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THE NUMBER OF OUTCOMES IN THE PARETO-OPTIMAL SET OF DISCRETE BARGAINING GAMES

by

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Abstract

A set of n players is to bargain over which one of m possible outcomes will be chosen. The payoff of outcome i to player j is a random variable, X_{ij}. Al X_{ij} are assumed to be independent, and for a fixed player j all X_{ij} have the same continuous distribution function F_j. The mean and distribution of the number of outcomes in the Pareto-optimal set are shown to be invariant with respect to the distributions F_j, and are calculated for finite m. As $m \to \infty$ the mean is asymptotic to $(\log m + .577)^{n-1}/(n - 1)!$, and for $n = 2$, $m \to \infty$, the distribution approaches the normal distribution. The results are also applied to a problem in multiattribute utility theory. Suppose we wish to select an individual with high values on two personal attributes that are independent and continuously measurable. Of a world population of four billion, the efficient set would have an expectation of 22.7 individuals.

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Introduction

The feasible set S of a bargaining game is the set of all possible agreements that are at least as preferred by both players as a disagreement. An outcome in S in in the Pareto-optimal set PO if and only if no other outcome in S is strictly preferred by both players.

What is the mean proportion of the feasible set that is Pareto-optimal? More generally what is the distribution of the number of Pareto-optimal outcomes? This paper will provide some answers to these questions, under the assumptions that S is finite with payoffs that are random variables. The random variable X_{ij} is defined as the payoff to player i of outcome j. All X_{ij} are assumed independent. For a fixed player j all X_{ij} have the same continuous distribution function, F_j.

As an example of such a problem consider two people who wish to attend a movie together. Assume movies arrive in town by some probabilistic process, such that the utilities of the two for seeing a movie are independent random variables. Assume that each is able to rank the movies without ties and that there are 20 theatres. It can then be shown that on the average 3.6 movies will be in the Pareto-optimal set.

It is surprising that an exact value can be calculated, but note that membership in PO depends only on the players' set of rankings, not on their cardinal utility evaluations. Since all sets of rankings are equiprobable and there are no ties, various statistics concerning the size PO can be calculated, independent of the exact shape of the distributions F_j.

If A is a set of outcomes, PO(A) will designate those outcomes of A that are Pareto-optimal with respect to A. The set PO(S), those outcomes Pareto-optimal with respect to the entire feasible set, will usually be abbreviated as PO. $K_{m,n}$ is a random variable whose value is the number of outcomes in PO, where the utilities of the m outcomes in S for n players are chosen independently from continuous distributions, one for each player. $P_n(k, m)$ will designate the probability that $K_{m,n} = k$.

The following theorem gives the distribution of $K_{m,2}$.

Theorem 1:

$$P_2(k, m) = \frac{m-1}{m} P_2(k, m-1) + \frac{1}{m} P_2(k-1, m-1), \qquad (1)$$

where $P_2(k, 0) = P_2(0, m) = 0$ for $k, m \geq 1$, and $P_2(0, 0) = 1$.

Proof: Let the outcomes in S be designated $O_1, O_2 \ldots O_m$ according to their rank of preference by player 1, such that player 1 prefers O_1 to $O_2 \ldots$ to O_m. Let π be a permutation of S giving the rank of preference by player 2, i.e., player 2 prefers $\pi(O_1)$ to $\pi(O_2) \ldots$ to $\pi(O_m)$.

Figure 1 about here

We can compare PO(S) with PO(S - $\{O_m\}$). The addition of O_m will not eliminate from PO(S) outcomes that were in PO(S - $\{O_m\}$) since O_m is player 1's least preferred outcome. PO(S) is a superset of PO(S - $\{O_m\}$), either identical to it or containing a further point, O_m. Therefore $P_2(k, m)$ is the sum of the probabilities of two events:

1) $O_m \in PO(S)$ and $|PO(S - \{O_m\})| = k - 1$

2) $O_m \notin PO(S)$ and $|PO(S - \{O_m\})| = k$.

Let π' be the permutation giving the rank of preference by player 2 over the set S - $\{O_m\}$. The event that $|PO(S)| = k$ depends entirely on π

and likewise the event that $|PO(S - \{O_m\}|$ = k or k - 1 depends entirely on π'. Since payoffs X_{ij} are independent, all permutations π of m outcomes are equiprobable, and all permutations π' are also equiprobable. Therefore the probabilities that $|PO(S - \{O_m\})| = k$, and $k - 1$, are $P_2(k, m - 1)$ and $P_2(k - 1, m - 1)$ respectively.

Outcome O_m is in PO(S) if and only if it is first in preference for player 2. Since all permutation π of m outcomes are equiprobable, this event occurs with probability 1/m and is independent of the size of $PO(S - \{O_m\})$.

Therefore,

$$P_2(k, m) = \frac{1}{m} P_2(k - 1, m - 1) + \frac{m-1}{m} P_2(k, m - 1)$$

which is the formula to be proved. The initial conditions stated in the theorem are easily verified. □

The mean and variance of $K_{m,2}$ are

$$E(K_{m,2}) = \sum_{i=1}^{m} \frac{1}{j}, \text{ and} \tag{2}$$

$$Var(K_{m,2}) = \sum_{i=1}^{m} \frac{1}{j} - \frac{1}{j^2} . \tag{3}$$

Both of these expressions can be derived from Theorem 1, but (2) can also be proved directly by using the argument of Theorem 1 with $P_2(k, m)$ replaced by $E(K_{m,2})$.

It follows from (1) that the probability generating function for $P_2(k, m)$ is

$$\varphi_{m,2}(t) \equiv t(t + 1) \ldots (t + m - 1) \tag{4}$$

$\varphi_{m,2}(t)$ is also the probability generating function of $s(m, k)$, the signless Stirling numbers of the first kind (Riordan, 1958, p. 71) and thus

$$P_2(k, m) = (-1)^{m+k} s(m, k), \tag{5}$$

The expression (4) is also the generating function for the probabilities that a permutation of length m has exactly k cycles. A number of results that have appeared in the literature of permutations and Stirling numbers can be applied directly. Formulae (1), (2) and (3) arise also in a sampling problem discussed by Vout (1973).

The following theorem gives the asymptotic distribution of $K_{m,2}$. It was proved in the context of permutation cycles by Feller (1968, p. 258).

Theorem 2. Let C be the Euler's constant, that is, $C = .577216^{-}$. Then, the distribution of

$$\frac{K_{m,2} - (\log m + C)}{(\log m + C - \pi^2/6)^{\frac{1}{2}}}, \tag{6}$$

approaches the standard normal distribution in the limit as $m \to \infty$.

Proof. The mean of $K_{m,2}$ is asymptotic to $\log m + C$, since by (2) it is the sum of the first m terms of the harmonic series, $1/i$. The variance is asymptotic to $\log m + C - \pi^2/6$ since it is the term by term difference of the two series $1/i$ and $1/i^2$.

To show that the random variable (6) has a limiting distribution that is normal, label the possible agreements O_1 to O_m in order of preference by player 1, agreement O_1 being the most preferred.

Associate with each outcome O_i, the random variable X_i:

$$X_i = 0 \qquad \text{if} \qquad O_i \notin PO$$

$$X_i = 1 \qquad \text{if} \qquad O_i \in PO.$$

The X_i are independently (although not identically) distributed binomial distributions with $\text{Prob}(X_i = 1) = 1/i$, and $\text{Var}(X_i) = (i - 1)/i^2$. The number $K_{m,2}$ of Pareto-optimal outcomes is $\sum X_i$. The random variables $(X_i - (\log m + C))/(\log m + C)^{\frac{1}{2}}$ satisfy the conditions of Lindeberg's central limit theorem (Feller, 1968), and thus (6) approaches the standard normal distribution. □

The following theorem, which was pointed out to the author by Roger Myerson, generalizes formula (2) for the mean to the case of three or more bargainers.

<u>Theorem 3</u>:

$$E(K_{m,n}) = \sum_{i=1}^{m} (-1)^i \binom{m}{i} \frac{1}{i^{n-1}} \tag{7}$$

<u>Proof</u>:

Let O_m be player 1's least preferred outcome and let X_m be the random variable:

$$X_m = 0 \qquad \text{if} \qquad O_m \notin PO$$

$$X_m = 1 \qquad \text{if} \qquad O_m \in PO.$$

Then $K_{m,n} = K_{m-1,n} + X_m$ where $K_{m-1,n}$ is the number of Pareto-optimal points in $S - \{O_m\}$. It follows that

$$E(K_{m,n}) = E(K_{m-1,n}) + E(X_m). \tag{8}$$

Now O_m will be Pareto-optimal if and only if it is Pareto-optimal among the other n - 1 players, excluding player 1. This event has expectation $\frac{1}{m}$ $E(K_{m,n-1})$, which yields:

$$E(K_{m,n}) = E(K_{m-1,n}) + \frac{1}{m} E(K_{m,n-1}) \tag{9}$$

where

$$E(K_{0,n}) = 0, \quad E(K_{m,1}) = 1.$$

Formula (7) satisfies the above initial conditions. To show that it also satisfies recursion (9), we begin by defining

$$f_{m,n} \equiv \sum_{i=1}^{m} (-1)^i \binom{m}{i} \frac{1}{i^{n-1}}$$

Then $f_{m,n} = \sum_{i-1}^{m} (-1)^i [\binom{m-1}{i} + \binom{m-1}{i-1}] \frac{1}{i^{n-1}}$

$$= f_{m-1,n} + \sum_{i=1}^{m} (-1)^i \frac{(m-1)!}{(i-1)!(m-i)!} \frac{1}{i^{n-1}}$$

$$= f_{m-1,n} + \frac{1}{m} \sum_{i=1}^{m} (-1)^i \frac{m!}{i!(m-1)!} \frac{1}{i^{n-2}}$$

$$= f_{m-1,n} + \frac{1}{m} f_{m,n-1}.$$

This proves the theorem. □

Since the random variables W_1 and $K_{m-1,n}$ are dependent formula (8) cannot be generalized to determine the entire distribution of $K_{m,n}$. Theorem 3 gives us a result for the mean only.

The following limit of $E(K_{m,n})$ can be calculated.

Theorem 4. As $m \to \infty$, $E(K_{m,n})$ is asymptotic to

$$(\log m + C)^{n-1}/(n-1)! \qquad (10)$$

Proof: Let

$$g_{m,n,j} = \sum_{i_1=1}^{m} \sum_{i_2=1}^{i_1} \cdots \sum_{i_j=1}^{i_{j-1}} \sum_{i_{j+1}=1}^{i_j-1} \frac{1}{i_1 i_2 \cdots i_{n-1}}$$

The role of index j is thus that for $k \le j$, the summation ranges from $i_k = 1$ to i_{k-1} (taking $i_0 = m$), whereas for $k > j$ the range is from $i_k = 1$ to $i_{k-1} - 1$.

The proof proceeds in three steps:

(i) $g_{m,n,1}$ is asymptotic to (10) as $m \to \infty$.

(ii) $g_{m,n,j+1}$ is asymptotic to $g_{m,n,j}$ as $m \to \infty$.

(iii) $g_{m,n,n-1} = E(K_{m,n})$.

(i) Jordan (1939, Ch. IV) has shown that

$$\begin{aligned} g_{m,n,1} &= (-1)^{m+n+1} s(m-1, n)/m! \\ &\sim (\log (m+1) + C)^{n-1}/(n-1)! \qquad \text{as } m \to \infty, \end{aligned}$$

where $s(m + 1, n)$ is a Stirling number of the first kind.

Since $\log(m + 1) \sim \log m$, it follows that $g_{m,n,1}$ is asymptotic to the expression given in (10).

(ii) Suppose $g_{m,n,j}$ is asymptotic to (10). $g_{m,n,j+1}$ can be expressed

$$g_{m,n,j+1} = g_{m,n,j} + \sum_{i_1=1}^{m} \cdots \sum_{i_{j+1}=i_j}^{i_j} \cdots \sum_{i_{n-1}=1}^{i_{n-2}-1} \frac{1}{i_1 i_2 \cdots i_{n-1}} .$$

Dividing each side by $(\log m + C)^{n-1}/(n-1)!$ gives

Then
$$g_{m,n,j+1} \times \frac{(n-1)!}{(\log m+C)^{n-1}} = g_{m,n,j} \frac{(n-1)!}{(\log m+C)^{n-1}} \tag{11}$$

$$+ (n-1)! \sum_{i_1=1}^{m} \frac{k}{i_1} \cdots \sum_{i_{j+1}=i_j}^{i_j} \frac{k}{i_{j+1}} \cdots \sum_{i_{n-1}=1}^{i_{n-1}-1} \frac{k}{i_{n-1}}$$

where $k = 1/(\log m + C)$.

In the right hand term the sum indexed by i_{n-1} must be less than 1 for all values of m since $1/k > \sum_{i=1}^{\ell} \frac{1}{i}$ for all finite ℓ. Thus the sum indexed by i_{n-2} must also be less than 1. This argument may be repeated up to the index i_{j+1}. This sum has only one term and therefore tends to zero, and both sides of (11) approach 1.

(iii) Replacing $K_{m,n}$ by $g_{m,n,n-1}$ in (a), $g_{m,n,n-1}$ clearly satisfies the initial conditions of (9). To show the recursive formula is also satisfied, $g_{m,n,n-1}$ may be expressed as

$$\sum_{i_1=1}^{m-1} \sum_{i_2=1}^{i_1} \cdots \sum_{i_{n-1}=1}^{i_{n-2}} \frac{1}{i_1 i_2 \cdots i_{n-1}} + \sum_{i_2=1}^{m} \cdots \sum_{i_{n-1}=1}^{i_{n-2}} \frac{1}{m i_2 \cdots i_{n-1}}$$

$$= g_{m-1,n,n-1} + \frac{1}{m} g_{m,n-1,n-2}$$

This proves the theorem. □

Calculations of the expected size of PO, based on Theorems 1, 3 and 4 are given in Table 1.

m \ n	1	2	3	4	5	10	15
1	1	1	1	1	1	1	1
2	1	1.50	1.75	1.88	1.94	1.998	1.9999
3	1	1.83	2.36	2.66	2.82	2.994	2.9998
4	1	2.08	2.88	3.38	3.67	3.988	3.9996
5	1	2.28	3.34	4.05	4.48	4.981	4.9994
6	1	2.45	3.75	4.67	5.26	5.972	5.9091
7	1	2.59	4.12	5.26	6.01	6.961	6.999
8	1	2.72	4.46	5.82	6.74	7.948	7.998
9	1	2.83	4.77	6.35	7.44	8.933	8.998
10	1	2.93	5.06	6.86	8.13	9.918	9.997
20	1	3.60	7.27	11.03	14.12	19.67	19.99
50	1	4.50	10.93	19.2	27.7	48.18	49.93
100	1	5.19	14.3	27.9	43.9	93.8	99.72
1000	1	7.49	28.8	76.5	157.	765.	980.2
10^4	1	9.79	48.7	164.	426.	4947.	9116.5
10^{6*}	1	14.39	103.6	497.	1788.	73030.	187,800.
$4 \times 10^{9*}$	1	22.7	257.3	1946.	11040.	4.39×10^6	1.098×10^8

* based on the approximation of Theorem 4.

m size of feasible set

n number of players

Table 1. Values of $E(K_{m,n})$

Approximations of $E(K_{m,n})$ based on Theorem 4 are compared with exact values in Table 2. It is clear that the approximation is accurate for small n but as n grows to 5 or larger, very high values of m are required to give acceptable accuracy.

m \ n	2	3	5	10
100	5.18 (5.19)	13.4 (14.3		
1000	7.48 (7.49)	28.0 (28.8)	131 (157)	
10000	9.79 (9.79)	47.9 (48.7)	382 (426)	2271 (4947)

m = size of feasible set

n = number of players

Table 2. Approximations of $E(K_{m,n})$ based on Theorem 4. Exact values appear in parentheses.

The determination of $P_n(k, m)$ for $n \geq 3$ is more difficult. A method will be outlined for $n \geq 3$. It can be applied to greater numbers of players but the calculations become exceedingly long.

Let A be a set of pairs of integers less than or equal to m: $A \subset I_m \times I_m$, and let $i_1, i_2 \in I_m$. We define a function onto the subsets of A as follows:

$$F[A, (i_1, i_2)] = \{(j_1, j_2) | (j_1, j_2) \in A;\ i_1 \neq j_1;\ i_2 \neq j_2;$$
$$(j_1 > i_1) \text{ or } (j_2 > i_2)\} \text{ where } i_1, i_2 \in I_m.$$

Let a be player 3's favorite outcome and let $r_1(a)$ and $r_2(a)$ be the ranks of preference for a by players 1 and 2 respectively, the most preferred outcomes being ranked m. The set of possible values for $(r_1(a), r_2(a))$ is $I_m \times I_m$. Outcome a will be Pareto-optimal, but further Pareto-optimal outcomes can arise with rank vectors only in the set

$$F[I_m \times I_m, (r_1(a), r_2(a))]$$

since other outcomes will be dominated by a.

Thus,

$$P_3(k, m) = \sum_{(i_1, i_2) \in I_m \times I_m} P(k - 1, F[I_m \times I_m, (i_1, i_2)])$$

where P(k, A) designates the probability of there being exactly k outcomes that both are Pareto-optimal and also have rank vectors for players 1 and 2 in the set A, where $k \in I_m$ and $A \subset I_m \times I_m$.

The values of P are determined by arguments similar to those just given but applied to A rather than $I_m \times I_m$. That is, considering all possible rankings for players 1 and 2, for player 1's favorite within A gives

$$P(k, A) = \sum_{(i_1, i_2) \in A} P(k - 1, F[A, (i_1, i_2)]),$$

where

$$P(0, A) = 1 \quad \text{if} \quad |A| = 0$$
$$= 0 \quad \text{otherwise.}$$

Various equivalences among the sets A can be exploited to speed computation. Standard deviations of $K_{m,3}$ determined by this method are given in Table 3.

m	1	2	3	4	5	6
$E(K_{m,3})$	1.00	1.75	2.36	2.88	3.34	3.75
Standard deviation	.000	.435	.673	.830	1.03	1.22

m = size of feasible set

Table 3. Means and standard deviations of $K_{m,3}$

Discussion:

It has been suggested that Pareto-optimality is a very weak limitation on the bargainers' possible agreements. Table 1 suggests the opposite in our view, especially for two bargainers and large m. For example if 1000 agreements are possible the expected size of PO is 7.49. By the normal approximation of Theorem 2, the actual size will be 11 or fewer with probability .92.

The size of PO rises sharply with n, reflecting the difficulties of decision-making encountered by large groups.

The concept of efficiency in multiattribute decision theory is equivalent to Pareto-optimality. Instead of two bargainers we have two attributes for each of m courses of action. A decision-maker wishes to have as high a value as possible of both attributes. Any choice will therefore lie in the efficient set.

As a fanciful application to a multiattribute decision problem, suppose that we wish to select an individual from the world's population based on two desirable qualities. We want an individual with as high values of the attributes as possible, but we have not decided on a rule to combine the values into a single measure of worth. The attributes are assumed to be precisely measurable and to be independent in the world's population. Of the four billion people on earth, the number of real contenders then has an expectation of 22.7, the rest being eliminated because they are not efficient choices since someone else is higher on both measures. With odds of better than 100 to 1 we would have to consider 34 individuals or fewer, according to Theorem 2.

It is very difficult to think of two qualities that are truly independent through the world's population, but this type of calculation can still be used as a guide. The expectation of 22.7 would be a rational estimate if we knew nothing about the form of the dependency. If a positive correlation were suspected, we could usually look forward to a smaller PO, and the expectation calculated above would give an upper bound.

References

Feller, William (1968). An Introduction to Probability Theory and Its Applications, Vol. I, 3rd edition. Wiley: New York.

Jordan, Charles (1939). Calculus of Finite Differences, reprint edition 1965. Chelsea: New York.

Riordan, John (1958). An Introduction to Combinatorial Analysis, Wiley: New York.

Vout, Colin (1973). Randomly Collecting Sets of Objects or . . . It's Another Spear-Thrower!, Eureka, No. 36, October 1973.

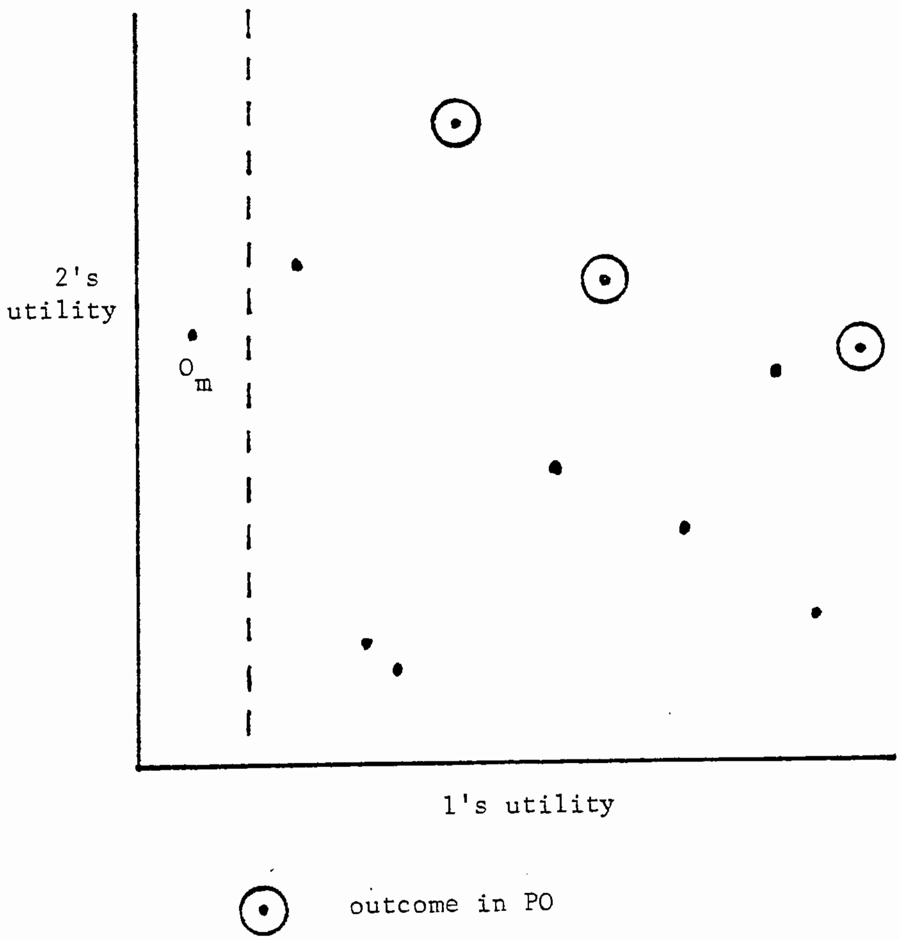

Figure 1.