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CHARACTERIZATION OF THE PRIVATE ALTERNATIVE DOMAINS ADMITTING ARROW SOCIAL WELFARE FUNCTIONS

by

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Abstract

A n-person social choice problem is considered in which the alternatives are n dimensional vectors with the i^{th} component of such a vector being the part of the alternative effecting individual i alone. Assuming that individuals are selfish (i is indifferent between any two alternatives that have the same i^{th} component) we characterize all the families of permissible individual preferences that admit nondictatorial Arrow type social welfare functions. We also show that the existence of such a function for a given family of preferences is independent of n provided that it is greater than one.

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1. Introduction

A theory of aggregations of individual preferences into group preferences should be in the core of areas involving group decision making. As such, it is surprising that social choice theory is not used more extensively in economics, game theory team theory, etc. Clearly incorporation of some of the many models and important results of social choice theory into these other areas may have interesting implications. We feel that one of the reasons for this phenomenon is the lack of structure of the alternatives' space in most of the models discussed in social choice, while, for example in economic aggregation problems, aggregating preferences over allocations of private alternatives may differ significantly from aggregating preferences over allocations of public alternatives. One would expect that this structure would be taken into consideration. We hope that the model and results presented in this paper are a step in this direction.

In his book ([1] Chapter II), Arrow discusses the "difference between the ordering (done by an individual) of the social states according to direct consumption of the individual and the ordering when the individual adds his general standards of equity." He refers to the former ordering as reflecting his "tastes" and the latter as reflecting his "values." When one deals with a social choice theory whose states are economic consumption bundles, ordering according to tastes, or what is sometimes referred to as a selfishness axiom, becomes a dominant factor in the individual's preferences. This distinction is discussed by Samuelson in his book [6] where he makes this assumption (assumption 6 on page 224) for economic environments. Motivated by this observation Arrow goes on to prove his "Possibility (impossibility) Theorem for Individualistic Assumptions" (Theorem 3 in his book). The difference between this theorem and the well-known "General Possibility Theorem" is in the fact that individuals care only about their component of the social state and that they are indifferent to changes effecting individuals other than themselves. The social choice model studied in this paper is designed to deal with the aggregation of preferences of selfish individuals.

We assume that each individual i in an n-person society has a set of conceivable preferences, Ω_i, over a set of his own private alternatives A_i. Society's goal is to aggregate these preferences into a social welfare function (SWF) which should rank $A_1 \times A_2 \times \ldots \times A_n$ - the set of n-person allocations in this society.

We assume that the individuals in this society are symmetric in that all the A_i's and Ω_i's are identical. We provide an answer to the question of which domains of preferences (Ω_i's) admit an Arrow type nondictatorial SWF. This is done in two stages. We first show that the answer to the question of existence of such a SWF for a given A and Ω is independent of the number of individuals in the society. Then we characterize the A's and Ω's that admit a 2-person nondictatorial SWF.

This work parallels Maskin [5] and Kalai-Muller [4]. They studied the same questions for the case of public alternatives. The results that we obtain are different than theirs and one model is not a special case of the other (because of the symmetry assumption made by them). In particular we show as an example of our characterization that for a large class of single peaked preferences over private alternatives every SWF is dictatorial. Thus "good" restricted domains of preferences over public alternatives may be "bad" for private alternatives. On the other hand the case of only one conceivable preference ordering ($|\Omega|=1$) admits non-dictatorial SWF's for private alternatives but for public alternatives it must be dictatorial (every individual must be a dictator by the unanimity assumption).

The model presented here has two drawbacks. One is that we assume that individuals have strong preference (indifference is not allowed). This is done for technical reasons, since we cannot solve the more general case. Secondly, we assume that all A_i's and all Ω_i's are the same. This again is done for technical reasons yet it is not very restrictive. Ω_i is the set of all conceivable preferences of individual i. It is an input into the model and has to be determined by means of some other theory or empirical observations. For example, in economic environments Ω_i may consist of the monotonic convex preferences. It is likely that if these restrictions hold for one individual then they should hold for the others as well. The choice of the A_i's is not critical since we will require our SWF to satisfy a condition of independence of irrelevant alternatives. Thus if the A_i's are too large and contain some nonfeasible alternatives, that will not affect the choice among the feasible ones.

Let A denote a set of alternatives with at least 2 elements. For an integer $n \geq 2$ let A^n represent the set of all n-tuples of alternatives from A. An element of A^n, $X = (x_1,\ldots,x_n)$, is called an <u>n-person allocation</u> of alternatives. Let Σ denote the set of all transitive antisymmetric total (i.e., if $p \in \Sigma$ and $x,y \in A$ then xpy or ypx or $x=y$) binary relations on A. An element of Σ is called a <u>preference relation</u>. For $n \geq 2$ let $\Delta^{(n)}$ represent the set of all transitive antisymmetric total binary relations on A^n. Let Ω be a nonempty subset of Σ; the elements of Ω represent the admissible preference relations in the society. For $n \geq 2$, Ω^n represents the set of all n-tuples of preferences from Ω and $P = (p_1,\ldots,p_n)$, an element of Ω^n, is called an <u>n-person profile</u>.

An <u>n-person social welfare function</u> (SWF) on Ω is a function $f^n:\Omega^n \to \Delta^{(n)}$ which satisfies the following two conditions.

1. <u>Unanimity</u>. For every $P \in \Omega^n$ and $X,Y \in A^n$, if for $i=1,\ldots,n$, whenever $x_i \neq y_i$ then $x_i p_i y_i$, then $Xf^n(P)Y$.

2. <u>Independence of irrelevant alternatives</u> (IIA). For $X,Y \in A^n$, and $P,Q \in \Omega^n$ if [$x_i p_i y_i$ iff $x_i q_i y_i$ for $i=1,\ldots,n$] then [$Xf^n(P)Y$ iff $Xf^n(Q)Y$].

f^n is <u>dictatorial</u> if there exists j, $1 \leq j \leq n$, such that for every $P \in \Omega^n$ and $X,Y \in A^n$, $Xf^n(P)Y$ whenever $x_j p_j y_j$. f^n is <u>nondictatorial</u> if f^n is not dictatorial.

An n-person Arrow-Social Welfare Function (ASWF) is a nondictatorial SWF.

Remark 1:

We could have formulated the problem by letting individuals have preferences over A^n (the same space over which the social preferences are defined) and require that for every $X,Y \in A^n$ with $x_i = y_i$ voter i should be indifferent between X and Y. This formulation is equivalent to the one that we use and it is more natural to people who are used to convention social choice papers. However this formulation would make our notations and statements longer and in this sense it is more natural to use the one formulated earlier.

Remark 2:

Campbell and Fishburn [3] investigated social welfare functions which are very symmetric in a social choice model that allowed them to distinguish between private and public alternatives. However their formulation, and the problems they study are very different from ours.

2. Independence of n

Theorem 1:

For $n \geq 2$

(a) if there exists a 2-person Arrow SWF on Ω then there exists an n-person Arrow SWF on Ω, and

(b) if $|A| > 2$ and there exists an n-person Arrow SWF on Ω then there exists a 2-person Arrow SWF on Ω.

Corollary 1: If $|A| > 2$ then for every $n > 2$, Ω admits an n-person Arrow SWF if and only if it admits a 2-person Arrow SWF.

Proof:

Part a: We assume that there exists f^m, an m-person ($m \geq 2$) Arrow SWF on Ω. We construct an m+1 ASWF on Ω, f^{m+1}, as follows. For $X,Y \in A^{m+1}$ and $P \in \Omega^{m+1}$, $X f^{m+1}(P) Y$ if and only if either $(x_1,\ldots,x_m) = (y_1,\ldots,y_m)$ and $x_{m+1}\ P_{m+1}\ y_{m+1}$ or $(x_1,\ldots,x_m) \neq (y_1,\ldots,y_m)$ and $(x_1,\ldots,x_m)\ f^m((p_1,\ldots,p_m))(y_1,\ldots,y_m)$.

It is straight forward to check that f^{m+1} is a well defined ASWF. Thus, by induction, if there is a 2-person ASWF on Ω then there is an n-person one for every $n \geq 2$.

Part b: Now we assume that $|A| > 2$ and for some $n > 2$ there exists f^n an ASWF on Ω. We have to show that there exists f^2 ASWF on Ω. We first prove it for the following special case.

Ω is said to contain an inseparable pair of alternatives if there exist $s,t \in A$ such that

i. for some $p \in \Omega$ spt and

ii. for no $p \in \Omega$ and $x \in A$ it is $spxpt$.

(s,t) is called an inseparable pair of alternatives.

Lemma 1:

If Ω contains an inseparable pair of alternatives (s,t), then there exists f^2 an ASWF on Ω.

Proof:

For every $X,Y \in A^2$ $(X \neq Y)$ and $P \in \Omega^2$ we define $f^2(P)$ as follows. If $x_1 = y_1$ or $x_2 = y_2$ then f^2 is defined by the unanimity rule according to the preferences of the nonindifferent voter. When $x_1 \neq y_1$ and $x_2 \neq y_2$ we distinguish the following special case $\{x_1, y_1\} = \{s,t\}$ and $s p_1 t$. In this case $f^2(P)$ on the pair X,Y is defined to agree with p_2. In all other cases we define $f^2(P)$ on the pair X,Y to coincide with the preferences of p_1. It is easy to check that f^2 is antisymmetric total relation. It satisfies IIA, nondicatorship, and unanimity by the way it was defined and it remains to be shown that it satisfies transitivity.

Let X,Y,Z be three distinct elements of A^2 and $P \in \Omega^2$ satisfying $Xf^2(P)Yf^2(P)Z$. We will show $Xf^2(P)Z$ by showing it for all the possible cases.

(i) $t p_1 s$ (in case there is such a p_1 in Ω).

1. $x_1 \neq y_1 \neq z_1$. Then $Xf^2(P)Yf^2(P)Z$ implies $x_1 p_1 y_1 p_1 z_1$. by transitivity it is $x_1 p_1 z_1$, which implies, by definition, $Xf^2(P)Z$.

2. $x_1 = y_1 \neq z_1$. Then $Yf^2(P)Z$ implies $y_1 p_1 z_1$ which is the same as $x_1 p_1 z_1$ and again by definition it implies $Xf^2(P)Z$.

3. $x_1 \neq y_1 = z_1$. Then $Xf^2(P)Y$ implies $x_1 p_1 y_1$ which is the same as $x_1 p_1 z_1$ and by definition it is $Xf^2(P)Z$.

4. $x_1 = y_1 = z_1$. Then $Xf^2(P)Yf^2(P)Z$ implies $x_2 p_2 y_2 p_2 z_2$. By transitivity it is $x_2 p_2 z_2$, and by definition it is $Xf^2(P)Z$.

(ii) $sp_1 t$.

1. At least two of the elements x_1, y_1, z_1 do not belong to $\{s,t\}$. Then we are in the same situation as in case (i) and therefore it is $Xf^2(P)Z$.

2. $x_1, y_1 \in \{s,t\}, z_1 \notin \{s,t\}$. Then $Yf^2(P)Z$ implies $y_1 p_1 z_1$, which implies, by the inseparability condition, both $sp_1 z_1$ and $tp_1 z$. Therefore it is $x_1 p_1 z_1$ and by definition it is $Xf^2(P)Z$.

3. $x_1 \notin \{s,t\}, y_1, z_1 \in \{s,t\}$. Then $Xf^2(P)Y$ implies $x_1 p_1 y_1$, which implies both $x_1 p_1 s$ and $x_1 p_1 t$ and therefore it is $x_1 p_1 z_1$. Then by definition it is $Xf^2(P)Z$.

4. $x_1, z_1 \in \{s,t\}$, $y_1 \notin \{s,t\}$. Because of the inseparability condition, this case is impossible.

5. $x_1, y_1, z_1 \in \{s,t\}$. If $x_2 \neq y_2 \neq z_2$ then $Xf^2(P)Yf^2(P)Z$ implies $x_2 p_2 y_2 p_2 z_2$. By transitivity it is $x_2 p_2 z_2$, and by definition this implies $Xf^2(P)Z$. If $x_2 \neq y_2 = z_2$ then $Xf^2(P)Y$ implies $x_2 p_2 y_2$ which is the same as $x_2 p_2 z_2$ and again by definition it is $Xf^2(P)Z$. If $x_2 = y_2 \neq z_2$ then $Yf^2(P)Z$ implies $y_2 p_2 z_2$, or $x_2 p_2 z_2$ and again it is $Xf^2(P)Z$. The case $x_2 = y_2 = z_2$ is impossible, since then $Xf^2(P)Y$ implies $x_1 = s$, $y_1 = t$, while $Yf^2(P)Z$ implies $y_1 = s$, $z_1 = t$ which is a contradiction. Q.E.D.

Continuing with the proof of Theorem 1 we now assume that Ω does not contain any inseparable pair. We also need to introduce some additional definitions and prove the following lemmas. For an integer $n \geq 2$, $z \in A$ and j, $1 \leq j \leq n$, let $A_j^n(z)$ denote the following subset of A^n, $A_j^n(z) = \{X \in A^n | x_j = z\}$. f^n, a SWF on Ω is called a z-dictatorial by j function if there exists a member k, $(1 < k < n, k \neq j)$ such that for every $X, Y \in A_j^n(z)$ and $P \in \Omega^n$, if $x_k p_k y_k$ then it is $Xf^n(P)Y$. k is called the z-dictator by j and is deonoted by $d_j(z)$. Notice that if $d_j(z)$ exists then it must be unique. Let $D_j = \{x \in A|$ there exists a $d_j(x)\}$.

Lemma 2:

Let f^n, for $n \geq 4$, be a SWF on Ω. If for every j $(1 \leq j \leq n)$, $D_j \neq \emptyset$ then there is a member k $(1 \leq k \leq n)$, such that for every $j \neq k$ and every $z \in D_j$, $k = d_j(z)$.

Proof:

Suppose that for some $x \in D_1$, $2 = d_1(x)$ and for some $y \in D_3$, $4 = d_3(y)$. Let $z, w \in A$ and $p \in \Omega$ be such that zpw. Let $P = (p, \ldots, p) \in \Omega^n$, $T = (x, z, y, w, \ldots, w) \in A^n$ and $S = (x, w, y, z, \ldots, z) \in A^n$. Since $T, S \in A_1^n(x)$ and $2 = d_1(x)$ therefore it is $Tf^n(P)S$, but since $T, S \in A_3^n(y)$ and $4 = d_3(y)$ it is also $Sf^n(P)T$, which is a contradiction. Hence we have proved, without loss of generality, that for every $j \notin \{1,2\}$ and every $y \in D_j$, $d_j(y) \in \{1,2\}$. Moreover, by applying the above method, it is easy to show that for every $j,k\notin\{1,2\}$, $y\in D_j$ and $z\in D_k$ it is $d_j(y) = d_k(z)$. Hence, if for every $j\notin\{1,2\}$ and $y\in D_j$ it is $2 = d_j(y)$, then by applying again the above technique, $2 = d_1(x)$ for every $x\in D_1$. On the other hand, if for every $j\notin\{1,2\}$ and $y\in D_j$ it is $1 = d_j(y)$, then again it has to be $1 = d_2(x)$ for every $x\in D_2$, which completes the proof of the lemma. Q.E.D.

In the following lemmas we extend the result of Lemma 2 to the case of $n = 3$. Notice that although the following lemmas are phrased for a certain order of the members 1, 2 and 3, it is obviously true for any permutation of these members.

Lemma 3:

Let f^3 be a SWF on Ω. If $D_j \neq \emptyset$, $j=1,2,3$, and if there exist distinct $x,y\in D_1$ such that $2 = d_1(x) = d_1(y)$ and distinct $z,w\in D_2$ such that $3 = d_2(z) = d_2(w)$ then $2 = d_3(t)$ for every $t\in D_3$.

Proof:

Suppose it is not the case and there exists $t\in D_3$ such that $1 = d_3(t)$. Without loss of generality let $P\in\Omega^3$ be such that xp_1y, zp_2w and suppose that for some $s\in A$ tp_3s. Since $(x,z,s),(x,w,t)\in A_1^3(x)$, $2 = d_1(x)$ and zp_2w, it is $(x,z,s)f^3(P)(x,w,t)$; since (x,w,t), $(y,z,t)\in A_3^3(t)$, $1 = d_3(t)$ and xp_1y, it is $(x,w,t)f^3(P)(y,z,t)$; since $(y,z,t),(x,z,s)\in A_2^3(z)$, $3 = d_2(z)$ and tp_3s, it is $(y,z,t)f^3(P)(x,z,s)$; hence, since f^3 is a SWF, by transitivity it is $(x,z,s)f^3(P)(x,z,s)$ which is a contradiction. Now suppose Ω is such that for every

$p \in \Omega$ and every $s \in A$ it is spt then, again without loss of generality, suppose $P \in \Omega^3$ is such that xp_1y, zp_2w and sp_3t for some $s \in A$. Then $(x,w,t),(y,z,t) \in A_3^3(t)$, $1 = d_3(t)$ and xp_1y imply $(x,w,t)f^3(P)(y,z,t)$; $(y,z,t),(y,w,s) \in A_1^3(y)$, $2 = d_1(y)$ and zp_2w imply $(y,z,t)f^3(P)(y,w,s)$; $(y,w,s),(x,w,t) \in A_2^3(w)$, $3 = d_2(w)$ and sp_3t imply $(y,w,s)f^3(P)(x,w,t)$ and since by transitivity it is $(x,w,t)f^3(P)(x,w,t)$ it is again a contradiction, which complete the proof. Q.E.D.

Lemma 4:

Let f^3 be a SWF on Ω. If $D_j \neq \emptyset$ for $j=1,2,3$ and there exist distinct $r,t \in D_3$ such that $2 = d_3(r) = d_3(t)$ and $x \in D_1$ such that $2 = d_1(x)$ then $2 = d_1(y)$ for every $y \in D_1$.

Proof:

Suppose it is not the case and there exists $y \in D_1$ such that $3 = d_1(y)$. Suppose that for some $p \in \Omega$ xpy then we can let $a,b,c \in A$ and $P \in \Omega^3$ be such that without loss of generality xp_1y, ap_2bp_2c and rp_3t. Then (y,a,t), $(x,b,t) \in A_3^3(t)$, $2 = d_3(t)$ and ap_2b imply $(y,a,t)f^3(P)(x,b,t)$; (x,b,t), $(x,c,r) \in A_1^3(x)$, $2 = d_1(x)$ and bp_2c imply $(x,b,t)f^3(P)(x,c,r)$. By unanimity it is $(x,c,r)f^3(P)(y,c,r)$ and (y,c,r), $(y,a,t) \in A_1^3(y)$, $3 = d_1(y)$ and rp_3t imply $(y,c,r)f^3(y,a,t)$. Then by transitivity it is $(y,a,t)f^3(y,a,t)$ which is a contradiction. If Ω is such that for every $p \in \Omega$ it is ypx, then let $P \in \Omega^3$ be such that yp_1x, ap_2bp_2c and rp_3t. Then by unanimity it is $(y,a,t)f^3(P)(x,a,t)$; (x,a,t), $(x,b,r) \in A_1^3(x)$,

$2 = d_1(x)$ and ap_2b imply $(x,a,t)f^3(P)(x,b,r)$; (x,b,r), $(y,c,r) \in A_3^3(r)$, $2 = d_3(r)$ and bp_2c imply $(x,b,r)f^3(P)(y,c,r)$; (y,c,r), $(y,a,t) \in A_1^3(y)$, $3 = d_1(y)$ and rp_3t imply $(y,c,r)f^3(P)(y,a,t)$ and again by transitivity, it is $(y,a,t)f^3(P)(y,a,t)$ which is a contradiction and this completes the proof. Q.E.D.

The following lemma, by using Lemmas 3 and 4, extends the result of Lemma 2 to the case of $n = 3$.

Lemma 5:

Let f^3 be a SWF on Ω. If for $j=1,2,3$ $D_j = A$ then there is a member k, $1 \leq k \leq 3$, such that for every $j \neq k$ and every $x \in D_j$, $k = d_j(x)$.

Proof:

Since $|A| \geq 3$ and since $D_j = A$ then obviously there is a member $i (i \neq 1)$ such that for at least two distinct alternatives, say x,y $i = d_1(x) = d_1(y)$. Let us assume without loss of generality that this i is 2. By the same reasoning there is a member $k (k \neq 3)$ for which there are at least two alternatives, say z,w such that $k = d_3(z) = d_3(w)$. Hence there are two possible cases.

Case 1. $k=2$. Then by Lemma 4, for every $z \in D_1$ and every $t \in D_3$, $2 = d_1(z) = d_3(t)$ and since $A = D_1 = D_3$, then for every $x \in A$ it is $2 = d_1(x)$ and $2 = d_3(x)$.

Case 2. $k=1$. Then by Lemma 3, for every $x \in D_2$ it is $1 = d_2(x)$, which implies, by Lemma 4, that for every $x \in D_3$ it is $1 = d_3(x)$, which completes the proof. Q.E.D.

All of the above lemmas lead to the following result.

Lemma 6:

If f^n, for $n \geq 3$, is a SWF on Ω such that for every member j $(1 \leq j \leq n)$, $D_j = A$, then f^n is dictatorial on Ω.

Proof:

Suppose $D_j = A$ for $j=1,2,\ldots,n$. Assume without loss of generality that 1 is the k of Lemmas 2 and 5. Suppose $X,Y \in A^n$ and $P \in \Omega^n$ with $x_1 p_1 y_1$. Since we assume that Ω does not contain an inseparable pair we can assume by IIA that for some $z \in A$ $x_1 p_1 z p_1 y_1$.
Let $Z = (z, x_2, \ldots, x_{n-1}, y_n)$ then we have $Xf^n(P)Z$ by $d(x_2) = 1$ and $Zf^n(p)Y$ by $d_n(y_n) = 1$. Thus $Xf^n(P)Y$ by transitivity.

Q.E.D.

We can now complete the proof of Theorem 1 by showing that for $n \geq 3$ if there is an n-person ASWF on Ω, f^n, then there exists an n-1 person nondictatorial SWF on Ω, f^{n-1}. Since f^n is assumed to be nondictatorial, Lemma 6 implies that there is a person i and an alternative x for which $d_i(x)$ does not exist. Assume without loss of generality that $i = n$. Let $p \in \Omega$ be any fixed preference for n and for $X,Y \in A^{n-1}$ define $Xf^{n-1}(P)Y$ if and only if $(x_1, x_2, \ldots, x_{n-1}, x) f^n (p_1, p_2, \ldots, p_{n-1}, p)(y_1, y_2, \ldots, y_{n-1}, x)$. It is easy to check that f^{n-1} defined this way is an ASWF on Ω^{n-1}.

3. The Characterization of the Nondictatorial Domains

In this section we characterize the domains of preferences Ω that admit 2-person Arrow SWF's. By Theorem 1 this is also a characterization of the domains that admit n-person (for every $n \geq 2$) nondictatorial SWF, provided that the number of alternatives is greater than 2 ($|A| > 2$). Throughout this section we let Ω be an artibtrary fixed nonempty subset of Σ.

A set $R \subset A^2 \times A^2$ is said to be closed under decisiveness implications if the following two conditions hold for every X, Y, $Z \in A^2$.

D11. If for some $p \in \Omega$ $x_1 p y_1 p z_1$, $(X,Y) \in R$, and $(Y,Z) \in R$, then $(X,Z) \in R$.

D12. If for some $p \in \Omega$ $x_2 p y_2 p z_2$ and $(Z,X) \in R$ then $(Z,Y) \in R$ or $(Y,X) \in R$.

We let $F = \{(X,Y) \in A^2 \times A^2 | \text{for some } p \in \Omega\ x_1 p y_1\}$. Thus these are the pairs (X,Y) for which it is feasible for voter 1 to prefer X to Y. We let $C = \{(x,y) \in F | \text{for some } p \in \Omega\ y_2 p x_2\}$. Thus these are the pairs (X,Y) which are feasible for voter 1 but also voter 2 can object to voter 1 and prefer Y to X.

We say that Ω has a decomposition if there is an R which is closed under decisiveness implications. We say that Ω has a nontrivial decomposition if it decomposes with an R such that

$$(F - C) \subsetneqq R \subsetneqq F.$$

The intuitive explanation for the above definition is as follows. Suppose there is a 2-person ASWF f on Ω. We let R consist of the pairs (X,Y) which are feasible to voter 1

(x_1py_1, for some $p \in \Omega$) and for which he is also decisive (if he prefers X to Y then so will f). Clearly $F - C \subset R \subset F$. The nondictatorship of voter 1 implies that $R \neq F$ and the nondictatoroship of voter 2 implies that $F - C \neq R$. The closure under decisiveness implications of R follows from the transitivity of f and turns out to be equivalent to it. Hence we obtain

<u>Theorem 2</u> Ω admits a 2-person (n person if $|A| > 2$) Arrow SWF if and only if Ω has a non-trivial decomposition.

<u>Proof</u>. Suppose Ω admits a 2-person ASWF f. Let $R = \{(X,Y) \in F \mid \text{for every } P \in \Omega^2 \text{ if } x_1p_1y_1 \text{ then } Xf(P)Y\}$.
Clearly $(F - C) \subset R \subset F$. Since 2 is not a dictator there is a pair (X,Y) and a $P \in \Omega^2$ with $x_2p_2y_2$ and $Yf(P)X$. Therefore $y_1p_1x_1$ (since f satisfies unanimity), and by IIA and unanimity $(Y,X) \in R$ yet $(Y,X) \in C$. Hence $(F - C) \subsetneqq R$. Since 1 is not a dictator there is a pair (X,Y) and a $P \in \Omega^2$ with $x_1p_1y_1$ but $Yf(P)X$. Hence $(X,Y) \in F$ and $(X,Y) \notin R$. So $R \subsetneqq F$. To show that R is closed under decisiveness implications we assume that $X,Y,Z \in A^2$. Suppose that for some $p \in \Omega$ $x_1py_1pz_1$, $(X,Y) \in R$, and $(Y,Z) \in R$. Clearly $(X,Z) \in F$. Let P be any profile with $x_1p_1z_1$. By IIA we can assume without loss of generality that $x_1p_1y_1p_1z_1$ without effecting the (X,Z) outcome. Since $(X,Y) \in R$ and $(X,Z) \in R$ we have $Xf(P)Y\ f(P)Z$ and by transitivity $Xf(P)Z$. Thus $(X,Z) \in R$ and R is closed under the first type of decisiveness implications. Now we assume that for some $p \in \Omega$ $x_2py_2pz_2$ and $(Z,X) \in R$. Assume that $(Z,Y) \notin R$ and $(Y,X) \notin R$. Since $(Z,X) \in R$ there is a $p_1 \in \Omega$ with

$z_1 p_1 x_1$. Choose p_2 to satisfy $x_2 p_2 y_2 p_2 z_2$ and let $P = (p_1, p_2)$. Then $Zf(P)X$ since $(Z,X) \in \Omega$. If it is not the case that $z_1 p_1 y_1$ then $Yf(P)Zf(P)X$ and $(Y,X) \in R$, a contradiction. So we assume that $z_1 p_1 y_1$. If $Zf(P)Y$ then $(Z,Y) \in R$, a contradiction. Hence $Yf(P)Z$. By transitivity $Yf(P)X$ hence $(Y,X) \in \Omega$ which is a contradiction. So we have that $(Z,Y) \in R$ or $(Y,X) \in R$, and R is closed under decisiveness implications of the 2nd type.

We now assume that R presents a nontrivial decomposition for Ω and demonstrate the existence of an Arrow SWF f. For every $(X,Y) \in A^2 \times A^2$ and every $P \in \Omega^2$ define $Xf(P)Y$ if

(a) $x_i p_i y_i$ for i=1,2, or

(b) $x_1 p_1 y_1$ and $(X,Y) \in R$, or

(c) $x_2 p_2 y_2$ and $(Y,X) \notin R$.

We first show that if $X,Y \in A^2$, $X \neq Y$, then $Xf(P)Y$ or $Yf(P)X$ but not both. If neither $Xf(P)Y$ nor $Yf(P)X$ then we have either (i) $x_1 p_1 y_1$ and $y_2 p_2 x_2$ or (ii) $y_1 p_1 x_1$ and $x_2 p_2 y_2$. If (i) holds yet not $Xf(P)Y$ then $(X,Y) \notin R$ so $Yf(P)X$ by part (c) of the definition of $f(P)$, a contradiction. If (ii) holds yet not $Yf(P)X$ then $(Y,X) \notin R$ but then we should have $Xf(P)Y$, a contradiction. If both $Xf(P)Y$ and $Yf(P)X$ then again there is no unanimity and one of the situations (i) (ii) described above holds. Situation (ii) implies that $(X,Y) \in R$ and $(X,Y) \notin R$, a contradiction. Situation (ii) implies that $(Y,X) \notin R$ and $(Y,X) \in R$ a contradiction again.

It is clear that f satisfies IIA and unanimity. It remains to be shown that f is transitive and nondictatorial.

To show that voter 1 is not a dictator choose $(X,Y) \in F - R$. Since $F - C \subset R$, then $(X,Y) \in C$. Thus for some $p_2 \in \Omega, y_2 p_2 x_2$. Since $(X,Y) \in F$, for some $p_1 \in \Omega$ $x_1 p_1 y_1$. Let $P=(p_1,p_2)$ then $Yf(P)X$, so 1 is not a dictator. To see that 2 is not a dictator choose $(X,Y) \in R-(F - C)$ For some $p_1 \in \Omega$ $x_1 p_1 y_1$ and for some $p_2 \in \Omega$ $y_2 p_2 x_2$. Let $P=(P_1,P_2)$ then $Xf(P)Y$, so 2 is not a dictator.

To show transitivity we assume to the contrary that for some $X,Y,Z \in A^2$, $Xf(P)Yf(P)Zf(P)X$.

Case 1: $z_1 p_1 x_1$ and $z_2 p_2 x_2$

<u>Subcase a</u>: $y_1 p_1 x_1$. Since $Xf(P)Y$ then $x_2 p_2 y_2$ and $(Y,X) \in R$. Since $Yf(P)Z$ then $y_1 p_1 z_1$ and $(Y,Z) \in R$. Since $(Z,(x_1,z_2)) \in R$ by decisiveness implications of the first type (DI1) $(Y,(x_1,z_2)) \in R$. By DI2 (with X being (x_1,z_2), Y being X and Z being Y) either $(Y,X) \in R$ or $(X,(x_1,z_2)) \in R$, a contradiction.

<u>Subcase b</u>: not $y_1 p_1 x_1$. Therefore $y_2 p_2 z_2$ which implies $x_1 p_1 y_1$. Hence $(X,Y) \in R$ and $(Z,Y) \notin R$. Since $((z_1,x_2),X) \in R$ we get by DI1 that $((z_1,x_2),Y) \in R$. Now by DI2 we get that when $((z_1,x_2),Z) \in R$ or $(Z,Y) \in R$, a contradiction.

<u>Case 2</u>: not $z_2 p_2 x_2$. Therefore $z_1 p_1 x_1$ and $(Z,X) \in R$. If not $y_2 p_2 z_2$ then $y_1 p_1 z_1$ and $(Y,Z) \in R$ which implies by DI1 that $(Y,X) \in R$ which contradicts $Xf(P)Y$. Therefore $y_2 p_2 z_2$. If not $x_2 p_2 y_2$ then $x_1 p_1 y_1$ and $(X,Y) \in R$ which implies by DI1 that $(Z,Y) \in R$ which contradicts $Yf(P)Z$ since $z_1 p_1 y_1$. So we are left with the situation that $x_2 p_2 y_2 p_2 z_2$ and $z_1 p_1 x_1$. If not $y_1 p_1 z_1$ and not $x_1 p_1 y_1$ then $(Z,Y) \notin R$ and $(Y,X) \notin R$. But then DI2 contradicts the facts that $(Z,X) \in R$. So we must

have $x_2p_2y_2p_2z_2$ and either $y_1p_1z_1p_1x_1$ or $z_1p_1x_1p_1y_1$. If $y_1p_1z_1p_1x_1$ then $(Y,X)\notin R$. But by DI2 $((z_1,y_2),X)\in R$ and by DI1 (with X being Y, Y being (z_1,y_2) and Z being X) $(Y,X)\in R$, a contradiction. So we must have $x_2p_2y_2p_2z_2$ and $z_1p_1x_1p_1y_1$. But now DI2 implies that $(Z,(x_1,y_2))\in R$ and DI1 (with X being Z, Y being (x_1,y_2) and Z being (y_1,y_2)) implies that $(Z,Y)\in R$. This contradicts $Yf(P)Z$ which shows that case 2 is impossible.

Case 3: not $z_1p_2x_1$. Therefore $z_2p_2x_2$ and $(X,Z)\notin R$. If not $x_1p_1y_1$ then $x_2p_2y_2$ and $(Y,X)\notin R$. By condition DI2 (with X being Z, Y being X, and Z being Y), $(Y,Z)\notin R$. Therefore $Zf(P)Y$, a contradiction. So $x_1p_1y_1$. If not $y_1p_1z_1$ then $y_2p_2z_2$ and $(Z,Y)\notin R$. By condition DI2 (with X being Y, Y being Z, and Z being X) we have that $(X,Y)\notin R$ which contradicts $Xf(P)Y$. So we must have $x_1p_1y_1p_1z_1$. If not $y_2p_2z_2$ and not $x_2p_2y_2$ then $(Y,Z)\in R$ and $(X,Y)\in R$ which contradicts by DI1 the fact that $(X,Z)\notin R$. So we have either $y_2p_2z_2p_2x_2$ or $z_2p_2x_2p_2y_2$. If $y_2p_2z_2p_2x_2$ then $(X,Y)\in R$, by DI1 $(X,(z_1,y_2))\in$ by DI2 (with X being (z_1,y_2) Y being (z_1,z_2) and Z being X) $(X,Z)\in R$, a contradiction. So we must have $x_1p_1y_1p_1z_1$ and $z_2p_2x_2p_2y_2$. Since $Yf(P)Z$, $(Y,Z)\in R$. By DI1 $((x_1,y_2),Z)\in R$, and by DI2 (with X being Z, Y being (x_1,x_2) and Z being (x_1,y_2)) $(X,Z)\in R$, a contradiction.

Q.E.D.

We summarize the question of existence in

Theorem 3: For $n \geq 2$

(a) If $|A| > 2$ then there exists an n-person Arrow SWF if and only if Ω has a nontrivial decomposition.

(b) If $|A|=2$ say $A=\{x,y\}$ then

(i) if $\Omega=\Sigma$ then there exists an n-person Arrow SWF for Ω, and

(ii) if $\Omega\neq\Sigma$ then there exists a nondictatorial SWF for Ω if and only if $n \geq 3$.

Proof. Part (a) is an immediate consequence or Theorems 1 and 2. For part (b)(i) for any pair of alternatives $X,Y\in A^2$ with $X \neq Y$ we define f(P) to be determined by the unanimity rule if it is applicable. Otherwise we define f(P) to coincide with p_1 if xp_1y and to coincide with p_2 if yp_1x. It is straightforward to check that this defines a 2-person Arrow SWF on A. The extension to n-person follows by Theorem 1. To show part (ii) assume w.l.o.g. that $\Omega=\{p\}$ with xpy. In this case the only possible profile is $p=(p,p,\dots,p)$. For n=2 if f is a SWF then $(x,y)f(P)(y,x)$ or $(y,x)f(P)x,y)$. In either case either voter 1 is a dictator or voter 2 is a dictator. For n=3 we define

$\rangle = f(P)$ by $(x,x,x)\rangle(x,x,y)\rangle(x,y,x)\rangle(y,x,x)\rangle(y,y,x)\rangle(y,x,y)\rangle(x,y,y)\rangle(y,y,y)$

It is easy to check that this is a well defined nondictatorial 3-person SWF. By Theorem 1 part a, (ii) holds true for every $n \geq 3$. Q.E.D.

4. Applications

To show the usefulness of Theorems 1, 2 and 3, we discuss some examples.

Example 1. Arrow impossibility theorem for private alternatives.

If $|A| > 2$ and $\Omega = \Sigma$ then Ω is dictatorial.

Proof: Suppose Ω is nondictatorial, then Ω has a nontrivial decomposition $F - C \subsetneqq R \subsetneqq F$, which means that there is $(X,Y) \in C$ such that $(X,Y) \in R$. Since $\Omega = \Sigma$ then $F = \{(X,Y) \in A^2 \times A^2 | x_1 \neq y_1\}$ and $C = \{(X,Y) \in F | x_2 \neq y_2\}$. Therefore by DI1 and DI2, $(X,Y) \in R$ implies that for every $(X,Z) \in C$ also $(X,Z) \in R$. But again by DI1 and DI2 this implies that for every $(W,Z) \in C$ it is also $(W,Z) \in R$, or that $R = F$, a contradiction. Q.E.D.

Example 2. Ω contains an inseparable pair of alternatives (s,t).

We define the set R which corresponds to f^2, the non-dictatorial SWF defined in Lemma 3, as follows.

$R = (F-C) \cup \{(X,Y) \in F|$ such that $x_1 = s$ and $y_1 = t\}$, the proof that Ω has a nontrivial decomposition is straightforward.

Example 3. Single peaked preferences.

The set of single peaked preferences is one of the most celebrated examples of domains of public alternatives, admitting Arrow SWF, independently of the size of theset of alternatives (see Black [2] and the other standard texts). Therefore the following result is somewhat of a surprise.

Let $q \in \Sigma$, and define the <u>set of single peaked preferences relative to the linear order q</u>, by

Ω_q = {$p \in \Sigma$| for every three distinct alternatives x,y,z if zqyqz then it is not the case that xpy and zpy}.

That Ω_q admits an ASWF when A, the set of alternatives, is finite is immediate because of Example 2. Even when A is infinite but has two alternatives $x,y \in A$ such that for every $z \in A$ it is the case of xqyqz (or for every z it is zqyqx), is immediate because of Example 2. However, in general, the single peak condition does not imply the existence of a SWF for private alternatives.

<u>Theorem 4</u>.

If A is a (infinite) set of alternatives and the linear order q is such that there is no maximal alternative x (for any other alternative z, xqz) and there is no minimal alternative y (for any other alternative w, wqy) then Ω_q admits only dictatorial SWFs.

<u>Proof</u>: Suppose Ω_q admits an Arrow SWF, then by Theorem 2 it has a nontrivial decomposition with an R such that $F - C \subsetneq R \subsetneq F$. Suppose $((a,b),(c,d)) \in R \cap C$, then there are four possible cases. That (aqc and bqd) or (aqc and dqb) or (cqa and dqb) or that (cqa and dqb). Since the proof for the different cases are almost

identical, we assume without loss of generality that it is the case of (aqc and bqd). In all of the following steps we use extensively the assumption of no minimal and maximal alternatives in A. Notice that $F = \{(X,Y) \in A^2 \times A^2 | x_1 \neq y_1\}$ amd $C = \{(X,Y) \in F | x_2 \neq y_2\}$, therefore if for some $p \in \Omega_q$ $x_1 p y_1 p z_1$ then by DI1

(1a) $(X,Y) \in R$ implies $((X,(z_1,y_2)) \in R$, and

(1b) $(Y,Z) \in R$ implies $((x_1,y_2),Z) \in R$; and

if for some $p \in \Omega_q$ $x_2 p y_2 p z_2$ then by DI2

(2a) $(Z,X) \in R$ implies $(Z,(x_1,y_2)) \in R$, and

(2b) $(Z,X) \in R$ implies $((z_1,y_2),X) \in R$.

<u>Step 1</u>: Let $y \in A$ such cqy, then there exist $p,p' \in \Omega_q$ such that apcpy and cp'ap'y. Therefore by (1a), $((a,b),(c,d)) \in R$ implies $((a,b),(y,d)) \in R$, which in turn, by (1b), implies $((c,b),(y,d)) \in R$.

<u>Step 2</u>: Let $y,z \in A$ such that cqy and zqy $(z \neq c)$. Then there exists $p \in \Omega_q$ such that zpcpy and by (1b), $((c,b),(y,d)) \in R$ implies $((z,b),(y,d)) \in R$.

<u>Step 3</u>: Let $z,y \in A$ such that cqy and zqy (z may also be c). Since for every $v \in A$ such that vqz or yqv, there exists $p \in \Omega_q$ such that zpypv, then by (1a), $((z,b),(y,d)) \in R$ implies $((z,b),(v,d)) \in R$.

<u>Step 4</u>: Let $z,s \in A$ such that zqsqc. There exists $v \in A$ and $p \in \Omega_q$ such that vqz and zpvps, then, by (1a), $((z,b),(v,d)) \in R$ implies $((z,b),(s,d)) \in R$.

Steps 1, 2, 3 and 4 together imply that for every $z,s \in A$ such that $((z,b),(s,d)) \in C$, then $((z,b),(s,d)) \in R$.

<u>Step 5</u>: Let $u \in A$ such that bqu then there exists $p \in \Omega_q$ such that dpupb and, by (2a), $((z,b),(s,d)) \in R$ implies $((z,b),(s,u)) \in R$.

<u>Step 6</u>: Let $u,w \in A$ such that bqu and bqw, then there exists $p \in \Omega_q$ such that upwpb and by (2b), $((z,b),(s,u)) \in R$ implies $((z,w),(s,u) \in R$.

<u>Step 7</u>: Let $x,u \in A$ such that xqb and bqu (or u is b), then there exist $w \in A$ and $p \in \Omega_q$ such that xquqw and upxpw and by (2b), $((z,W),(s,u)) \in R$ implies $((z,x),(s,u)) \in R$.

<u>Step 8</u>: Let $r,t \in A$, then there exist $x,u \in A$ such that xqr,xqt and bqu. By Step 7, $((z,x),(s,u)) \in R$. Considering alternative x as the alternative b, Step 6 implies that $((z,r),(s,t)) \in R$.

This completes the proof that for every $r,s,t,z, \in A$ such that $((z,r),(s,t)) \in C$ it is also $((z,r),(s,t)) \in R$, hence $R = F$, a contradiction.

Q.E.D.

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