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INTERSECTORAL INTERDEPENDENCE AND DOMINANCE IN INPUT-OUTPUT SYSTEMS

by

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ABSTRACT

Starting from a critical appraisal of the current measure of linearity of an economic system, based on the triangular structure of its input-output matrix, this paper illustrates the difficulties involved in such linearity measures. (See in particular Korte and Oberhofer [8] and Helmstadter [5].) To avoid such problems it is shown that two different concepts are involved: one of interdependence stricto sensu which is best apprehended via the sectoral multipliers of the input-output matrix, or more specifically via the study of its inverse; the other concept is that of dominance which is simply a binary asymmetric relation defined over the set of industries. Its degree of transitivity can be measured by a class of functions viz "indices of dominance" (partial or global). Finally, two examples are provided in order to show how these dominance functions are not affected by variations in the level interdependence - which was one of the difficulties overlooked by the Helmstädter index and the Korte-Oberhofer study.

I - Introduction: A review of the standard notion of triangularization

Scientific controversies often survive long after an improved technology has made them purely academic. Input - output economics is no exception to this rule. In the early days of input - output studies it was soon realized that a large number of entries of an actual input - output matrix were zeros. This empirical fact coupled with the rather primitive and inefficient computing equipment available led the investigators to attempt to exploit this feature in order to make the problem of inverting the matrix computationally tractable. If the input - output system is to be used in determining the output levels of the various industries for a given vector of final demand, a matrix inversion operation has to be performed on the input - output coefficient matrix. As any student of linear algebra knows today, the Gauss - Doolittle algorithm, being iterative in nature, can be considerably simplified if the coefficient matrix happens to be in triangular form. This meant that those who had to compute this inverse tried to achieve as much "triangular" structure as possible by changing the order in which the sectors were listed to put as many zeros as possible either below the diagonal or above the diagonal, using heuristic techniques. In point of fact, however, the computing abilities of today's third generation computers have little in common with the desk calculators which Leontief used to solve his first systems. Even cost considerations are no longer sufficient to justify the use of triangular input - output matrices on that basis alone.

What originally started as a technique to alleviate computational headaches, however, also draws some of its significance from the history of economic thought. The so-called "pyramid of production" concept as proposed by the Austrian economists also belongs to that class of triangular structures that have been found to be computationally convenient. In trying to quantify these notions of "earlier" and "later" stages of production the Austrian School ran into insurmountable problems for lack of an adequate mathematical apparatus. Almost a century later a number of authors have provided tentative solutions to these problems through the use of the input - output framework. Here it suffices to mention the well-known works of Chenery and Watanabe [2], Aujac [1] and Masson [11], Simpson and Tsukui [16] to cite but just a few. One of the most recent contributions to this line of thought has provided a precise mathematical formulation of the triangularization problem: Korte and Oberhofer [8] have formulated it as a very special quadratic assignment problem and proposed several types of algorithms that appear particularly well suited for its solution. The real issue, however, is to decide whether or not the triangularization operation as it is currently performed really addresses the problem of the proffered existence of linear of circular dependencies between industries. For illustrative purposes let us consider the following simple input - output coefficient matrix */

*/ Note that this matrix has already been triangularized following the Korte - Oberhofer formulation.

$$(1) \quad A = \begin{bmatrix} 0 & 3/4 & 0 & 0 \\ 0 & 0 & 3/4 & 0 \\ 0 & 0 & 0 & 3/4 \\ 3/4 & 0 & 0 & 0 \end{bmatrix}$$

If we now use the Helmstädter [5] linearity measure:

$$(2) \quad \ell(A) = \frac{\sum_{i<k}^{n} a_{ik}}{\sum_{\substack{i,k=1 \\ i \neq k}}^{n} a_{ik}}$$

where a_{ik} = ratio of the amount of commodity k used by sector i to the total production of sector i.*/

We can easily verify that $\ell(A) = .75$.

Extending the pattern of the A matrix to a (100 X 100) matrix yields a higher value of the linearity index viz. $\ell(A) = .99$! Let us also note that the actual level of dependence in this matrix as measured by the ℓ index is independent of the values of the diagonal elements. If we have an exogenous

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*/ As the reader can readily notice, this is the transpose of the coefficient used in most input - output studies. The reason for this becomes clear later on.

increase in demand in any one sector by one dollar, the impact on "distant" sectors depends on how much of that dollar's worth of additional demand is absorbed by profit of intermediate sectors between the initial sector and the "distant" sectors. For instance if we had all diagonal elements in the first matrix (1) equal to 1/4, then we would have every sector equally tied to every other sector: more specifically, an exogenous dollar increase in final demand in any one sector induces the same dollar amount of production in each sector. This is a characteristic of stochastic matrices. If we limit ourselves to the simple notion of triangularity and its natural corollary, the linearity index, we are in fact discarding some potentially useful information. What is called for are much more refined notions of triangularity and of the relationships among sectors. In the example discussed so far, even though the linearity index is quite high, this is highly misleading. In point of fact the economy represented by such a matrix (1) is highly interdependent if the diagonal elements are near 1/4, as we show in the next section. This points out that two concepts are involved here and that the current measure of linearity is inadequate to discriminate between them. If we reflect upon the Austrian notion of triangularity, we notice that the concept is actually one of <u>dominance</u> . This notion will be elaborated upon and quantitatively defined in the next section; at this point on purely intuitive grounds, it can be said that one industry "dominates" another industry whenever it uses a "large" amount of the output of this industry, i.e. "larger" than the amount used by the second industry from the first industry. For instance we expect the steel industry to "dominate" the iron mining industry since steel consumes the largest portion of ore production and iron mining uses only a small portion of the steel output. By the same token the auto industry dominates the steel industry. In the next section we shall develop the framework in which this notion of dominance will be used.

II - A framework of analysis

2.1 At this point let us look back at the original purpose of triangularization. Given any (n+2) x (n+2) nonnegative transactions matrix T, it is always possible to ask whether a simultaneous relabelling of some (or all) of its rows and columns would yield a maximal value for the sum of its upper diagonal elements. If it turned out that in fact the T matrix could be made perfectly triangular i.e., all of its lower diagonal entries being zero then we would claim that we have found a sector ordering along which all goods flow in only one direction. A closer look however should dampen our enthusiasm for this traditional and elementary exercise in input - output economics. Any nonzero lower diagonal entry that may appear in a quasi-triangular production structure may be more troublesome than the avid practitioners of triangularization exercises would have us believe! This T matrix is, in fact, an incomplete view of the system: latent information is actually hidden underneath and to bring it out in the open necessitates an appropriate mathematical operation.

We must look at the original purpose of triangularization. This is to measure the amount of interdependency among the sectors of an economy. The failure of the method presented above is that it measures only the immediate impact on all sectors of a change in output of a sector. What we have to do is measure the <u>cumulative</u> impact on all sectors of a change in demand in one sector. To do this we must formulate the input - output matrix as a Markov matrix.

We look at the production coefficients matrix; that is, a normalization of the input - output matrix where the coefficients in a row $(a_{i0}, \ldots, a_{in+1})$ represent the amounts of input from each sector to produce a dollar of output in sector i. Now, the sum of all the elements in a row is one, and we assume $a_{ij} \geq 0$. Let a_{i0} represent the profit per dollar of output from sector i, and a_{in+1} represent the labor cost per dollar. Removing the a_{i0}, a_{in+1} for $i=1, \ldots, n$, $(a_{00}, a_{01}, \ldots, a_{0n+1})$ and $(a_{n+10}, a_{n+11}, \ldots a_{n+1n+1})$ we have retained the demands between sectors only. Let

$$(3) \quad A = \begin{pmatrix} a_{11} \cdots a_{1n} \\ a_{n1} \cdots a_{nn} \end{pmatrix} = [a_{ij}].$$

An increase in demand in sector i by one dollar has the immediate effect of increasing the demand in sector j by a_{ij}. This is the first level effect as treated by Korte and Oberhofer [8]. However, this increase in demand in sector j causes an increase in demand in sector k by $a_{ij}a_{jk}$. That is, the total effect on sector k of an increase in demand for every sector j by a_{ij} dollars can be simply written:

(4) $$b_{ik} = \sum_{j=1}^{n} a_{ij}a_{jk}.$$

This measures the secondary effects of demand on sector k from an increase in demand at sector i by one dollar; that is, the extra increase in demand caused by the rise in production to produce the inputs directly needed to produce one dollar's worth of goods in sector i for the increase in final demand. If we set

(5) $$B_2 = [b_{ij}]$$

we have a matrix where b_{ij} represents the secondary effect on sector j of an increase in demand from sector i. From (2) we see that

(6) $$B_2 = A \cdot A.$$

As with the primary increases in demand causing secondary increases in demand, the secondary increases cause tertiary increases. The amount is

(7) $$B_3 = A \cdot A \cdot A.$$

These effects build up ad infinitum.

The cumulative effect of the dollar increase is the sum of the amounts at each level of demand including the initial dollar increase.

This is

(8) $N = I + A + A^2 + A^3 + A^4 + \ldots = [n_{ij}]$

Multiplying N by (I-A) we have

(9) $(I-A)N = (I-A)+(A-A^2) + (A^2-A^3) + \ldots = I.$

If $(I-A)^{-1}$ exists we have

(10) $N = (I-A)^{-1}$.

Since $a_{i1}+a_{i2}+\ldots+a_{in}<1$ as we removed a_{i0} and a_{in+1}, we know $(I-A)^{-1}$ exists .

Now that we have measured the total impact on demand of a change in demand in any one sector, we can begin to develop an effective measure of dominance.

2.2 Furthermore if we consider any pair of vectors (i,j), for $i \neq j$, a different question can be asked: is industry (i) input-dominated by industry (j) or vice versa? In other words we could ask whether $a_{ij} > a_{ji}$ or $a_{ji} > a_{ij}$. Actually, for the same reasons that prompted us to use the N matrix instead of just the A matrix, another form of this question would be: is $n_{ij} > n_{ji}$ or else $n_{ji} > n_{ij}$? This amounts to asking whether industry (i) is directly and indirectly input-dominated by industry (j) or vice versa. Clearly this dominance relation we just defined is a binary asymmetric and irreflexive relation on the entries of the N matrix. Its transitivity is an open question to be answered only after an empirical investigation. More will be said on this point in the next section. For now we can readily see that the N matrix of our original example (see equation 12 below) shows no pattern of overall dominance i.e. even though each individual arc $i \to j$ reflects a certain dominance, the direction of these arcs cancel each other; in this case the dominance relation is totally intransitive. Let us now examine this example more closely.

III - A measure of dominance

3.1 For illustration purposes let us return, for a moment, to the original example presented earlier viz.

$$(11) \qquad A = \begin{bmatrix} 0 & 3/4 & 0 & 0 \\ 0 & 0 & 3/4 & 0 \\ 0 & 0 & 0 & 3/4 \\ 3/4 & 0 & 0 & 0 \end{bmatrix}$$

As we argued in the previous section in order to take into account all indirect interdependencies besides direct interdependencies we must first consider the $N = (I-A)^{-1}$ matrix. In this case the N matrix reads:

$$(12) \quad N = (I-A)^{-1} = \begin{bmatrix} \frac{256}{175} & \frac{192}{175} & \frac{144}{175} & \frac{108}{175} \\ \frac{108}{175} & \frac{256}{175} & \frac{192}{175} & \frac{144}{175} \\ \frac{144}{175} & \frac{108}{175} & \frac{256}{175} & \frac{192}{175} \\ \frac{192}{175} & \frac{144}{175} & \frac{108}{175} & \frac{256}{175} \end{bmatrix}$$

If we now calculate the Helmstädter linearity index for this matrix we obtain

$$(13) \qquad \ell(N) \cong .55$$

The reader can easily check that if we had chosen an A matrix with the same entry pattern but, say, 1/10 instead of 3/4 and then computed its inverse, the ℓ index for that inverse would be very close to .75. Upon closer examination of the N matrix above one can notice the following fundamental fact: the economy represented by the A matrix is totally interconnected in that each sector eventually uses some output of every other sector.

The example we have just discussed demonstrates the need for more refined concepts than the linearity index proposed earlier. Two independent notions are involved: one of interpendence and the other of overall dominance. The Helmstäder index is insufficient to distinguish between the two Yet in order to be meaningful we must exhibit a measure which singles out the variation in the dominance structures of various economies and yet is not affected by the level of interdependence between industries. Such a measure will now be offered.

3.2 Consider, first the inverse N of an input - output matrix A. Following Korte and Oberhofer [8], we can first obtain the triangulated structure of this N matrix. This amounts to solving the following quadratic assignment problem.

Let

$$(14) \qquad R(N) = \sum_{i<j} n_{ij}$$

Let P denote an (nxn) permutation matrix ($P \in \mathcal{P}$, the set of permutation matrices.) Find $P \in \mathcal{P}$ which maximizes $R(PNP^T)$. Two possible solution techniques have been provided by Korte and Oberhofer. Once this N matrix has thus been triangulated, we can proceed to measure the level of overall dominance in the economy described by this N matrix. Define the following sets of coefficients n_{ij} from the N matrix:

$$(15) \qquad D^U_1 = \{n_{12}, n_{23}, \ldots, n_{i,i+1}, \ldots, n_{n-1,n}\}$$

$$(16) \qquad D^L_n = \{n_{n,1}\}$$

or more generally

$$(17) \qquad D^u_k = \{n_{1k}, n_{2,k+1}, \ldots, n_{n-k+1,n}\}$$

$$(18) \qquad D^L_{n-k+1} = \{n_{n-k+1,1}, n_{n-k,2}, \ldots, n_{n,n-k+1}\}$$

(These sets refer respectively to the k-th upper and lower main diagonals in the N matrix).

The first dominance index δ, can be defined as follows:

Letting

$$(19) \qquad n_{i^*j^*} = \max_{n_{ij} \in \{D_1^u \cup D_n^L\}} \{n_{12}, \ldots, n_{ij} \ldots\}$$

we have

$$(20) \qquad \delta_1 = \frac{n_{i^*j^*} - n_{n,1}}{n_{i^*j^*}}$$

Clearly δ_1 ranges over the closed interval [0,1]. In words it gives a relative measure of the amount of pairwise dominance from the nth sector to the first sector while taking into account the reverse flow (n_{n1}) from 1 to n. Actually, this dominance index δ_1 is just one among (n-1) such possible indices. For instance

$$(21) \qquad \delta_k = \frac{n_{i^*j^*} - n_{\bar{i}\bar{j}}}{n_{i^*j^*}}$$

where

$$(22) \qquad n_{i^*j^*} = \max_{n_{ij} \in \{D_k^u \cup D_{n-k+1}^L\}} \{n_{1k}, n_{2k+1} \ldots ; n_{n-k+1}, n_{n-k,2} \ldots\}$$

In other words we find the maximal element in the kth upper and (n-k+1) lower diagonal of N, say $n_{i^*j^*}$, and subtract from the maximal element in the (n-k+1) lower diagonal $n_{\bar{i}\bar{j}}$ and normalize the difference. We can also define an overall index of dominance δ as the mean (arithmetic or otherwise) of the (n-1) δ_k indices thus obtained. An illustration of this technique for measuring economic dominance among sectors will now be provided.

3.3 Consider again the economy represented by the N matrix (equation (12))

$$(12) \quad \begin{bmatrix} \frac{256}{175} & \frac{192}{175} & \frac{144}{175} & \frac{108}{175} \\ \frac{108}{175} & \frac{256}{175} & \frac{192}{175} & \frac{144}{175} \\ \frac{144}{175} & \frac{108}{175} & \frac{256}{175} & \frac{192}{175} \\ \frac{192}{175} & \frac{144}{175} & \frac{108}{175} & \frac{256}{175} \end{bmatrix}$$

Then

$$(23) \quad \delta_1 = \frac{\frac{192}{175} - \frac{192}{175}}{\frac{192}{175}} = 0$$

Similarly because of the identical structure of the other upper and lower main diagonals we can see that: $\delta_k = 0 \quad \forall\ k = 1,2,\ldots,n-1$. Of course the overall dominance index δ will also be zero. With the help of this example it is clear that this δ index removes the effect of the level of interdependence between sectors which the Helmstadter index was measuring along with the dominance effect. The δ index is a measure of dominance <u>only</u> - irrespective of the amount of interdependence. This point is further explained in the following example. Consider an economy with the following structure:

$$(24) \qquad A = \begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have

$$(25) \qquad N = (I-A)^{-1} = \begin{bmatrix} 1 & 1/4 & 5/12 & 5/6 \\ 0 & 1 & 1/3 & 2/3 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and now

$$(26) \qquad \delta_1 = \delta_2 = \delta_3 = 1 = \delta$$

while the Helmstadter index $\ell(N) = 1$ also! At this point the reader might wonder why the two indices take on the same value in this case. The reason is simple enough. There are no circular interdependencies exhibited by the N matrix so that no such perturbing factor distorts the ℓ index. Such a situation however is purely accidental. Clearly no realistic input-output table can have this form although it is necessary to look at examples of this sort. In fact it is well known that in empirical input - output studies such circularities occur rather often. The use of the δ indices offers the fundamental advantage that it is by construction, invariant under any circularity increasing (or decreasing) transformation of the productive structure.

As such, of course, it provides a non ambiguous summary measure for international and/or intertemporal comparisons. On the contrary any such comparison based on the simple ℓ index is bound to yield very ambiguous results; it could only be justified in two extreme cases: either no circularity exists in the production structures to be compared or exactly the same circularities exist. This is hardly likely to be verified in practice and at any rate, the whole issue can be bypassed by using the δ indices.

3.4 Some concluding comments are now in order. First of all one might remark that if we deem it appropriate to neglect as a first approximation the chain of indirect reactions between sectors following exogenous change in final demand, the same argument that led to the adoption of the index can be applied and remains valid: The δ measure works equally well on the A matrix instead of the N matrix except, of course, that the sectoral multiplier effects are ignored. On the other hand and from a historical standpoint, it is interesting to note that the two separate notions of "dominance" and "interdependence" have underlied many of the previous studies. The fact that they are quite independent and are in no way linked by a one-way or two-way implication relation was never noticed. If anything at all can be said, it is that the implicit assumption seemed to be that one was a broader notion than the other. Such is not the case however, as we have shown. And finally, from a purely policy standpoint, it might seem natural to ask which notion is most useful. The answer to this question clearly depends upon the kind of policy issue which is under review: the interdependence notion allows us to isolate clusters of industries that are linked in a pairwise fashion with each other whereas the notion of dominance establishes a priority between sectors based on the size of their transactions.

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