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PRICES AND MARKET IMBALANCE INDEXES IN A GENERAL EQUILIBRIUM MODEL[1]

by

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Abstract. Prices and market imbalance indexes are used as the coordinating market signals in a general equilibrium model of an exchange economy. These imbalance indexes represent rates of excess capacity or shortage in the markets for commodities other than money. In equilibrium, total consumption should equal supply for every commodity, and the prices should depend on the imbalance indexes through some exogenous Phillips-curve relations. Depending on how these generalized Phillips curves are constructed, this model can include both the Arrow-Debreu model and some unemployment equilibrium models as special cases.

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1. Introduction

Clower [1965] brought into clear focus a central question in economic theory: at the level of the individual decision problems faced by the economic agents in our models, what basic differences distinguish the Keynesian from the classical Walrasian equilibrium models? Clower argued that the basic difference must be that, in the Keynesian models, individuals may perceive quantity constraints on their transactions, in addition to prices, when they make their economic decisions. Certainly, any equilibrium model which does not admit quantity constraints into the individual decision problems would have already assumed away involuntary unemployment, and could not provide a theory of unemployment.

The Arrow-Debreu model, presented by Debreu [1959], is the modern mathematical formulation of Walrasian equilibrium theory. So our basic theoretical problem is how to modify or extend the Arrow-Debreu general equilibrium model to make it consistent with Keynesian theories of unemployment and inflation. Several such modified equilibrium models have been offered, notably by Benassy [1975], Drèze [1975], Grandmont and Laroque [1976], Heller and Starr [1976], and others. (A good survey is in Grandmont [1977].) This paper will present another approach to this problem.

In constructing the model to be presented here, a basic goal has been to admit unemployment equilibria while otherwise remaining as faithful as possible to the essential spirit of the Arrow-Debreu model. Thus, this research has been guided by the following four principles.

First, every agent in our model should make his economic decisions so as to solve an optimization problem which depends only on his own fixed personal parameters and on some market signals which he passively accepts. This principle leads us diverge from the model of Benassy [1975], which has the consumers solving multiple decision problems. Of course, assuming that all traders passively accept market signals means that we can not try to describe rational price-making behavior in this model, and we will have to resort to an "auctioneer" story to explain the origin of prices.

Second, every agent should get the same market signals and the number of signals should be independent of the number of agents in the economy. This condition is necessary if our model is to involve only market signals which are (at least in principle) empirically observable.

Third, every unit of a commodity offered for sale ought to get the same treatment in the market, as should every unit of money offered to purchase the commodity, without regard to who made the offers. (On the supply side, this principle can be made almost tautological by suitably redefining our "commodities." For example, if high-seniority workers are hired before low-seniority workers, then we should simply redefine high-seniority-labor and low-seniority-labor as different commodities.) This anonymity principle is implicit in the Arrow-Debreu model's assumption that all trades should occur at the same price ratios, throughout the economy. This principle only permits rationing schemes of the proportional type; that is, all traders on the same side of a given commodity market must receive the same proportion of their offered trades. Thus we will use proportional rationing schemes on both sides of every market, which will distinguish this model from that of Drèze [1975].

Finally, although our model should include the possiblity of equilibria with unemployment, the model should also include the Arrow-Debreu model as a special case.

2. The Basic Auctioneer Model

We consider an exchange economy with I consumers numbered $i = 1,2,\ldots,I$. There are $H + 1$ commodities, numbered $h = 0,1,2,\ldots,H$, which these consumers can trade. The commodity 0 represents money, the medium of exchange in this economy, and will merit special treatment in our story.

For each consumer i and each commodity h, let

$$\omega_h^i = (\text{consumer } i\text{'s initial endowment of commodity } h) \quad .$$

For simplicity, let us assume that the consumption set of consumer i is $\mathbb{R}_+^{H+1}$, the nonnegative orthant of $\mathbb{R}^{H+1}$. That is, any bundle including nonnegative quantities of all commodities would be an acceptable consumption bundle for i. Let

$$u^i\colon \mathbb{R}_+^{H+1} \to \mathbb{R}$$

be consumer i's <u>utility function</u>. In planning his trades, consumer i will try to end up with the bundle $x^i = (x_0^i, x_1^i, \ldots, x_H^i)$ in R_+^{H+1} giving him the highest utility $u^i(x^i)$ among all the bundles which he can buy.

To formulate our model, we begin by following the well-known Walrasian auctioneer story. Let us imagine an economy with some central auctioneer who coordinates all trading. In the usual story, the auctioneer announces prices for the commodities, but in this model we want to also allow quantity constraints on the consumer's trading. To be consistent with the spirit of the traditional story, we should have all market information communicated to the consumers only by auctioneer's announcements. So we let the auctioneer announce three nonnegative indexes for every commodity h, as follows:

$$p_h = (\text{price of commodity } h) \geq 0$$

$$e_h^S = (\text{excess supply index for commodity } h) \geq 0$$

$$e_h^D = (\text{excess demand index for commodity } h) \geq 0 \quad .$$

We will assume that $p_0 = 1$ and $e_0^D = e_0^S = 0$, because money is the numeraire and the medium of exchange. Thus, the auctioneer's announcement will consist of the three vectors $p = (p_1,\ldots,p_H)$, $e^S = (e_1^S,\ldots,e_H^S)$, and $e^D = (e_1^D,\ldots,e_H^D)$. Also, we will assume that the auctioneer cannot claim to have both excess supply and excess demand simultaneously on any market, so:

$$(1) \qquad e_h^S \cdot e_h^D = 0 \quad , \quad \text{for every } h = 1,2,\ldots,H \quad .$$

Given p, e^S, and e^D, each consumer i must respond to the auctioneer by send his requests for selling and buying in the commodity markets. For any commodity h, let:

s_h^i = (quantity of h offered for sale by i) ≥ 0

d_h^i = (quantity of h which i asks to buy) ≥ 0

represent these selling and buying requests for consumer i.

We assume that consumer i cannot offer to sell more of a commodity than he actually has, so his sales offers are contrained by:

(2) $$s_h^i \leq \omega_h^i \quad , \quad \text{for all } h = 1,\ldots,H \quad .$$

Unfortunately for i, he cannot necessarily sell all that he offers, because there may be excess supply on some markets. If the excess supply in each market must be distributed proportionately to the sales offers, then i will only succeed in selling $(1 - e_h^S)s_h^i$ units of h. That is, the excess supply index e_h^S is to be interpreted as the ratio of excess supply to total supply in the market for h, and this excess supply will be shared proportionately among the sellers.

Taking these excess supply (or unemployment) constraints into account, the auctioneer should credit consumer i with sales income $\sum_{h=1}^{H} p_h(1 - e_h^S)s_h^i$. Because all his demands to buy must be backed up by money or by real available income, consumer i faces the following budget constraint on his demands:

(3) $$\sum_{h=1}^{H} p_h d_h^i \leqq \omega_0^i + \sum_{h=1}^{H} p_h(1 - e_h^S)s_h^i \quad .$$

Unfortunately for i (again), if there is excess demand in some markets, his actual deliveries received may be less than those demanded. If e_h^D is the ratio of excess demand to total demand in the market for h, then under proportional rationing i can only get $(1-e_h^D)d_h^i$ units of h actually delivered. Consumer i will then have unusable sales income totalling $\sum_{n=1}^{H} p_h e_h^D d_h^i$, representing the value of his unfilled demand orders. This credit will be returned to him in the form of money, at the end of the trading session, together with any voluntarily unspent income (the slack in (3), if any). Thus, the final quantity of h which i will get to consume is

$$x_h^i = \omega_h^i - (1 - e_h^S)s_h^i + (1 - e_h^D)d_h^i \quad , \tag{4}$$

for any commodity $h \neq 0$, and i's final holdings of money will add up to

$$x_0^i = \omega_0^i + \sum_{h=1}^{H} p_h(1 - e_h^S)s_h^i - \sum_{i=1}^{H} p_h(1 - e_h^D)d_h^i \quad . \tag{5}$$

Consumer i's basic decision problem is to choose the vectors $x^i = (x_0^i, x_1^i, \dots, x_H^i)$, $d^i = (d_1^i, \dots, d_H^i)$ and $s^i = (s_1^i, \dots, s_H^i)$ so as to maximize $u^i(x^i)$ subject to (2), (3), (4), (5), and:

$$x^i \in \mathbb{R}_+^{H+1} \quad , \quad d^i \in \mathbb{R}_+^H \quad , \quad s^i \in \mathbb{R}_+^H \quad . \tag{6}$$

As in the usual Walrasian auctioneer story, the auctioneer's problem is to choose his announced quantities so that he can actually carry out his promised transactions. Thus, in an <u>auctioneer's solution</u>, the following equations must be satisfied:

$$(1 - e_h^S)\sum_{i=1}^{I} s_h^i = (1 - e_h^D)\sum_{i=1}^{I} d_h^i \quad , \quad \text{for all} \quad h = 1,2,\dots,H \quad . \tag{7}$$

We do not have to worry about the auctioneer's balance of money payments, once his other accounts balance, since (5) and (7) together imply that

$$\sum_{i=1}^{I} x_0^i = \sum_{i=1}^{I} \omega_0^i \quad .$$

When we combine equation (1) with (7), we get the equivalent conditions:

$$e_h^S = \max\,(0, \sum_{i=1}^{I} s_h^i - \sum_{i=1}^{I} d_h^i)/(\sum_{i=1}^{I} s_h^i) \tag{8}$$

$$e_h^D = \max\,(0, \sum_{i=1}^{I} d_h^i - \sum_{i=1}^{I} s_h^i)/(\sum_{i=1}^{I} d_h^i) \quad . \tag{8'}$$

That is, in an auctioneer's solution, the announced excess demand and supply ratios must be verified by the consumers' optimal supply and demand requests.

3. A Reformulation of the Basic Model

The s_h^i and d_h^i quantities defined in the preceding section play a somewhat strange role. They represent trading offers announced by consumer i which he must be able to carry out, even if the relevant market constraints are relaxed, because of constraints (2) and (3). But consumer i does not necessarily expect to actually carry out all these trades, since he knows that his consumption bundle will actually be determined by (4) and (5). These s_h^i and d_h^i quantities were introduced in the preceding section mainly to clarify the logic behind our auctioneer story. We can now simplify our analysis by reformulating the model directly in terms of the only quantities which really matter to the consumers, the final consumption quantities x_h^i.

No consumer would ever want to trade on both sides of any market (there are no additional net trades which he could accomplish by doing so), and so we may assume without loss of generality that $s_h^i \cdot d_h^i = 0$, for all i and h. Then equation (4) becomes:

$$s_h^i = \frac{1}{(1 - e_h^S)} \cdot \max\,(0, \omega_h^i - x_h^i) \quad , \tag{9}$$

$$d_h^i = \frac{1}{(1 - e^D)} \cdot \max\,(0, x_h^i - \omega_h^i) \quad . \tag{9'}$$

Substituting (9) into (2) gives us

$$x_h^i \geq e_h^S \omega_h^i \quad . \tag{10}$$

Substituting (5) into (3) implies that

$$0 \leq x_0^i - \sum_{h=1}^{H} p_h e_h^D d_h^i \quad .$$

Then (9') gives us

$$x_0^i \geq \sum_{h=1}^{H} p_h \cdot \frac{e_h^D}{1 - e_h^D} \cdot \max\,(0, x_h^i - \omega_h^i) \quad . \tag{11}$$

Notice that constraint (11) includes as an implication that:

$$\text{if } \; e_h^D = 1 \; \text{ then } \; x_h^i - \omega_h^i \leq 0 \quad , \tag{11'}$$

because

$$\frac{e_h^D}{1 - e_h^D} = \frac{1}{1 - 1} = +\infty \quad \text{when} \quad e_h^D = 1 \quad .$$

Substituting (9) and (9') into (5) also gives us

$$x_0^i + \sum_{h=1}^{H} p_h x_h^i = \omega_0^i + \sum_{h=1}^{H} p_h \omega_h^i \quad . \tag{12}$$

Thus, any consumption vector x^i which i can achieve when (p,e^S,e^D) is the auctioneer's announcement must satisfy (10), (11), and (12). Conversely, it is straightforward to check that any x^i satisfying (10), (11), and (12) will also be consistent with the trading constraints (2), (3), (4), and (5), when s_h^i and d_h^i are chosen as in (9) and (9').

Let us now introduce two assumptions which we will need.

<u>Assumption 1</u>: For each consumer i, $\omega^i \in \mathbb{R}_+^{H+1}$, and $\omega_0^i > 0$. Also, $\sum_{i=1}^{I} \omega_h^i > 0$ for every commodity h.

<u>Assumption 2</u>: For each consumer i, the utility function $u^i\colon \mathbb{R}_+^{H+1} \to \mathbb{R}$ is continuous and strictly quasi-concave. Also $u^i(\cdot)$ is strictly increasing. (That is, for any two vectors $x^i \neq y^i$ in $\mathbb{R}_+^{H+1}$, if $y_h^i \geq x_h^i$ for all h then $u^i(y^i) > u^i(x^i)$).

With these two assumptions, we can define consumer i's <u>consumption function</u> $X^i(p,e^S,e^D)$ so that;

(13) $$x^i = X^i(p,e^S,e^D) \text{ maximizes } u^i(x^i)$$

$$\text{subject to } \begin{cases} x_i^h \geq e_h^S \omega_h^i \quad \text{for all } h = 1,2,\ldots,H; \\ x_0^i \geq \sum_{i=1}^{H} \dfrac{p_h e_h^D}{1 - e_h^D} \max(0, x_h^i - \omega_h^i); \\ x_0^i + \sum_{h=1}^{H} p_h x_h^i = \omega_0^i + \sum_{h=1}^{H} p_h \omega_h^i \quad . \end{cases}$$

That is, $X^i(p,e^S,e^D)$ is the best feasible consumption bundle for consumer i when the price vector is p and the market excess supply and demand indexes are e^S and e^D. Asssumptions 1 and 2 are sufficient to imply that $X^i(p,e^S,e^D)$ is always well-defined, as long as

(14) $$p_h > 0 \quad , \quad 0 \leq e_h^S \leq 1 \quad , \quad \text{and} \quad 0 \leq e_h^D \leq 1$$

for every commodity $h = 1,2,\ldots,H$.

For the auctioneer's problem we can make at least one definition to streamline the notation. Since, by (1), at least one of e_h^S and e_h^D must always be zero, the auctioneer really only needs to announce one <u>market imbalance index</u> e_h satisfying

(15) $$-1 \leq e_h \leq +1 \quad , \quad \text{for every } h = 1,2,\ldots,H \quad .$$

Then the excess supply and excess demand indices can be defined from e_h by

(16) $$e_h^S = \max(0,e_h) \quad , \quad e_h^D = \max(0,-e_h) \quad .$$

Then we can redefine an auctioneer's solution to be any $(p,e) = (p_1,\ldots,p_H, e_1,\ldots,e_H)$ such that every $p_h > 0$, every $e_h \in [-1,+1]$, and

$$(17) \qquad \sum_{i=1}^{I} \omega_h^i = \sum_{i=1}^{I} X_h^i(p,e^S,e^D) \qquad (\forall\ h = 0,1,\dots,H)$$

where e^S and e^D are vectors defined according to (16).

4. Boundedness of Prices

There may be, in general, many auctioneer's solutions to an economy, as will be shown later in this paper. However these solutions do satisfy an important boundedness property: the price of a commodity cannot be raised arbitrarily high without having extreme excess supply. Similarly, the price of a commodity cannot be lowered to zero without having extreme shortage.

Theorem 1: Given $(\omega^i,u^i)_{i=1}^I$ satisfying Assumptions 1 and 2, let $(p(k),e(k))_{k=1}^\infty$ be any sequence of auctioneer's solutions, satisfying (17). Consider any commodity $h \neq 0$. If $\lim_{k\to\infty} p_h(k) = +\infty$, then $\lim_{k\to\infty} e_h(k) = 1$. If $\lim_{k\to\infty} p_h(k) = 0$, then $\lim_{k\to\infty} e_h(k) = -1$.

Proof: Let $x^i(k) = X^i(p(k),e^S(k),e^D(k))$ (where $e_h^S(k) = \max(0,e_h(k))$ and $e_h^D(k) = \max(0,-e_h(k))$). Observe that $0 \leq x_h^i(k) \leq \sum_{j=1}^{I} \omega_h^j$ since $(x^1(k),\dots,x^I(k))$ is a feasible allocation, and $-1 \leq e_h(k) \leq 1$ for all h. So $((e(k),x^1(k),\dots,x^I(k)))_{k=1}^\infty$ is a bounded sequence of vectors and must have a convergent subsequence. It suffices to prove that the theorem holds for every such subsequence. So henceforth we may assume that $(e(k),x^1(k),\dots,x^I(k))$ converges to some $(e,x^1,\dots,x^I)$.

To prove the first assertion, let us assume to the contrary that $\lim_k p_h(k) = +\infty$ but $\lim_k e_h(k) = e_k < 1$. We will show that this leads to a contradiction.

Select player i so that $x_h^i > e_h\omega_h^i$. (Such a player must exist since $\sum_i x_h^i = \sum_i \omega_h^i > e_h \sum_i \omega_h^i > 0$.) Define a sequence of vectors $(z^i(k))$ and z^i so that:

$$z_0^i(k) = x_0^i(k) + 1 \quad , \quad z_0^i = x_0^i + 1 \quad ;$$

$$z_h^i(k) = x_h^i(k) - \frac{1}{p_h(k)} \quad , \quad z_h^i = x_h^i \quad ; \text{ and}$$

$$x_g^i(k) = x_g^i(k) \quad , \quad z_g^i = x_g^i \quad , \text{ for all } g \text{ other than } 0 \text{ and } h \quad .$$

Observe that $\lim_k u^i(z^i(k)) = u^i(z^i) > u^i(x^i) = \lim_k u^i(x^i(k))$ because u^i is continuous and strictly increasing. But for k sufficiently large, $z_h^i(k) > e_h(k)\omega_h^i$, because $z_h^i = x_h^i > e_h\omega_h^i$. So for k sufficiently large, $z^i(k)$ is feasible for i with respect to $(p(k),e(k))$, and $u^i(z^i(k)) > u^i(x^i(k))$. This contradicts $x^i(k) = X^i(p(k),e^S(k),e^D(k))$, and proves that the first assertion must hold.

To prove the second assertion, let us again assume to the contrary that $\lim_k p_h(k) = 0$ but $\lim_k e_h(k) = e_h > -1$, so $e_h^D < 1$.

Select any player i, and define

$$\lambda(k) = p_h(k)/((1 - e_h^D(k))\omega_0^i) \quad .$$

Observe that $\lambda(k) \to 0$, since $\omega_0^i > 0$, $p_h(k) \to 0$, and $e_h^D(k) \to e_h^D < 1$. Let $y^i(k) = \lambda(k)\omega^i + (1 - \lambda(k))x^i(k)$. It is straightforward to check that $y^i(k)$ is feasible for i at $(p(k),e(k))$ (as long as $\lambda(k) \leq 1$)

and that the slack in constraint (11) must be at least $p_h(k)/(1-e_h^D(k))$. Thus it is possible to buy one more unit of h than at $y^i(k)$ and still satisfy the liquidity constraint. So now we define $(z^i(k))$ and z^i so that

$$z_0^i(k) = y_0^i(k) - p_h(k) \quad , \quad z_0^i = x_0^i \quad ;$$

$$z_h^i(k) = y_h^i(k) + 1 \quad , \quad z_h^i = x_h^i + 1 \quad ; \quad \text{and}$$

$$z_g^i(k) = y_g^i(k) \quad , \quad z_g^i = x_g^i \quad \text{for all } g \text{ other than } 0 \text{ and } h \ .$$

As before, we get $\lim_k u^i(z^i(k)) = u^i(z^i) > u^i(x^i) = \lim_k u^i(x^i(k))$ and $z^i(k)$ is feasible for i at $(p(k),e(k))$ when k is sufficiently large. This contradicts $x^i(k) = X^i(p(k),e^S(k),e^D(k))$, and proves the second assertion. Q.E.D.

5. Market Equilibria with Phillips Curves

The auctioneer's problem, as presented in the last section, gives him $2H$ independent parameters to choose (in p and e), while he has only H independent constraints to satisfy (in equation (17)). By the equation-counting rule, then, we are still H equations short of a fully determined model. But our story thus far has ignored the most basic postulate of price theory: that prices respond to imbalance between supply and demand. Translating this assumption into our model will give us the H additional equations.

Let us part with our auctioneer story, and consider how the equations of Section 3 look if interpreted as describing a short-run equilibrium of a real market economy. For each commodity we have a price and a market imbalance index. If the imbalance index is positive then it represents an unemployment rate or excess capacity rate for sellers of the commodity, and it tells then that they cannot expect to sell all of their endowment of this commodity (constraint (10)). If the imbalance index is negative, then it represents a shortage rate to buyers of the commodity, and it tells them that they may have to increase their liquidity to make any substantial purchases of this commodity (constraint (11)).

Let us suppose that, in the short run (say, during a one year period), prices are sticky but not completely rigid. Suppose that q_h was last years price for commodity h, and that the experiences of the last several years have led people to anticipate a rate of inflation ρ. If there is no major imbalance in the market for h, neither extreme excess capacity nor extreme shortage, then we should expect the price p_h this year to be close to $q_h(1 + \rho)$. If there is high unemployment, however, we would expect somewhat lower prices. If there is major shortage, then p_h should be much higher, as prices rise in response to the shortage. That is, we should expect to see some relationship between market imbalance and price of the form:

$$p_h = q_h(1 + \rho)\phi_h(e_h) \quad , \tag{18}$$

where $\phi_h\colon [-1,1] \to \mathbb{R}$ is some decreasing function. This function ϕ_h simply an adaptation of the Phillips curve [1958] to our model. We may take such Phillips curve functions ϕ_h as part of the exogenous structure of the economy, telling us how flexible prices are in response to market imbalance. Some prices may be very flexible and have steep ϕ_h curves, while other prices may be stickier and have flatter ϕ_h curves.

If we are only interested in studying short-term behavior of the economy in one period, then we may take last period's prices and inflation expectations as fixed parameters of the economy. (I would argue that, if prices are sticky over our time horizon, then inflation expectations should be fixed. Surely inflation expectations cannot be more flexible than the prices themselves.) Then we can replace (18) by a more general condition of the form

$$(19) \qquad p_h \in \Phi_h(e_h) \quad , \quad (h = 1,2,\ldots,H)$$

where $\Phi_h\colon [-1,1] \Rightarrow \mathbb{R}_+$ is a point-to-set correspondence representing the generalized Phillips curve for commodity h. (To get (18) back, simply let $\Phi_h(e_h) = \{q_h(1+\rho)\phi_h(e_h)\}$.) Once the H generalized-Phillips curves $\Phi_1,\ldots,\Phi_H$ have been specified, then (19) gives us the H additional conditions promised.

To summarize, then, we can define our exchange economy by

$$(20) \qquad E = ((\omega^i,u^i)_{i=1}^{I}, (\Phi_h)_{h=1}^{H})$$

where each $\omega^i \in \mathbb{R}_+^{H+1}$, each u^i is a function from $\mathbb{R}_+^{H+1}$ to $\mathbb{R}$, and each Φ_h is a point-to-set correspondence from $[-1,1]$ to $\mathbb{R}_+$. We can now define a <u>market equilibrium</u> for E to be any $(p,e,x^1,\ldots,x^I)$ in $\mathbb{R}^H \times \mathbb{R}^H \times (\mathbb{R}^{H+1})^I$ such that

$$(21) \qquad -1 \le e_h \le 1 \text{ and } 0 < p_h \in \Phi_h(e_h) \quad , \quad \text{for every } h = 1,2,\ldots,H \quad ;$$

$$(22) \qquad \sum_{i=1}^{I} \omega_h^i = \sum_{i=1}^{I} x_h^i \quad , \quad \text{for every } h = 0,1,\ldots,H \quad ; \text{ and}$$

(23) $\quad x^i = X^i(p,e^S,e^D)$, for every $i = 1,2,\ldots,I$,

where $X^i(\cdot)$ is as in (13), and e^S and e^D are derived from e as in (16). That is, at a market equilibrium, the optimal consumption bundles demanded by the consumers should add up to the aggregate endowment available, and prices should respond to shortages or excess capacity according to the generalized Phillips curve.

6. The Class of Acceptable Phillips Curves

For each commodity h, $\Phi_h(\cdot)$ describes the relative adjustment speeds of the price and the imbalance index for h, so we want to allow the widest possible class of such curves, from vertical (representing fixed imbalance) to horizontal (representing fixed price). However, we will need to impose one regularity condition on the $\Phi_h(\cdot)$, to prove that equilibria actually exist.

Assumption 3: For every commodity $h = 1,\ldots,H$, there exists a continuous function $G_h\colon \mathbb{R}_+ \times [-1,1] \to \mathbb{R}$ such that, for some numbers $\varepsilon > 0$ and $\delta > 0$, the following three statements are true at every (p_h,e_h) in $\mathbb{R}_+ \times [-1,1]$:

(a) if $p_h \geq \frac{1}{\delta}$ and $e_h \geq 1 - \varepsilon$ then $G_h(p_h,e_h) > 0$;

(b) if $p_h \leq \delta$ and $e_h \leq -1 + \varepsilon$ then $G_h(p_h,e_h) < 0$; and

(c) $p_h \in \Phi_h(e_h)$ if and only if $G(p_h,e_h) = 0$.

This assumption is satisfied for a wide class of correspondences Φ_h. For example, if $\Phi_h(e_h) = \{\phi_h(e_h)\}$, where $\phi_h\colon [-1,1] \to \mathbb{R}_+$ is any continuous positive-valued function, then letting $G_h(p_h,e_h) = p_h - \phi_h(e_h)$ will verify Assumption 3. So any continuous bounded Phillips curve will be acceptable.

Another interesting generalized Phillips curve is defined by $\Phi_h^W(0) = \mathbb{R}_+$ and $\Phi_h^W(e_h) = \emptyset$ (the empty set) for $e_h \neq 0$. This correspondence may be called the Walrasian Phillips curve, because if every Φ_h in the economy is of this form then all our market equilibria must have all $e_h = 0$ and therefore must coincide with ordinary Walrasian equilibria of the Arrrow-Debreu model. Letting $G_h(p_h,e_h) = e_h$, we can see that Assumption 3 is satisfied for the Walrasian Phillips curve.

The Keynesian Phillips curves are of the form $\Phi_h^K(e_h) = \{q_h\}$ if $e_h > 0$, $\Phi_h^K(e_h) = \emptyset$ if $e_h < 0$, and $\Phi_h^K(0) = [q_h,\infty)$. This kind of Phillips curve represents a price which is downwardly rigid at q_h if there is excess capacity or unemployment, but which is infinitely flexible upwards if there is shortage. Letting $G_h(p_h,e_h) = \min\,(p_h - q_h, e_h)$ verifies Assumption 3 for these Keynesian Phillips curves.

One other interesting class of Phillips curves are of the form:

$$\Phi_h(e_h) = \{q_h\} \text{ if } a_h < e_h < b_h\ ,\quad \Phi_h(a_h) = [q_h,\infty)\ ,\quad \Phi_h(b_h) = [0,q_h]$$

$$\text{and } \Phi_h(e_h) = \emptyset \text{ if } e_h < a_h \text{ or } e_h > b_h\ .$$

(This assumes that $-1 \leq a_h < b_h \leq 1$ and $q_h > 0$.) This kind of step-shaped Phillips curve describes a price which is fixed at q_h for moderate market imbalance, $a_h < e_h < b_h$. But if increasing shortage (or decreasing unemployment) pushes the imbalance down to a_h then the price can rise freely; and if unemployment reaches the high level of b_h then the price can fall freely. Letting $G_h(p_h,e_h) = \min\,(e_h - a_h, \max(e_h - b_h, p_h - q_h))$ will verify Assumption 3 for this Φ_h. These step-shaped

Phillips curves only permit market imbalance to be in the limited range $[a_h, b_h]$, which suggests they may be useful for describing economies in which shortage and unemployed capacity do not reach the extreme levels thus far allowed in our general model. (In particular, our constraint (11) may look more realistic if we know that e_h^D will never go above 1/2, that is, $e_h \geq -1/2$.)

7. Existence of Market Equilibria

We can now state our basic existence theorem.

Theorem 2: Under Assumptions 1, 2, and 3, there must exist at least one market equilibrium $(p,e,x^1,\ldots,x^I)$, satisfying (21)-(23) for the economy $E = ((\omega^i,u^i)_{i=1}^I, (\Phi_h)_{h=1}^H)$.

Before proving Theorem 2, we state and prove a lemma.

Lemma: $X^i(p,e^S,e^D)$ is a continuous function on the set of all (p,e^S,e^D) which satisfy (14).

Proof of the Lemma: First, to establish that $X^i(\cdot)$ is a function, observe that the set of all points satisfying the constraints in (13) is compact and convex. (10) consists of linear inequalities in x^i, (11) constrains x_0^i to be greater than or equal to an expression which is convex in x^i, and (12) is a linear equation in x^i, so the set of feasible points is convex. As long as all prices are positive, (12) also forces the feasible set to be compact, since (10) and (11) imply that all x_h^i are nonnegative. The feasible set is nonempty, since $x^i = \omega^i$ satisfies all the constraints. By Assumption 2, $u^i(\cdot)$ is strictly quasi-concave and continuous, so it must have a unique maximum point in the feasible set of (13). So $X^i(p,e^S,e^D)$ is well-defined.

Let $(p(k),e^S(k),e^D(k))_{k=1}^{\infty}$ be any sequence of points satisfying (14) and converging to (p,e^S,e^D). Let $x(k) = X^i(p(k),e^S(k),e^D(k))$. We will show that $\lim_{k\to\infty} x(k) = X^i(p,e^S,e^D)$, which will prove continuity.

Since $(p_h(k))_{k=1}^{\infty}$ is a sequence of positive numbers converging to $p_h > 0$, there must exist $\bar{p}_h$ and $\underline{p}_h$ such that $\bar{p}_h \geq p_h(k) \geq \underline{p}_h$. Then, for all k,

$$0 \leq x_h(k) \leq (\omega_0^i + \sum_{g=1}^{H} \bar{p}_g \omega_g^i)/\underline{p}_h \quad .$$

So the $x(k)$ points remain within a compact set. To show that $\lim_{k\to\infty} x(k) = X^i(p,e^S,e^D)$, it suffices to show this for any convergent subsequence. Since we are only concerned with convergent subsequences, we may as well asssume that $(x(k))_{k=1}^{\infty}$ itself is convergent. Let $x^i = \lim_{k\to\infty} x(k)$. We want to show that $x^i = X^i(p,e^S,e^D)$.

We show first that x^i satisfies all the constraints of (13) for (p,e^S,e^D). If $e_h^D = 1$, then we must have $x_h^i \leq \omega_h^i$, because otherwise $((p_h(k)e_h^D(k)/(1 - e_h^D(k)) \cdot \max\,(0,x_h(k) - \omega_h^i))$ would diverge to $+\infty$, contradicting (11) for $x(k)$. Thus (11') will hold. All the other terms in (11), and all the terms in (10) and (12) are continuous at (x^i,p,e^S,e^D) so these constraints must hold at (x^i,p,e^S,e^D), since they hold at all $(x(k),p(k),e^S(k),e^D(k))$. So x^i is feasible for i at (p,e^S,e^D).

Let $\bar{x}^i = X^i(p,e^S,e^D)$. We still must show that $x^i = \bar{x}^i$. Define a sequence $(y(k))_{k=1}^{\infty}$ of vectors in $\mathbb{R}^{H+1}$ by:

$$y_h(k) = \max\,(\bar{x}_h^i, e_h^S(k)\omega_h^i) \quad \text{for all} \quad h = 1,\ldots,H \quad ,$$

$$y_0(k) = \omega_0^i + \sum_{h=1}^{H} p_h(k)\cdot(\omega_h^i - y_h(k)) \quad .$$

Clearly $y(k)$ satisfies (10) and (12) for $(p(k),e^S(k),e^D(k))$. Also notice that $y_h(k) \leq \omega_h^i$ if $e_h^D = 1$, because $e_h^S(k)\omega_h^i \leq \omega_h^i$ and $\bar{x}_h^i(p,e^S,e^D) \leq \omega_h^i$ if $e_h^D = 1$. So $y(k)$ also satisfies (11') for $(p(k),e^S(k),e^D(k))$. (We may assume that $e_h^D(k) \neq 1$ unless $e_h^D = 1$, since $e_h^D(k) \to e_h^D$.) But $y(k)$ need not satisfy (11), so we define:

$$\gamma(k) = \sum_{h=1}^{H} \frac{p_h(k)\, e_h^D(k)}{1 - e_h^D(k)}\cdot\max\,(0, y_h(k) - \omega_h^i) \quad ;$$

$$\lambda(k) = \max\,(0, \frac{\gamma(k) - y_0(k)}{\gamma(k) - y_0(k) + \omega_0^i}) \quad ; \quad \text{and}$$

$$z(k) = \lambda(k)\omega^i + (1 - \lambda(k))y(k) \quad .$$

Then $z(k)$ satisfies (10), (11), and (12) for $(p(k),e^S(k),e^D(k))$. And so $u^i(z(k)) \leq u^i(x(k))$, since the $x(k)$ were optimal choices for $(p(k),e^S(k),e^D(k))$.

Now observe that $\lim_k e_h^D(k)\omega_h^i = e_h^D\omega_h^i \leq \bar{x}_h^i$. So $\lim_k y(k) = \bar{x}^i$. Then $\lim_k (\gamma(k) - y_0(k)) \leq 0$, since $\bar{x}^i$ satisfies (11) for (p,e^S,e^D). So $\lim \gamma(k) = 0$ (we use $\omega_0^i > 0$ here), and so $\lim_k z(k) = \bar{x}^i$. Then from the preceding paragraph,

$$u^i(\bar{x}^i) = \lim_k u^i(z(k)) \leq \lim_k u^i(x(k)) = u^i(x^i) \quad .$$

But x^i is feasible for i at (p,e^S,e^D) (see two paragraphs preceding) and $\bar{x}^i$ is optimal for i at (p,e^S,e^D). So $x^i = \bar{x}^i$, as we needed to prove the lemma. Q.E.D.

Proof of Theorem 2: Choose numbers $\varepsilon > 0$ and $\delta > 0$, and choose functions $G_h(\cdot)$ verifying Assumption 3 for every commodity h. If we choose ε and δ small enough, they will work for all H commodities at once. We may assume that $\delta < 1$. By Theorem 1 (choosing δ even smaller if necessary), we can also assume that, for any solution (p,e) satisfying (17), for every h, if $p_h \leq \delta$ then $e_h \leq -1 + \varepsilon$, and if $p_h \geq 1/\delta$ then $e_h \geq 1 - \varepsilon$.

Now we define a function

$$F: ([\delta,\tfrac{1}{\delta}]^H \times [-1,1]^H) \to ([\delta,\tfrac{1}{\delta}]^H \times [-1,1]^H)$$

so that $F(p,e) = (p',e')$ when

$$p_h' = \max\,(\delta,\min\,(\tfrac{1}{\delta}, p_h - G_h(p_h,e_h))) \quad ,$$

$$e_h' = \max\,(-1, \min\,(1, e_h + \sum_{i=1}^{I} (\omega_h^i - X_h^i(p,e^S,e^D))))$$

for every $h = 1,\ldots,H$. $F(\cdot)$ is continuous, since the $X^i(\cdot)$ are continuous, and e^S and e^D depend continuously on e by (16). Thus, by the Brouwer Fixed Point Theorem, there must exist some (p,e) in $[\delta,1/\delta]^H \times [-1,1]^H$ such that:

(24) $\qquad (p,e) = F(p,e) \quad .$

So let us choose (p,e) to satisfy (24), and let $x^i = X^i(p,e^S,e^D)$, using (16). We will show that $(p,e,x^1,\ldots,x^I)$ is a market equilibrium.

For every $h \neq 0$, (24) implies that one of the following three conditions must hold:

$$\text{either} \quad \sum_{i=1}^{I} (\omega_h^i - x_h^i) = 0 \quad ,$$

$$\text{or} \quad \sum_{i=1}^{I} (\omega_h^i - x_h^i) > 0 \quad \text{and} \quad e_h = 1 \quad ,$$

$$\text{or} \quad \sum_{i=1}^{I} (\omega_h^i - x_h^i) < 0 \quad \text{and} \quad e_h = -1 \quad .$$

But the second condition would violate (10) (since $e_h = 1$ means $e_h^S = 1$), and the third condition would violate (11'). Thus (22) (or (17)) must be satisfied for every $h = 1,\ldots H$. Then, by the usual Walras Law argument, (12) assures us that (22) holds for $h = 0$ as well.

For every $h \neq 0$, (24) also implies that one of the following three conditions must hold:

$$\text{either} \quad G_h(p_h,e_h) = 0 \quad ,$$

$$\text{or} \quad G_h(p_h,e_h) > 0 \quad \text{and} \quad p_h = \delta \ ,$$

$$\text{or} \quad G_h(p_h,e_h) < 0 \quad \text{and} \quad p_h = \frac{1}{\delta} \quad .$$

But (p,e) is an auctioneer's solution (by the preceding paragraph), so $p_h = \delta$ would imply $e_h \leq -1 + \varepsilon$ and $G_h(p_h,e_h) < 0$. Similarly, $p_h = 1/\delta$ would imply $e_h \geq 1 - \varepsilon$ and $G_h(p_h,e_h) > 0$. So $G_h(p_h,e_h) = 0$ and $p_h \in \Phi_h(e_h)$ for every $h = 1,\ldots,H$. Thus $(p,e,x^1,\ldots,x^I)$ is indeed a market equilibrium.

References

Benassy, J.P. [1975], "Neo-Keynesian Disequilibrium Theory in a Monetary Economy," Review of Economic Studies 42, 503-523.

Clower, R.W. [1965], "The Keynesian Counterrevolution: A Theoretial Appraisal," in Theory of Interest Rates, ed. by F.H. Frank and F.P.R. Brechling. London: Macmillan.

Debreu, G. [1959], Theory of Value, New York: Wiley.

Dreze, J. [1975], "Existence of an Equilibrium under Price Rigidity and Quantity Rationing," International Economic Review, 16, 301-320.

Grandmont, J.M. [1977], "Temporary General Equilibrium Theory," Econometrica, 45, 535-572.

Grandmont, J.M. and G. Laroque [1976], "On Temporary Keynesian Equilibria," Review of Economic Studies, 43, 53-67.

Heller, W.P. and R.M. Starr [1976], "Unemployment Equilibrium with Rational Expectations," Economics Series, Technical Report No. 229, Institute for Mathematical Studies in the Social Sciences, Stanford University.

Phillips, A.W. [1958], "The Relationship between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861-1957," Economica, 25, 283-299.