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# A MODEL OF TECHNOLOGY WITH INNOVATION

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This paper presents a model of technology encompassing innovation either in the form of new products or of new methods. The intent is to provide a framework within which technological change can be more adequately represented and from which the familiar representations of technology used in economic models can be derived. Technological change or innovation is increasingly regarded as an endogenous economic process. It should therefore be useful to have a model of technology within which the processes of technological change can be explicitly expressed.

Production activities transform material substances. A fully detailed description of such an activity would in general involve specifying all the properties, including location in time and space, of the substances or entities involved, perhaps an infinite description. A production activity generally operates on an aggregation of substances, transforming them into another aggregation in a way which depends on the properties of the substances and on the methods of combining them, or acting on them.

Let $\mathcal{X}$ denote the space of all fully described aggregates of substances. The points of $\mathcal{X}$ represent the elementary entities, aggregates of fully described substances, that could be involved in production. Points of $\mathcal{X}$ will be called <u>fully described bundles</u> or briefly, <u>bundles</u>.

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Let $\mathcal{M}$ be the space of <u>fully</u> <u>described</u> <u>methods</u> of production. An element $m \in \mathcal{M}$ is interpreted as specifying in complete detail a sequence of actions to be performed in order to transform an element of $\mathcal{X}$ into another one.

The fully specified technology--Nature's laws--is given by a mapping

$$T : \mathcal{X} \times \mathcal{M} \to \mathcal{X} \quad .$$

For each $x \in \mathcal{X}$ and $m \in \mathcal{M}$, $T(x,m) = y$ is the bundle that results from applying the method $m$ to the bundle $x$.

In general, a method $m \in \mathcal{M}$ is applicable to some collection $X(m) \subset \mathcal{X}$. However, we can take $X(m) = \mathcal{X}$ for each $m \in \mathcal{M}$ as follows. Suppose $X(m) \neq \mathcal{X}$, then for $x \in \mathcal{X} - X(m)$ define $T(x,m) = x$. In this way $m$ is formally applicable to all of $\mathcal{X}$.

I make the following assumptions about the fully described technology.

Assumption I. $\mathcal{X}$ is a complete, normed linear space. $\underset{\sim}{\mathcal{X}} \subset 2^{\mathcal{X}}$ is the class of measurable (Borel) subsets of $\mathcal{X}$; $\mu$ is a measure on $\mathcal{X}$, so that $(\mathcal{X}, \underset{\sim}{\mathcal{X}}, \mu)$ is a measure space. Let $|\cdot|$ denote the norm in $\mathcal{X}$.

The assumption that $\mathcal{X}$ is a linear space is used only in studying the relation between the underlying technology and the conventional production set and production function properties. It would be possible to omit any algebraic structure in $\mathcal{X}$ or to impose only a weaker group operation, say, addition, reflecting the fact that bundles can be aggregated to form 'larger' bundles. The norm in $\mathcal{X}$ reflects the assumption that the properties of substances are fully quantified. The assumption regarding measure will be discussed below.

Assumption II 1) There is a method $o \in \mathcal{M}$ such that $T(x,o) = x$ for all $x \in \mathcal{X}$.

2) $\mathcal{M}$ is closed under composition. I.e., there is a binary operation $\cdot$ defined on $\mathcal{M}$ such that $m^1 \in \mathcal{M}$, $m^2 \in \mathcal{M}$ implies

$m^1 \cdot m^2 \in \mathcal{M}$.

The method $m^1 \cdot m^2$ is interpreted as applying the method $m^1$ to the bundle that results from applying $m^2$ to a bundle. Thus,

$$T(x,m^1 \cdot m^2) = T(T(x,m^2),m^1)$$

3) Let $\underset{\sim}{\mathcal{M}}$ denote the class of measurable subsets of $\mathcal{M}$ and let $\nu$ be a measure on $\mathcal{M}$. Thus, $(\mathcal{M},\underset{\sim}{\mathcal{M}},\nu)$ is a measure space.

<u>Definition 1</u> 1) For $m \in \mathcal{M}$, let $g(m) = \{(x,y) \in \mathcal{X} \times \mathcal{X} \mid T(x,m) = y\}$ ; $g(m)$ is the graph of $m$ in $\mathcal{X} \times \mathcal{X}$ .

2) Let the subsets of $\mathcal{X} \times \mathcal{X}$ have the metric topology of closed convergence based on the norm in $\mathcal{X}$. Let $\rho$ denote that metric. Define the distance between method $m'$ and $m''$ in $\mathcal{M}$ to be

$$\rho(m'.m'') = \rho(g(m'),g(m'')) .$$

Thus, given $\varepsilon > 0$, the $\varepsilon$-neighborhood $\mathcal{N}_\varepsilon(m)$ of $m \in \mathcal{M}$ consists of all methods $m'$ whose graphs are within $\varepsilon$ of the the graph of $m$ . This says that if $(x,y)$ is an input-output pair producible by the method $m$ and if $m^1$ is within $\varepsilon$ of $m$, then there is an input-output pair $(x^1, y^1)$ such that $|(x,y) - (x^1,y^1)| < \varepsilon$, where the norm $|\cdot|$ here denotes the extension of $|\cdot|$ in $\mathcal{X}$ to $\mathcal{X} \times \mathcal{X}$.

<u>Assumption III</u> T is jointly continuous on $\mathcal{X} \times \mathcal{M}$ (and hence measurable).

Let $g(T) = \{(x,m,y) \in \mathcal{X} \times \mathcal{M} \times \mathcal{Y} \mid T(x,m) = y\}$. Let A be a subset of the product space $Z_1 \times \ldots \times Z_n$. Then $P_{r_{i_1, i_2, \ldots, i_n}}(A)$

denotes the projection of A into the factors $Z_{i_1} \times Z_{i_2} \times, \ldots, \times Z_{i_1}$ $i_j \in \{1, \ldots, r\}$ for $j = 1, \ldots, k$.

Assumption III 2) $P_{r_{12}} g(T) = \mathcal{X} \times \mathcal{M}$. I.e., the domain of T is all of $\mathcal{X} \times \mathcal{M}$. And, $P_{r_3}(g(T)) \subset \mathcal{X}$. I.e. not every bundle is producible.

The mapping T represents the 'true' underlying technology which is for the most part unknown. However, known or not, every act of production is ultimately described as a point of g(T), however it may be perceived. The set of productions which have been experienced is a certain subset $\gamma \subset g(T)$. Generally, $\gamma$ depends on the history of production and hence changes over time, $\gamma_t$ being the set of fully described productions that have been experienced prior to time t.

Let

$$\mathcal{X}(\gamma) = P_{r_1}(\gamma) \cup P_{r_3}(\gamma)$$

and let

$$\mathcal{M}(\gamma) = P_{r_2}(\gamma) .$$

Technological Information

The information or knowledge about technology accessible to an economic agent at any time $t$ is limited. First, the perception of bundles is limited. Only some properties of substances are recognized at all, and only relatively coarse discriminations among bundles is possible. The same is true of methods of production. Those discriminations are made in part on the basis of existing scientific knowledge and observational processes and on the basis of knowledge and experience of production. Beyond this, at a particular time the economy operates with a particular set of commodities, which amounts to a classification of substances into equivalence classes. Similarly, there is a set of production methods describing production in terms of the given classification of commodities.

Of course, the discriminations possible on the basis of scientific knowledge and observational technique go somewhat beyond existing technological knowledge. This situation may be formalized as follows. Since we refer to the state of affairs at a particular moment of time, the subscript $t$ is the same for all entities depending on time and can be omitted.

The commodity space at the moment of time under consideration is $\mathbb{R}^{\ell}$, the euclidean space of $\ell$ dimensions. The classification of fully described bundles into commodity vectors is given by a function

$$\xi : \mathcal{X} \to \mathbb{R}^{\ell} .$$

However, special relevance attaches to the classification of substances within the range of economic experience, i.e., to the restriction of $\xi$ to $\mathcal{X}(\gamma)$. As $\underline{a}$ varies over $\mathbb{R}^{\ell}$, the sets

$$\xi_{\gamma}^{-1}(\underline{a}) = \{x \in \mathcal{X} \mid \xi(x) = a\} \cap \mathcal{X}(\gamma)$$

form a partition of $\mathcal{X}(\gamma)$.

Denote by $\frac{\mathcal{X}(\gamma)}{\xi}$ the space whose elements are these equivalence classes. Similarly, current knowledge of production methods is represented by a function

$$\eta : \mathcal{M} \to \eta(\mathcal{M})$$

where $\eta(\mathcal{M})$ is a (measurable) space of descriptions of production methods.

The sets

$$\eta_{\gamma}^{-1}(z) = \{m \in \mathcal{M} \mid \eta(\mathcal{M}) = z\} \cap \mathcal{M}(\gamma)$$

for $z \in \eta(\mathcal{M})$, form a partition of $\mathcal{M}(\gamma)$; these sets are elements of the quotient space $\frac{\mathcal{M}(\gamma)}{\eta}$.

Assumption IV. $\xi$ and $\eta$ are measurable.

Hence the sets $\xi_{\gamma}^{-1}(\underline{a})$ and $\eta_{\gamma}^{-1}(z)$ for $\underline{a} \in \mathbb{R}^{\ell}$ and $z \in \eta(\mathcal{M})$ are elements of $\underset{\sim}{\mathcal{M}}$ and $\underset{\sim}{\mathcal{X}}$, respectively; hence also $\underset{\sim}{\mathcal{M}}(\gamma)$ and $\underset{\sim}{\mathcal{X}}(\lambda)$, respectively. Next we define $T : \underset{\sim}{\mathcal{X}} \times \underset{\sim}{\mathcal{M}} \to \mathcal{P}(\mathcal{X})$ where $\mathcal{P}(\mathcal{X})$ is the set of probability measures on $(\mathcal{X}, \underset{\sim}{\mathcal{X}})$. If $X \in \underset{\sim}{\mathcal{X}}$ and $M \in \underset{\sim}{\mathcal{M}}$, then for $y \in \underset{\sim}{\mathcal{X}}$

$$T(X,M)(Y) = \frac{\int_M \mu\{x \in \mathcal{X} \mid T(x,m) \in Y\} \, d\nu(m)}{\mu(X) \cdot \nu(M)}$$

i.e. $T(X,M)(Y)$ is the conditional probability that the output bundle $T(x,m)$ is an element of $Y$ given that $(x,m)$ is in $(X \times M)$ .

We write $\lambda(Y|X,M) = T(X,M)(Y)$ for $(X,M) \in \underset{\sim}{\mathcal{X}} \times \underset{\sim}{\mathcal{M}}$ and $Y \in \underset{\sim}{\mathcal{X}}$

The interpretation of $T(X,M)$ is that if the production action $X,M$ is attempted, Nature chooses a fully described action $(x,m) \in X \times M$ , according to the probability measures $\mu$ and $\nu$ , and the resulting production is $T(x,m) = y$ . Thus, choice of actions $X , M$ determines a probability measure on the output of production and hence on events $Y$ in $\underset{\sim}{\mathcal{X}}$ .

<u>Definition</u> A pair $(X,M) \in \frac{\mathcal{X}(\gamma)}{\xi} \times \frac{\mathcal{M}(\gamma)}{\eta}$ , called a (<u>perceived</u>) <u>production process</u> ,or <u>process</u>, is <u>developed relative to $\gamma, \xi, \eta$</u> if and only if there exists $Y \in \frac{\mathcal{X}(\gamma)}{\xi}$ such that

$$\lambda(Y|X,M) = 1 .$$

Thus a perceived process is developed relative to existing technological knowledge if its perceived output is certain.

<u>Assumption V</u> If $(X,M) \in \frac{\mathcal{X}(\gamma)}{\xi} \times \frac{\mathcal{M}(\gamma)}{\eta}$ then $(X,M)$ is developed relative to $\gamma$, $\xi$ , and $\eta$ .

This is a simplifying assumption made in order to avoid introducing another subset of $\mathcal{X} \times \mathcal{M}$ consisting of the developed processes.

Production Sets in the Commodity Space and Production Functions

Definition  Given $\gamma, \xi$, and $\eta$, for $M \in \frac{\mathcal{M}(\gamma)}{\eta}$ there is a collection $\underline{\bar{X}}(M)$ of elements $X \in \frac{\mathcal{X}(\gamma)}{\xi}$ ( a subset $\underline{\bar{X}}(M) \subset \frac{\mathcal{X}(\gamma)}{\xi}$ ) such that the process $X,M$ is developed, i.e., let

$$\Gamma = \{(X,M,Y) \in \frac{\mathcal{X}(\gamma)}{\xi} \times \frac{\mathcal{M}(\gamma)}{\eta} \times \frac{\mathcal{X}(\gamma)}{\xi} \mid (x,m,y) \in \gamma \text{ for each } (x,m,y) \in (X \times M \times Y)\} .$$

$\Gamma$ of course depends on $\gamma$ as well as $\xi$ and $\eta$.

Then $\underline{\bar{X}}(M) = \{X \in \frac{\mathcal{X}(\gamma)}{\xi} \mid (X,M,Y) \in \Gamma$ for some $Y \in \frac{\mathcal{X}(\gamma)}{\xi}\}$ .

(By Assumption V, $XM$ is then developed.)

To each $M \in \frac{\mathcal{M}(\gamma)}{\eta}$ we can associate a function

$$F_M: \xi(\underline{\bar{X}}(M)) \rightarrow \mathbb{R}^{\ell}$$

where, if $X \in \underline{\bar{X}}(M)$ and $\xi(X) = a \in \mathbb{R}^{\ell}$

$F_M(a) = \xi(Y) \in \mathbb{R}^{\ell}$, where $Y$ is the set such that $\lambda(Y \mid XM) = 1$ .

The function $F_M$ is the perceived production function corresponding to $M$ .

The <u>production set in the commodity</u> space is

$\{\underline{a} \in \mathbb{R}^{\ell} \mid \underline{a} = \xi(Y) - \xi(X)$ for some $X,Y \in \frac{\mathcal{X}(\gamma)}{\xi}$
for which there exists $M \in \frac{\mathcal{M}(\gamma)}{\eta}$ such that $(X,M,Y)$
is developed relative to $\gamma, \xi$ and $\eta$.$\}$

As was remarked above, existing information based on scientific knowledge and observational techniques goes beyond the information embodied in the existing technological structure given by $\gamma, \xi, \eta$. This may take the form of finer discriminations among substances, i.e., a finer partition in $\mathcal{X}$, or finer discrimination among methods in $\mathcal{M}$ or discriminations in $\mathcal{X}$ and $\mathcal{M}$ which go beyond the known domain of experience $\gamma$.

This situation can be represented as follows.

Let $\gamma^* \supset \gamma$, let $P$ and $Q$ be partitions of $\mathcal{X}(\gamma^*)$ and $\mathcal{M}(\gamma^*)$ respectively.

$P$ is a refinement of the partition determined by $\xi$ and $Q$ is a refinement of that determined by $\eta$. The partitions $P, Q,$ based on the more fundamental scientific observation, together with the domains $\mathcal{X}(\gamma^*)$ and $\mathcal{M}(\gamma^*)$ provide the basis for conjectured new processes or products. If a conjectured production process $(X^*, M^*, Y^*)$ is selected in $\frac{\mathcal{X}(\gamma^*)}{P} \times \frac{\mathcal{M}(\gamma^*)}{Q} \times \frac{\mathcal{X}(\gamma^*)}{P}$ the resulting probability measures $\lambda(\cdot \mid X^*, M^*)$ is generally such that

$$\lambda(Y \mid X^* M^*) < 1$$

for all $Y \in \frac{\mathcal{X}(\gamma^*)}{P}$. Such a conjectured process may not produce any existing commodity. Its outputs may be new substances, i.e., new combinations of properties not classified among the existing commodities. Moreoever, in general its outputs as expressed in terms of the classification of properties given by $P$ are not certain. Indeed, the distribution of outputs may be so diffuse that no orderly or coherent outcome results from an experimental attempt to carry out $(X^*, M^*)$. Such an experiment would result in a fiasco. However, if some recognizable outcome were to occur with sufficiently high probability, the theory, expressed in $P$ and $Q$, together with the availability of observations renders the experimental process $(X^*, M^*)$ suitable for development. If a process is developable then a sufficiently thorough development effort, perhaps a sufficiently long sequence of experiments, would result in a developed process. Thus, the two necessary properties of a candidate for development are that

1) the experimental acts of production $(X^*, M^*)$ can be carried out, and

2) the results are sufficiently coherent to yield some information of value to the experimenter. These are embodied in the following definition.

<u>Definition</u> A process $X^*, M^*, Y^* \in \frac{\mathcal{X}(\gamma^*)}{P} \times \frac{\mathcal{M}(\gamma^*)}{Q} \times \frac{\mathcal{X}(\gamma^*)}{P}$ is <u>developable</u> (relative to $\gamma, \xi, \eta$ and $\gamma^*, P, Q$ and $\varepsilon_1$, $\varepsilon_2$, and $\delta$) if and only if (i) there exists $(X,M) \in \frac{\mathcal{X}(\gamma)}{\xi} \times \frac{\mathcal{M}(\gamma)}{\eta}$ such that $|X - X^*| < \varepsilon_1$ and $\rho(M,M^*) < \varepsilon_2$ and (ii) $Y^* \in \frac{\mathcal{X}(\gamma^*)}{P}$ is such that

$$\lambda(Y^* \mid X^*, M^*) \geq \delta$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\delta > 0$ are given scalars.

Condition (1) of the definition will be satisfied if $\gamma^*$ is in the $\varepsilon$-neighborhood of $\gamma$ where $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$. The constraints $\varepsilon_1$ and $\varepsilon_2$ tell us how near the new process must be to existing experience in order to ensure that an experiment can actually be made with the conjectured $X^*$ and $M^*$, while $\delta$ gives the level of probability needed to insure that the experiment gives some reproducible result.

<u>Assumption VI.</u> If $(X^*, M^*, Y^*) \in \frac{\mathcal{X}(\gamma^*)}{P} \times \frac{\mathcal{M}(\gamma^*)}{Q} \times \frac{\mathcal{X}(\gamma^*)}{P}$ is developable, then (A) there exist partitions $P'$ and $Q'$ of $\mathcal{X}(\gamma^*)$ and $\mathcal{M}(\gamma^*)$ respectively, such that (i) $X^1 \subset X^*$ and $M^1 \subset M^*$, (ii) if $x^1 \notin X^*$ then $x = x^1 (\text{mod } P') \Leftrightarrow x = x^1 (\text{mod } P)$ and if $m \notin M^*$ then $m = m' \ (\text{mod } Q') \Leftrightarrow m = m' (\text{mod } Q)$ and (iii) $X^1$ and $X^* \setminus X^1$ are elements of $P'$ and

$M^1$ and $M^* \setminus M^1$ are elements of $Q^1$.

Thus, $P^1$ and $Q^1$ are refinements of $P$ and $Q$ respectively, such that the sets $X^*$ and $M^*$ are split so that $X^* = (X^1 \cup X^* \setminus X^1)$ and $M^* = (M^1 \cup M^* \setminus M^1)$, while $P^1$ and $P$ are the same elsewhere, as are $Q^1$ and $Q$.

And

(B) $\lambda(Y^1 \mid X^1, M^1) = 1$

It is possible that $(X^*, M^*, Y^*)$ and $(X^*, M^*, Y^{**})$ are both developable. In that case there may exist $(X^1, M^1, Y^1)$ and $(X^{11}, M^{11}, Y^{11})$ satisfying Assumption VI.

Once a new process $(X^1, M^1, Y^1)$ is developed it can become part of the (new) known technology. Thus, if $\gamma = \gamma_t$, $\xi = \xi_t$, $\eta = \eta_t$, then $\gamma_{t+1} = \gamma_t \cup (X^1, M^1, Y^1)$ and $\xi_{t+1}$ is a representative of $P^1$ in a Euclidean space $R^{\ell+k}$ where $k$ is the number of new commodities introduced by the process $(X^1, M^1, Y^1)$, and $\eta_{t+1}$ is the representation of $Q^1$ in the space of descriptions of known methods. Corresponding to the change in the known technology the production set and production function representations also change, as described above.

In this model, the existing knowledge of technology is represented by $\gamma, \xi, \eta$, and the conjectures regarding possible new technological innovations by $\gamma^*, P$ and $Q$. As we have seen, the development process, starting from a developable process

$$(X^o, M^o, Y^o) \in \frac{\chi(\gamma^*)}{P} \times \frac{m(\gamma^*)}{Q} \times \frac{\chi(\gamma^*)}{P}$$

(representing a conjectured new process or product), and the corresponding measure $\lambda(\cdot \mid X^o, M^o)$ goes through a sequence of experiments

$(X_1 M_1 Y_1) \rightarrow \ldots \rightarrow (X_k, M_k, Y_k) = (X', M', Y')$ when $X^o = X_1$ and $M^o = M_1$ and where $\lambda(Y' \mid X', M') = 1$.

Suppose for simplicity that the initially known technology $\gamma^*, P, Q$ is known to everyone, and suppose that some agent, I, has carried out a development process $\theta$, resulting in a new domain of experience $\gamma^1$, and new partition $P'$ and $Q'$ reflecting new knowledge of products and methods. In the first instance, this new knowledge accrues only to I. Everyone else's knowledge is given by $\gamma^*, P, Q$. I's new knowledge can be acquired by others only as a result of some process of communication. That process may be a simple one, such as publication of a description of the new technology or it may be a complex and costly one, such as might be the case when the new knowledge can be acquired effectively only through direct observation and experience. Thus, after I's innovation technological knowledge becomes dispersed, and all the more so if several innovators are in action at the same time. Then an adequate description of the state of technological information would require specifying who knows what. I.e., there would be a triple $\gamma_i, P_i, Q_i$ for each agent i in the economy.

Thus, even though technological knowledge is a public good in the sense that its use by one agent does not reduce the amount of it available, technological knowledge, especially new knowledge, can be privately held and traded among the agents.

An innovation, such as I's, alters the course of economic history. The newly developed processes, resulting from $\theta$, make developable other processes which were not developable before that. The full social value, ex post, of the development $\theta$ is in principle the difference between the value of the economic history following $\theta$ and the history that would have occured without $\theta$. How much of this value accrues to I depends on the institutions governing trading of technological information, e.g., patent laws, as well as on the structure of the economy generally, and, of course, on the effects of uncertainty since the trading is done ex ante rather than ex post.

Thus, when a potential developer I contemplates undertaking the development of a process $(X^o,M^o,Y^o)$ he contemplates a random variable $\theta$, whose terminal experiment(s) $(X',M',P')$ is also a random variable. The knowledge generated by $\theta$ has a market value as technological knowledge. We shall suppose that a potential developed of $(X^o,M^o,Y^o)$ can form his own estimate of the value that would accrue to him if the actual sequence of experiments starting from $(X^o,M^o,Y^o)$ turned out to be $\theta$.

We postulate the function

$$V: \frac{\mathcal{X}(Y^*)}{P} \times \frac{M(Y^*)}{Q} \times \frac{\mathcal{X}(Y^*)}{P} \times \Theta \to \mathbb{R}$$

whose value at $(X^o,M^o,Y^o,\theta)$,

$$V(X^o,M^o,Y^o,\theta)$$

is the value of the innovation resulting from starting the development process at $(X^o,M^o,Y^o)$ and pursuing the sequence of experiments determined by $\theta$ to the end. This is of course a random variable whose distribution depends on the distribution of sequences $(X_1,M_1,Y_1 \to \cdots \to (X_k M_k Y_k)$ where

$X_1 = X^o$ and $M_1 = M^o$ and

$$\lambda(Y_k \mid X_k M_k) = 1.$$

Furthermore, the cost of carrying out that sequence of experiments can be assumed to be known given the sequence. Thus, the random variable

$$\mathcal{C} : \frac{\mathcal{X}(\gamma^*)}{P} \times \frac{\mathcal{M}(\gamma^*)}{Q} \times \frac{\mathcal{X}(\gamma^*)}{P} \times \Theta \longrightarrow R$$

where $\mathcal{C}(X^\circ, M^\circ, Y^\circ, \theta)$ is the cost of performing the sequence of experiments, $\theta$ , beginning from $(X^\circ, M^\circ, Y^\circ)$. Then

$$W(X^\circ, M^\circ, Y^\circ; \theta) = V(X^\circ, M^\circ, Y^\circ, \theta) - \mathcal{C}(X^\circ, M^\circ, Y^\circ, \theta)$$

is the net present value of starting a development process $\theta$ with a developable process $(X^\circ, M^\circ, Y^\circ)$ . The expected value $\mathcal{E}W$ of this random variable is a function whose value $\mathcal{E}W(X^\circ M^\circ Y^\circ)$ is defined at every developable project $X^\circ Y^\circ M^\circ$ in $\frac{\mathcal{X}(\gamma^*)}{P} \times \frac{\mathcal{M}(\gamma^*)}{Q} \times \frac{\mathcal{X}(\gamma^*)}{P}$, given $\gamma^*$, P and Q .

This function discriminates among directions in $\frac{\mathcal{X}(\gamma^*)}{P} \times \frac{\mathcal{M}(\gamma^*)}{Q} \times \frac{\mathcal{X}(\gamma^*)}{P}$, since it attaches a possibly different expected value to each point of that space. Hence it is capable of guiding the choice of processes to be developed. Specification of the probability distribution over development sequences, $\theta$ , would lead to a problem of decision making under uncertainty whose solution would be the set of developable processes whose development would be undertaken. I will not here go into this further.

Properties of the Production Set in the Commodity Space

In general equilibrium models existing technological possibilities are represented by the (perceived) production set in the commodity space. Properties such as convexity and constant or decreasing returns to scale are frequently assumed. What properties of the underlying technology correspond to these properties of the perceived production set? Let

$$S \subset \mathbb{R}^{\ell}$$

denote the perceived production set. Then

$$S = \{\underline{r} \in \mathbb{R}^{\ell} \mid r = \xi(Y) - \xi(X) \text{ , where } Y = T(X,M) \text{ for some}$$
$$(X,M,Y) \in (\frac{\mathcal{X}(Y)}{\xi} \times \frac{\mathcal{M}(Y)}{\eta} \times \frac{\mathcal{X}(Y)}{\xi}) \} .$$

$Y$ affords constant returns to scale if $v \in S$ and $\lambda \geq 0$ imply $\lambda \cdot v \in S$ .

Now, $v \in S$ implies that there exists a perceived production method $M \in \frac{\mathcal{M}(Y)}{\eta}$ and perceived inputs $X \in \frac{\mathcal{X}(Y)}{\xi}$ such that

$T(X,M) = Y$ , where $(X,M,Y) \in {}^{Y}/_{(\xi,\eta)}$ and $\xi(Y) = r$ . Now, $\lambda v \in S$ ,

for $\lambda \geq 0$ , if and only if there exists $(X^1,M^1,Y^1) \in \frac{Y}{(\xi,\eta)}$ such that

$$\lambda v = \xi(Y^1) - \xi(X^1)$$

where

$$v = \xi(Y) - \xi(X) \quad .$$

Thus, $\lambda v = \lambda\xi(Y) - \lambda\xi(X)$ and hence,

$$\lambda\xi(Y) - \lambda\xi(X) = \xi(Y^1) - \xi(X^1) \quad .$$

Now, if $\xi$ is homogeneous of degree 1 */ , (using the assumption that $\mathcal{X}$ is a linear space) ,

$$\lambda\xi(Y) - \lambda\xi(X) = \xi(\lambda Y) - \xi(\lambda X)$$

Now $\xi(\lambda Y) - \xi(\lambda X) \in S$ if and only if there exists $M' \in \frac{\mathcal{M}(Y)}{\eta}$ ,

such that

$$(\lambda X, M^1, \lambda Y) \in \frac{Y}{(\xi, \eta)}$$

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*/ If $\xi$ is not homogeneous of degree 1 it is still possible that T might have an offsetting homogeneity, e.g., if $\xi(x) = x_1 + x_2^2$ , where $x = (x_1, x_2, \ldots)$ and $T(X, M^\circ)$ is such that $T(\lambda X, M^\circ) = (\lambda Y_1, \lambda^{\frac{1}{2}} Y_2)$, $\lambda \geq 0$ , then $\xi \cdot T(\cdot M)$ is homogeneous of degree 1 in $x$ .

such that

$$T(\lambda X, M^1) \in \lambda Y .$$

This would be the case if, for example, the method M afforded constant returns, i.e., if

$$(X,M,Y) \in \frac{\gamma}{(\xi,\eta)} \Rightarrow (\lambda X, M, \lambda Y) \in \frac{\gamma}{(\xi,\eta)}$$

for all $\lambda \geq 0$ . However, this condition is not necessary.

If $\gamma$ is bounded, as would be the case if $\gamma$ consists only of productions actually experienced, then S cannot yield constant returns to scale. At most there would exist some $\bar{\lambda} > 0$ such that

$$v \in S \text{ and } \lambda \in [0,\bar{\lambda}] \text{ implies } \lambda v \in S .$$

Convexity of S is another property of interest. S is convex if $v'$ and $v''$ in S imply

$$\lambda v' + (1-\lambda)v'' \in S \quad \text{for } \lambda \in [0,1] .$$

But $v' \in S$ if and only if there exists $(X',M',Y') \in \frac{\gamma}{(\xi,\eta)}$ such that

$$v' = \xi(Y') - \xi(X')$$

Similarly, $v'' \in S$ if there exists $(X'',M'',Y'') \in \frac{\gamma}{(\xi,\eta)}$ such that

$$= \xi(Y'') - \xi(X'') .$$

If $M'$ and $M''$ afford decreasing returns to scale, i.e., if

$$(X',M',Y') \in \frac{\gamma}{(\xi,\eta)} \Rightarrow (\lambda X',M',\lambda Y') \in \frac{\gamma}{\xi\eta} \quad \text{for } \lambda \in [0,1]$$

Then,

$$\lambda r' + (1-\lambda)r'' = \lambda[\xi(Y') - \xi(X')] + (1-\lambda)(\xi(Y'') - \xi(X'')]$$

$$= \xi(\lambda Y') - \xi(\lambda X') + \xi((1-\lambda)Y'') - \xi((1-\lambda)X'')$$

(using homogeneity of degree 1 of $\xi$ and linearity of $\mathcal{X}$ ).

A stronger property of the underlying technology sufficient for convexity is that T be quasi-concave in the following sense.

If $(x,m,y) \in \gamma$ and $(x',m',y') \in \gamma$, i.e.,

$$T(x,m) = y$$

and

$$T(x',m') = y' ,$$

Then, for $\lambda \in [0,1]$, there exists $m_\lambda \in \mathcal{M}$ such that

$$T(\lambda x + (1-\lambda)x', m_\lambda) = \lambda y + (1-\lambda)y' .$$

There may, of course, be $\bar{m} \in \mathcal{M}$ such that

$$T(\lambda x + (1-\lambda)x', \bar{m}) \geq T(\lambda x + (1-\lambda)x', m_\lambda$$

where $\geq$ may be understood in terms of the norm in $\mathcal{X}$ and as the partial pre-ordering on $\mathcal{X}$ induced by the vectorial pre-ordering in $\mathbb{R}^\ell$, and the mapping $\xi$.

Technological Change in a Production Function Model

We have seen that to each perceived method $M \in \frac{\mathcal{M}}{\eta}$ we can associate a perceived production function $F_M$ as follows.

$$F_M(\xi(X)) = \xi[T(X,M)] \ .$$

Thus, $F_M$ maps perceived commodity vectors in $\mathbb{R}^\ell$ into vectors in $\mathbb{R}^\ell$ .

The simplest non-trivial case is that in which $F_M$ is linear, i.e. there is a constant matrix $\alpha(m)$ , depending on $M$ , such that

$$F_M(v) = \alpha(m) \cdot v$$

for all

$$v \in \xi(\mathcal{X}(\gamma)) \subset \mathbb{R}^\ell \ .$$

A special case of interest is that in which $\ell = 3$ , the first two commodities are capital and labor respectively , denoted K and L , and the third one is an output, whose quantity is denoted $Z$ . I shall assume that if

$(X,M,Y) \in \frac{\gamma}{(\xi,\eta)}$ , then

$$\xi(X) = (K,L,0)$$

and

$$\xi(Y) = (0,0,Z)$$

and consequently $F_M$ can be written in the form

$F_M(K,L) = Z$ .

Furthermore, assume that on $\gamma$ $F_M$ is perceived to have the form

$F_M(K,L) = \alpha(M) \cdot K \cdot L = \alpha(m) \cdot K \cdot L$ where $\alpha(m)$ is a scalar.
Thus, on $\gamma$ the production function is of the Cobb-Douglas type with exponents all equal to unity.

Technical Change

I. Suppose there is a point $(x',m',y') \in \gamma^* - \gamma$ , i.e. an untried process, and a perceived production process $(X',M,Y') \in \frac{\gamma^*}{(\xi,\eta)}$ which contains $(x',m',y')$ such that $\xi(X') = (K',L',0)$ and $\xi(Y') = (0,0,Z)$ , where $K' \cdot L' < K \cdot L$ . Then,

$$\frac{Z}{K'L'} = \frac{F_M(K',L')}{K' \cdot L'} > \alpha(M) = \frac{Z}{K \cdot L} .$$

Thus, if the process $(X',M,Y')$ were developed, it would be perceived as a change in the production coefficient $\alpha(M)$ and possibly attributed to a change in $M$ , though it might be entirely the result of the application of the given methods $M$ to a new input vector perceived as a smaller capital labor combination, hitherto unknown or untried.

II. A similar change in the perceived production function might result from the application of a newly perceived method $M' \in \frac{\mathcal{M}(\gamma^*)}{\eta}$ to an existing input combination perceived as a known capital labor combination. Thus, suppose

$$(X,M',Y') \in \frac{\gamma^*}{(\xi,\eta)} \quad \text{where} \quad (X,M',Y') \in \gamma^* - r$$

while $(x,m,y) \in \gamma$ and $(X,M,Y) \in \dfrac{\gamma}{(\xi,\eta)}$ .

Thus, $T(XM') = Y'$ and $T(XM) = Y$ and $\xi(Y') = Z' > Z = \xi(Y)$ .

Then $\dfrac{Z'}{(K\cdot L)} = \alpha(M') > \alpha(M) = \dfrac{Z}{K\cdot L}$

where $\xi(X) = (K,L,0)$ .