdot \|\{i \in \mathcal{C}_j(t)) \mid b_t^i$ is accepted$\}\|$, then accepted bids must satisfy the condition

$$\Sigma y_t^i \leqq - c^j \ \| i \in \mathcal{C}_j(t) \mid y_t^i \neq 0\}\| \cdot \underline{1}$$

hence c^j is the cost per transaction].

The arguments used in the proofs of Propositions 1 and 2 can, with appropriate modification, be applied to the case where $c^j \geqq 0$ for $j \in \mathcal{J}$ to establish analogous results. In essence, let s be a state such that the bidding distribution of consumers is based on a point from which an r-lateral trade, $r \leq K$, is possible which yields at least $- c^j \cdot \underline{1}$, where $\underline{c} = \min_{j \in \mathcal{J}} c^j$, the analogue of Proposition 1 states that s is a transient state.

Similarly, in place of Proposition 2 is a result to the effect that states which do not permit the type of improvement just described form an absorbing class.

The quantities c^j, $j \in \mathcal{J}$ can be interpreted in another way. Each trader may regard himself as having monopoly power, i.e., the power to misrepresent to consumers the "natural" terms of trade. One way of doing this is for j to choose a value $c^j > 0$ and accept with non-zero probability only combinations of bids that yield a profit at least $(- c^j \cdot \underline{1})$.

It is easy to see that when $\mathcal{J}$ consists of more than one trader and when consumers bid defensively, if for some $j \in \mathcal{T}$, $c^j > c = \min_{j \in \mathcal{T}} c^j$, then the average profit per period of trader j will fall to zero. This follows from Proposition 1 in the case $c^j \neq 0$, because states such that profits of any trader are more than c are transient.

Suppose that the number of traders is fixed, and that the c^j, $j \in \mathcal{T}$ are chosen once and for all at the beginning of trading. Then, the long-run behavior of this process will be one in which all traders j whose $c^j = c$, the minimum profit level, will share equally, on average, in the monopoly profits thereby determined, while those traders with $c^j > 0$ will get no shares.

If we imagine that the choice of c^j can be modified, then "competitive pressure" exists tending to drive c^j and hence c to the level of the minimum cost of being a trader, say c^*. However, with a fixed number of traders, many stable situations can exist in which all traders choose $c^j = c \geqq c^*$. We suppose that if the number of traders increases relative to the number of consumers, then while each period in which a transactions takes place yields the transacting trader a profit of $- c^j \cdot \underline{1}$, the average profit per period tends to zero as the number of traders increases.

It would be useful to explore the consequences of different structures of information (representing different institutional arrangements in the market). Public information about transactions or about bids and transactions might be made available to consumers and traders. For example, market reports similar to the stock exchange reports published in newspapers might be made available. One interesting question is whether making additional information available to consumers can by itself overcome the effects of anonymous trading and insufficient capacity of individual traders in restricting the optimality of the allocations achievable by the process to the K-core. (Allowing supplementary trading among traders is another interesting possibility.)

It appears to be the case that in general if $K < n$ this process cannot be guaranteed to achieve core allocations. In general multilateral trades involving all n consumers are needed to achieve core allocations. Furthermore, this situation is not improved by having a large number of traders. Only if the capacity of at least one trader exceeds the number of consumers can we be sure that the process does not get stuck at states such that the (joint) bidding distribution of consumers is based on an allocation not in the core.

Even if more information is made available to consumers, for example, if the structure of observation is such that consumers observe all transactions that take place (or all bids and transactions, for that matter), we cannot be sure that the process does not get stuck at a non-core state. Based on the greater information consumers may simultaneously aspire to utility levels which cannot be achieved as a result of K-lateral trades which yield a non-negative arbitrage profit. It appears to be essential that there be an institution which permits n-lateral trades to be achieved via several simultaneous K-lateral ones. This is, of course, what trading at constant known prices permits. */

Another approach to studying this process, though in a different spirit, would be to set it up as a game in which the learning functions or perhaps the mappings φ^i are strategies of the players and to study the Nash equilibria of that game.

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*/ Vernon Smith pointed out to me that the international gold market is similar to the process presented here. In that market the institution which permits full multilaterial clearing to be achieved with traders who deal with only part of the market is a final stage consisting of a tatonnement among the dealers.

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