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ON THE DESIGN OF OPTIMAL MECHANISMS FOR THE ARROW-HAHN-McKENZIE ECONOMY

by

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ABSTRACT

Considered is a neoclassical economy with depletable and non-depletable commodities where individual utilities and production possibilities may vary with the allocation. Presented for this economy are necessary and sufficient conditions for design of a first-order optimal allocation mechanism that allocates all resources in the economy, and a particular mechanism that satisfies these conditions. The basic mechanism design is the Groves-Ledyard mechanism generalized to include a rule for computing social prices for the depletable commodities and a rule for computing each consumer's initial wealth. The particular mechanism proposed is examined with three different price rules. The first involves an implicit auctioneering process relative to which the mechanism is individually incentive compatible and thus optimal. With the second price rule, income effects occur as a result of price manipulation by agents, and the mechanism is not optimal in general. The third price rule is an averaging rule employed so as to circumvent the problem of income effects. With this price rule, the mechanism is first-order optimal, but is incentive compatible if and only if all agents communicate the same vector of prices in equilibrium. Generally, this occurs only when the economy <u>and</u> the enforcement structure implicit in the mechanism are of special forms.

1. Introduction.

In this paper, we consider the problem of designing mechanisms that will allocate resources efficiently in the Arrow-Hahn-McKenzie (AHM) economy, an economy where individual utilities and production possibilities may vary with the allocation, agents being allowed to pursue their self-interest. The general mechanism design is similar to that of Groves and Ledyard [5]. However, here the interest centers on the allocation of depletable commodities (commodities that satisfy a scarcity constraint) that appear as initial resources rather than the allocation of non-depletable or public consumption commodities.

The consumption or production of any commodity may induce externalities in the AHM economy. Hence, an optimal allocation mechanism for this economy cannot include Walrasian markets unless agents are assumed to reveal correctly their preferences and production possibilities. Yet, prices somehow must enter the system if consumers' wealths are to be a well-defined function of their initial endowments. This has an obvious implication for the general mechanism design. The specification of a Groves-Ledyard type mechanism must be extended to include a rule that computes prices for commodities that appear as initial endowments. Thus, we must attend to the possibility and consequences of price manipulation by agents.

The mechanism that Groves and Ledyard [5] considered was interpreted as a government that communicates with consumers, and then represents them in existing markets for public goods, interacting with producers at Walrasian prices to determine an efficient allocation of public goods; consumers and producers interact in Walrasian markets to determine the allocation of private goods. Suchanek [10,11] extended the Groves-Ledyard government to include communication between the government and producers, doing away with the need for markets for public goods. The Groves-Ledyard-Suchanek governments are restricted to economies with public good externalities, and do not encompass the allocation of private goods. Suchanek [11] conjectured that his extension of the Groves-Ledyard government could be generalized to an optimal mechanism for the AHM economy.

Brock [3] has addressed the more general question of what conditions such allocation mechanisms must satisfy if they are to generate Pareto optimality. He limits his analysis to a pure exchange economy with consumption externalities, and he particularizes both the message space and the allocation rule.[1] Brock has conjectured also that his results can be extended to the AHM economy.

One purpose of this paper is to provide these extensions. In Section 2, we formulate a neoclassical economy that captures the essence of the AHM economy. Adding a special agent called government, we consider in Section 3 the general incentive design problem: characterize governments so that a Nash non-cooperative equilibrium allocation is Pareto optimal for the AHM economy. We derive a first-order design rule for optimality of allocation mechanisms without particularizing either the message space or the allocation rule. In Section 4, we consider a specific government

G* and show that it satisfies our design rule. For this government, the price rule is an implicit auctioneering process. Our major results are presented in Section 5 where we examine alternative price rules for the government G*. In particular, we suggest a price rule such that G* is incentive compatible and therefore optimal if the depletable commodities are pure private goods and a specific (reasonable) enforcement structure is assumed. Relaxing either of these restrictions voids the incentive compatibility of G* in general, but G* is still a first-order optimal mechanism.

2. The Model.

Let $\mathbb{R}^{L+Q}$ denote commodity space for L depletable and Q non-depletable commodities. Suppose there are I utility maximizing consumers indexed $i = 1,\ldots,I$ and J profit maximizing producers indexed $i = I + 1,\ldots, I + J = N$. Let $\omega_i \in \mathbb{R}^L$ be the initial endowment of the ith consumer, consisting of depletable commodities only, and suppose that a consumer's consumption of non-depletable commodities induces no externalities, reducing the complexity of the analysis without great loss of generality. Define $X = (x_1,\ldots,x_N) \in \mathbb{R}^{NL}$ and $Y = (y_{I+1},\ldots,y_N) \in \mathbb{R}^{JQ}$ where $x_i = (x_{i1},\ldots,x_{iL})$ denotes the ith agent's consumption or input-output vector of depletable commodities and $y_i = (y_{i1},\ldots,y_{iQ})$ denotes the ith producer's output vector of non-depletable commodities. For notational convenience, we assume that inputs are positive and outputs are negative.[2] Let $W = \{(X,Y) \in \mathbb{R}^{NL+JQ} | \Sigma_{i \le N} x_i \ge 0\}$. Then the set of feasible allocations for the economy is $W_0 = \{(X,Y) \in W | x_i \ge 0,\ y_j \le 0,\ i \le I,\ j > I,$ $\Sigma_{i \le N} x_i = \Sigma_{i \le I} \omega_i\}$.

Now suppose that $u^i\colon W \to \mathbb{R}$ is a $\mathcal{C}^2$ function that represents the utility function of the ith consumer, and $f^j\colon W \to \mathbb{R}^{K_j+Q}$, $K_j < L$ is a $\mathcal{C}^2$ function given in implicit form representing the production possibilities of the jth firm. Observe that $f^j = (f^{j1},\ldots,f^{jK_j},\ldots,f^{j(K_j+Q)})$, $f^{js(j)}\colon W \to \mathbb{R}$ such that $f^{js(j)}(X,Y) = 0$ for each $s(j) = 1,\ldots,K_j + Q$, $j = I + 1,\ldots,N$, $(X,Y) \in W$. Finally, let π_i represent the ith consumer's share of firms' aggregate profits. The economy is given by

$E = \{(u^i, \omega_i, \pi_i)_{i \leq I}, (f^j)_{j>I}\}$.

To the economy E, we append a special agent called government whose task it is to ensure that equilibrium allocations are first-order Pareto optimal. To accomplish this objective, the government administers a mechanism consisting of a price rule P that specifies a vector of social prices for the depletable commodities $p \in \mathbb{R}^L$, an allocation rule F that specifies the equilibrium allocation, tax rules $\{C_i, i \leq N\}$ that specify the contributions (perhaps negative) to be paid by agents, and consumer wealth rules that inform each consumer i of his profit share π_i and of the prices at which his initial endowment is to be valued. The government depends on information transmitted by agents in the form of messages m to administer the mechanism in a socially desirable manner. The government may restrict the messages agents may choose to send by announcing allowable message spaces $\{M^i, i \leq N\}$. Formally, the government is given by

<u>Definition 1</u>. $G = \{M, P, F, \{C^i, i = 1, \ldots, N\}, \{w^i, i = 1, \ldots, I\}\}$

where

(i) $M = \mathop{\times}\limits_{i=1}^{N} M^i$,

(ii) $P: M \to \mathbb{R}^L$, $P(m) = p$, where $P \in \mathcal{P}$, the space of possible price rules,

(iii) $F: M \times \mathcal{P} \to \mathbb{R}^{NL+JQ}$, $F(m,P) = (X,Y)$,

(iv) $C^i: M \times \mathcal{P} \to \mathbb{R}$, $C^i(m,P) = c^i$,

(v) $w^i: M \times \mathcal{P} \times \mathbb{R}^{L+1} \to \mathbb{R}$, $w^i(m,P,\omega_i,\pi_i) = w^i$.

When selecting a message to send to the government, agents are allowed to pursue their self-interest. That is, consumers may select messages that misrepresent their preferences, and producers may select messages that misrepresent their production costs. The ability of an agent to perpetrate a misrepresentation depends on whether it is detectable, and this depends on the existing enforcement structure. Here, one can think of the enforcement structure as being modeled implicitly in terms of the allowable misrepresentations. Thus, restrictions on the message space M would constitute an implicit enforcement structure, encapsulating what the government <u>knows</u> or <u>believes</u> to be true <u>a priori</u>.

<u>Remark 1.</u>

The price vector p serves, as usual, to evaluate the worth of each consumer's initial endowment; it does not appear in the model <u>explicitly</u> as a determinant of the terms of trade. Thus, to require the government to determine these prices is the only reasonable way (of which we are aware) that they may be introduced into the model considered here. Further to this point, our formulation also makes it necessary to not introduce explicitly prices for the non-depletable commodities, y. The significance of this remark is apparent when we observe that Brock's treatment of a pure exchange economy with consumption externalities may be extended to include production in two distinct ways, one being the model we examine here, and a second being a less general economy with markets and an auctioneer. The latter would introduce firms in the manner of Groves-Ledyard [5]; producer-producer and consumer-producer externalities could not be allowed, losing the spirit of the full generality of the AHM economy.

3. A First-Order Design Rule

Our goal is to determine sufficient conditions for a Nash equilibrium allocation in the economy E relative to a government G to be a first-order Pareto optimum. We first solve for the sufficient conditions for an allocation in E to be a Pareto optimum.

Following Smale [9], we say that $(X,Y)\in\theta$, the first-order Pareto set, if there does not exist an admissable curve $\varphi: (a,b) \subset \mathbb{R} \to W$ passing through (X,Y). A $\mathcal{C}^1$ curve φ is said to be admissable if

(a) $\frac{d}{dt}u^i(\varphi(t)) > 0$ all $t\in(a,b)$, $i = 1,\ldots,I$,

(b) $\frac{d}{dt}f^{js(j)}(\varphi(t)) \geq 0$ all $t\in(a,b)$, $s(j) = 1,\ldots,K_j + Q$, $j = I + 1,\ldots,N$,

(c) $\frac{d}{dt}g_r(\varphi(t)) \geq 0$ all $t\in(a,b)$, $r = 1,\ldots,L$ where $g_r(\varphi(t)) = \Sigma_{i\leq I}\omega_i - \Sigma_{i\leq N}x_i$.

Theorem 1 is an important application of Smale [9, Thm. B, p. 214] for it gives conditions sufficient for an allocation (X,Y) of the economy E to be in the Pareto set θ.[3]

Theorem 1. Let $(X,Y)\in W_0$. If (1) (non-degeneracy of constraints) $\{Df^{js(j)}(X,Y), s(j) = 1,\ldots,K_j + Q, j = I + 1,\ldots,N; Dg_r(X,Y), r = 1,\ldots,L\}$ is a linearly independent set for each $(X,Y)\in W_0$, and (2) there exist non-negative numbers $\lambda_i, \alpha_{js(j)}, \beta_r$, $i = 1,\ldots,I$,

$s(j) = 1,\ldots,K_j + Q$, $j = I + 1,\ldots,N$, $r = 1,\ldots,L$, not all zero such that

$$(3.1) \qquad \Sigma_{i\leq I}\lambda_i Du^i(X,Y) + \Sigma_{j>I}\Sigma_{s(j)}\alpha_{js(j)}Df^{js(j)}(X,Y) + \Sigma_r\beta_r Dg_r(X,Y) = 0,$$

then $(X,Y)\in\theta$.

Remark 2. If $(X,Y)\in W_0$ is such that there does not exist $(X',Y')\in W_0$ with $u^i(X',Y') \geq u^i(X,Y)$ all $i \leq I$ and $u^k(X',Y') > u^k(X,Y)$ some $k \leq I$, then $(X,Y)\in\theta$ provided the non-degeneracy condition is satisfied. The non-degeneracy condition is trivially satisfied if depletable commodities $x_{\cdot r}$ are pure private goods and non-depletable commodities $y_{\cdot q}$ are pure public goods. Finally, the multipliers β_r will turn out to be the social prices for commodities $x_{\cdot r}$.

We next define a Nash non-cooperative equilibrium for E relative to G, and derive the first order conditions. Let $m^{)i(} = (m^1,\ldots,m^{i-1},m^{i+1},\ldots,m^N)$ and $m/\bar{m}^i = (m^1,\ldots,m^{i-1},\bar{m}^i,m^{i+1},\ldots,m^N)$. A Nash non-cooperative equilibrium for our economy is a set of messages $m^* = (m^{1*},\ldots,m^{N*})$ such that

(3.2) for each $i = 1,\ldots,I$, $u^i(F(m^*,P))$ maximizes $u^i(F(m^*/m^i,P))$ subject to $C^i(m^*/m^i,P) \leq w^i$,

(3.3) for each $j = I + 1,\ldots,N$, $C^j(m^*,P)$ maximizes $C^j(m^*/m^j,P)$ subject to $f^j(F(m^*/m^j,P)) = 0$, (3.4) $\Sigma_{i\leq N} x_i^* = \Sigma_{i\leq I} \omega_i$ where $(X^*,Y^*) = F(m^*,P)$,

(3.5) $\Sigma_{i\leq N} C^i(m^*,P) = 0.$

To derive intelligible necessary conditions on the tax functions, we will assume they are of a special form. For a given (fixed) i, $m^{)i(}$, P, define $g^i(\bar{m}^i) = F(m/\bar{m}^i,P)$. We make the following assumptions on the behavior of agents and the tax and allocation functions:

<u>Assumption 1.</u> (Local Non-satiation) For each $i = 1,\ldots,I$, $(X,Y)\in W$ and $\varepsilon > 0$, there exists $(X',Y')\in b_\varepsilon(X,Y)$ such that $u^i(X',Y') > u^i(X,Y)$ where $b_\varepsilon(X,Y)$ is a ball of radius ε centered at (X,Y).

<u>Assumption 2.</u> Each agent i considers $\{\pi_k\}_k$ and $m^{)i(}$ to be fixed parameters while maximizing his or her utility function or subsidy function if a producer.

<u>Assumption 3.</u> Each agent's tax (subsidy) function is of the form $C^i(m/\bar{m}^i,P) = \hat{C}^i(g^i(\bar{m}^i),m^{)i(},P)$.

<u>Assumption 4.</u> For each $i = 1,\ldots,N$, $m^{)i(}$ and $P\in\mathcal{P}$, the function $g^i\colon M^i \to W$ is onto W_0; that is, for each feasible allocation $(X,Y)\in W_0$ and given $m^{)i(}$, P, there exists $\bar{m}^i\in M^i$ such that $g^i(\bar{m}^i) = (X,Y)$.

<u>Assumption 5.</u> For each $i = 1,\ldots,N$, m, P, the function $g^i\colon M^i \to W$ is surjective (infinitesimally onto all of W); that is,

given any vector $v \in W$, there is a choice of messages m_t^i with $m_0^i = m^i$ and $\frac{d}{dt} g^i(m_t^i)|_{t=0} = v$.[4]

Assumption 1 ensures that each consumer exhausts his wealth. Assumption 2 is a weak competitive assumption standard to the literature on optimal mechanisms. Specifically, it rules out otherwise possible multiplayer coalition and wealth effects due to manipulation of individual profits shares by consumers.[5] Assumption 3 means that the ith agent can influence the amount he or she is taxed or subsidized only indirectly through the allocation rule. This assumption allows us to write the tax functions as functions of the allocation (X,Y). Assumption 4 means that the allocation rule is such that the ith agent, given the messages of the other agents and the price rule specified by the government, can achieve any possible allocation by selecting an appropriate message. Assumption 5 implies that all possible partial derivatives of $\hat{C}^i$ on W can be obtained by infinitesimal changes in the ith message.[6]

Remark 3. One conceptual difficulty is that we assume infinitesimal changes of allocations even outside of the feasible set W_0. The mechanism we consider in Section 4 is of this form. To alleviate this difficulty, one could adjust assumption 5 to assume only infinitesimal changes in the feasible set, and then express everything in terms of gradients of the functions restricted to the feasible set. Still, only those points in the feasible set are relevant for most things.

Because of assumptions 1 through 5, solving (3.2) and (3.3) is equivalent to solving

(3.6) for each $i = 1,\ldots,I$, maximize $u^i(X,Y)$ subject to $\hat{C}^i(X, Y, m^{)i(}, P) \leq w^i$,

(3.7) for each $j = I + 1,\ldots,N$, maximize $\hat{C}^j(X, Y, m^{)j(}, P)$ subject to $f^j(X,Y) = 0$.

This game yields the following first-order conditions for an interior maximum:

$$
(3.8) \quad \begin{cases} \text{for each } i = 1,\ldots,I \text{ and } j = I + 1,\ldots,N, \\ \gamma_i^{-1} Du^i(X,Y) = D_{(X,Y)}\hat{C}^i(X,Y,m^{)i(},P) \\ \text{and} \\ D_{(X,Y)}\hat{C}^j(X,Y,m^{)j(},P) = \gamma_j DF^j(X,Y) \end{cases}
$$

where $\gamma_j DF^j(X,Y) = \Sigma_{s(j)=1}^{K_j+Q} \gamma_{js(j)} Df^{js(j)}$, and where $\gamma_i \neq 0$, $i = 1,\ldots,I$ and γ_j, $j = I + 1,\ldots,N$ are Lagrange multipliers.[7] From Theorem 1 and (3.8), we get Theorem 2 which provides conditions on the tax functions sufficient to imply that a Nash equilibrium allocation is first-order Pareto optimal.

Theorem 2. (First-Order Design Rule) Let $m^* \in M$ be a Nash equilibrium for E relative to G, and suppose $(X^*,Y^*) = F(m^*,P)$. If the non-degeneracy condition is satisfied, and if there exist real numbers $\beta_r \neq 0$, $r = 1,\ldots,L$ such that

$$(3.9) \qquad \Sigma_{i\leq N} D_{(X,Y)} \hat{C}^i(X^*,Y^*,m)^i(^*,P) = (B,0)\in\mathbb{R}^{NL+JQ}$$

where $B = (\beta,\ldots,\beta)\in\mathbb{R}^{NL}$, $\beta = (\beta_1,\ldots,\beta_L)$, then $(X^*,Y^*)\in\theta$.

To interpret Theorem 2, define p^i_{hr} to be the unit price (possibly negative) that agent h must pay agent i when h consumes x_{hr} to compensate for the external impact of his or her consumption on i. p^i_{ir} is to be thought of as the net unit price that i must pay when he or she consumes x_{ir}. Similarly, let $-t^j_{jq}$ denote the price of the qth non-depletable commodity that firm j must receive to produce y_{jq} (so that t^j_{jq} is the price firm j is willing to pay), and let t^i_{jq} denote the individualized price that agent i is willing to pay firm j to produce y_{jq}. Then, condition (3.1) is equivalent to

$$(3.10) \qquad \left.\begin{aligned} &\lambda_i u^i_{x_{hr}}(X,Y) = p^i_{hr},\ i = 1,\ldots,I \\ &\alpha_j f^j_{x_{hr}}(X,Y) = p^j_{x_{hr}},\ j = I+1,\ldots,N \end{aligned}\right\} h = 1,\ldots,N$$

where $\Sigma_{i\leq N} p^i_{hr} = \beta_r$ is the social price of commodity r, $r = 1,\ldots,L$.,

$$(3.11) \qquad \left.\begin{aligned} &\lambda_i u^i_{y_{kq}}(X,Y) = t^i_{kq},\ i = 1,\ldots,I \\ &\alpha_j f^j_{y_{kq}}(X,Y) = t^j_{kq},\ j = I+1,\ldots,N \\ &\text{where } \Sigma_{i\neq k} t^i_{kq} = -t^k_{kq},\ q = 1,\ldots,Q. \end{aligned}\right\} k = I+1,\ldots,N$$

The well-known first-order conditions for a private ownership economy with only pure private and pure public goods are apparent special cases of (3.10) and (3.11). Finally, condition (3.9) merely asserts that for (X*,Y*) to be Pareto optimal, the tax functions must be such that $\hat{C}^i_{x_{hr}} = p^i_{hr}$ and $\hat{C}^i_{y_{kq}} = t^i_{kq}$ when evaluated at (X*,Y*).

4. An Optimal Government for the AHM Economy.

Assume that the economy E satisfies the non-degeneracy condition, and define the government G* by

Definition 2. $G^* = \{M, P, F, \{C^i\}_i\}$ where

(i) $M = \times_{i=1}^{N} M^i$, $M^i = \{m^i : W \to \mathbb{R} \mid m^i \text{ is } \mathcal{C}^2\}$,

(ii) $P(m) = p \in \mathbb{R}^L$ for all $m \in M$ (the constant function p),

(iii) $F(m,p) = (X^*, Y^*)$ where (X^*, Y^*) maximizes
$\Sigma_{i \leq N} m^i(X,Y) + p(\Sigma_{i \leq I} \omega_i - \Sigma_{i \leq N} x_i)$,

(iv) $C^i(m,p) \equiv \hat{C}^i(X,Y,m^{)i(},p) = (\rho,0) \cdot (X,Y) - \Sigma_{\substack{k \neq i \\ k \leq N}} m^k(X,Y)$
$+ R^i(m^{)i(})$, for $i = 1,\ldots,N$ where $\rho = (p,\ldots,p) \in \mathbb{R}^{NL}$
and $R^i(\cdot)$ is an arbitrary real-valued function constant in m^i,

(v) $w^i = p \cdot \omega_i + \pi_i$, $i = 1,\ldots,I$.

The government G* is a generalization of the government considered in Suchanek [10] that was shown to satisfy the first-order conditions for Pareto optimality in an economy with only pure private and pure public goods. A message $m^i(\cdot)$ may be interpreted as the ith agent's reported willingness to pay function. With this interpretation given to the messages, it follows that the allocation rule specifies the allocation that, in equilibrium, maximizes reported social surplus since, by (3.4), $\Sigma_{i\leq N}\omega_i - \Sigma_{i\leq N}x_i = 0$ at an equilibrium allocation. The tax rule $\hat{C}^i(\cdot)$ taxes the ith consumer and subsidizes the ith producer an amount equal to the deviation of the reported willingness to pay of the other agents from the government's declared value of the allocation plus a lump sum transfer $R^i(\cdot)$. The transfer functions $\{R^i(\cdot)\}_i$ are balance terms used to guarantee condition (3.5), and are incentive neutral in view of assumption 2.

For these tax functions, we get

$$(4.1) \qquad D_{(X,Y)}\hat{C}^i(X,Y,m)^{i}(,p) = (\rho,0) - \Sigma_{k\neq i}Dm^k(X,Y),$$

$$i = 1,\ldots,N.$$

Since definition 2 (iii) implies that $\Sigma_i Dm^i(X^*,Y^*) = (\rho,0)$ in equilibrium, it follows that the ith agent will select a message $m^i(\cdot)$ so that, in equilibrium,

$$(4.2) \qquad D_{(X,Y)}\hat{C}^i(X^*,Y^*,m)^{i}(,p) = Dm^i(X^*,Y^*).$$

In short, each agent will, in equilibrium, select a message that communicates his or her <u>true marginal willingness to pay</u>. Summing on i, we get

$$(4.3) \quad \Sigma_{i\leq N}D_{(X,Y)}\hat{C}^i(X^*,Y^*,m)^{i}(,p) = \Sigma_{i\leq N}Dm^i(X^*,Y^*) = (\rho,0),$$

and this is just (3.9) as desired.

We also need to check that G* satisfies assumptions 2 to 5. Assumption 2 is built into the machinery, and the tax rules clearly satisfy assumption 3. To see that assumption 4 is satisfied, we note that, given p, $m^{)i(}$ and a feasible (X',Y'), the ith agent could send a message m^i that so heavily weighted (X',Y') that (X',Y') would be the maximum of $\Sigma_{i\leq N} m^i(X,Y) + p(\Sigma_{i\leq I} \omega_i - \Sigma_{i\leq N} x_i)$. Assumption 5 follows from a generalized implicit function theorem (the local submersion theorem [8, p. 20]). The equation that defines the allocation rule is

$$DH(m,X,Y) = \Sigma_i Dm^i(X,Y) - (\rho,0) = 0.$$

At a given point $(X^*,Y^*),m^*$, the nth agent can change the nth message so as to change the components of the gradient $Dm^n(X^*,Y^*)$ in an arbitrary way. This implies that we can realize all infinitesimal changes in H by making infinitesimal changes in m^n. By the local form of submersion, we can solve for m^n as a differentiable function of (X,Y), not uniquely: $m^n = \mu^n(X,Y)$ such that $g^m(\mu^n(X,Y)) = F(m^*/\mu^m(X,Y),p) = (X,Y)$. Then, given $v \in W$, we let $m_t^n = \mu^n((X^*,Y^*) + tv)$, so $g^n(m_t^n) = (X^*,Y^*) + tv$ and $\frac{d}{dt} g^n(m_t^n)|_{t=0} = v$.

5. Price Rules

Implicit in the behavior of the government G* is an auctioneering process to determine the equilibrium vector of prices, p. In this section, we consider alternative, one-step price rules that the government G* may use to compute prices.

It seems natural to suggest initially that the government use a price rule similar in spirit to the allocation rule. Such a price rule would be of the form $P(m) = \hat{P}(X,Y,m^{)i(})$. However, it is easy to verify that if the price rule is of this form, then Nash equilibrium allocations relative to G* are not generally in θ. Price manipulation by consumers via their messages introduce wealth effects through the terms $P(m)\cdot\omega_i$ making the mechanism sensitive to the distribution of initial endowments. Moreover, for each $i = 1,\ldots,N$, the gradients $D_{(X,Y)}\hat{C}^i$ would contain the extra term $D_{(X,Y)}\hat{P}(X,Y,m^{)i(})\cdot(X,Y)$. Since it is these characteristics that nullify the Pareto efficiency properties of G* established in Section 4, the price rule must be such that the mechanism operates independently of the distribution of initial endowments and $D_{(X,Y)}\hat{P}(X,Y,m^{)i(}) = 0$.

To satisfy these conditions, the price rule must have two components, one component being a function of all messages $P_1(m)$ that enters the allocation rule, and the other component being a function of all but the ith message $P_2(m^{)i(})$ that enters the ith agent's tax function and is used to determine the value of the ith agent's initial endowment if a consumer. A plausible conjecture is that, with such a price rule, the government G* would generate Pareto optimal equilibrium allocations if and only if $P_1(m) = P_2(m^{)i(}) = P_2(m^{)j(})$ for all i,j, especially if the depletable commodities are pure private goods. The basis for this conjecture is that if $P_2(m^{)i(}) \neq P_2(m^{)j(})$, then the ith and jth agents are valuing depletable commodities differently in the margin, and their initial endowments are also being valued differently. However, the conjecture is false in general. The "if" part of the conjecture is always true, but the "only if" part holds only when a non-trivial (although reasonable) enforcement structure is assumed and the depletable commodities are pure private goods.

Let the government G' be given by

<u>Definition 3</u>. $G' = \{M, P, F, \{C^i\}_i\}$ where

(i) $M = \times_{i \leq N} M^i, M^i = \{m^i = (\mu^i, p^i) \in \mathcal{C}^2(W, \mathbb{R}) \times \mathbb{R}^L\}$,[8]

(ii) $P = (P_1, P_2)$ where

(a) $P_1(m) = (\Sigma_{i \leq N} p^i / N) \in \mathbb{R}^L$

(b) $P_2(m^{)i(}) = [\Sigma_{\substack{k \leq N \\ k \neq i}} p^k / (N-1)] \in \mathbb{R}^L, i = 1, \ldots, N,$

(iii) $F(m, P_1) = (X^*, Y^*)$ where (X^*, Y^*) maximizes

$\Sigma_{i \leq N} \mu^i(X, Y) + P_1(m) \cdot (\Sigma_{i \leq I} \omega_i - \Sigma_{i \leq N} x_i)$,

(iv) $C^i(m,P_2) = \hat{C}^i(X,Y,m^{)i(},P_2) = (\rho^i,0)(X,Y) - \Sigma_{k\neq i}\mu^k(X,Y) + R^i(m^{)i(}),\ i = 1,\ldots,N$ where $\rho^i = (P_2(m^{)i(}),\ldots,P_2(m^{)i(}))\in\mathbb{R}^{NL}$,

(v) $w^i(m^{)i(},P_2,\omega_i,\pi_i) = P_2(m^{)i(})\cdot\omega_i + \pi_i,\ i = 1,\ldots,I.$

Theorem 3 shows that all agents communicate the <u>same</u> vector of prices in equilibrium if the depletable commodities are pure private goods and the government <u>believes</u> that any externality associated with the consumption or production of a depletable commodity by an agent i is either non-beneficial or non-detrimental but not both for all agents $j \neq i$. From this, it follows that, in equilibrium, each agent also communicates his true marginal willingness to pay for each commodity. Thus, the mechanism is incentive compatible and therefore optimal.

<u>Theorem 3</u>. Let $m^* = (\mu^*,p^*)$ be a Nash equilibrium for E relative to G', and suppose $(X^*,Y^*) = F(m^*,P_1)$. Assume E and G' are such that

(1) $D_X u^i(X,Y) = (0,\ldots,0,D_{X_i}u^i(X,Y),0,\ldots,0)$ and
$D_X f^j(X,Y) = (0,\ldots,0,D_{X_j}f^j(X,Y),0,\ldots,0)$,
$i = 1,\ldots,I,\ j = I+1,\ldots,N$,

(2) $D_Y f^{js(j)}(X,Y) \neq D_Y f^{ks(k)}(X,Y)$ for $(j,s(j)) \neq (k,s(k))$ and $(X,Y)\in W_0$, and

(3) if $m\in M$, then

(a) $\mu^i_{x_{hr}}(X,Y) < 0$ for some $i \neq h$ implies $\mu^j_{x_{hr}}(X,Y) \leq 0$ for all $j \neq h$, and

(b) $\mu^i_{x_{hr}}(X,Y) > 0$ for some $i \neq h$ implies $\mu^j_{x_{hr}}(X,Y) \geq 0$ for all $j \neq h$.

Then

(a) $p^{i*} = p^{j*}$ for all $i,j = 1,\ldots,N$,

(b) (incentive compatibility) $D_{(X,Y)}\hat{C}^i(X^*,Y^*,m)^{i(*},P_2) = D\mu^{i*}(X^*,Y^*)$, $i = 1,\ldots,N$, and

(c) $(X^*,Y^*)\in\theta$.

<u>Proof</u>. Let $\rho = (\Sigma_{n\leq N}p^{n*}/N,\ldots,\Sigma_{n\leq N}p^{n*}/N)\in\mathbb{R}^{NL}$, $\rho^i = (\Sigma_{k\neq i}p^{k*}/(N-1),\ldots,\Sigma_{k\neq i}p^{k*}/(N-1))\in\mathbb{R}^{NL}$ and $\tau^i = ((\Sigma_{n\leq N}p^{n*}-Np^{i*})/N(N-1),\ldots,(\Sigma_{n\leq N}p^{n*}-Np^{i*})/N(N-1))\in\mathbb{R}^{NL}$ for $i = 1,\ldots,N$.

By hypothesis, condition (3.8) gives

(5.1)
$$(0,\ldots,0,\gamma_i^{-1}D_{x_i}u^i(X^*,Y^*),0,\ldots,0) = (\rho^i,0) - \Sigma_{k\neq i}D\mu^{k*}(X^*,Y^*),\ i = 1,\ldots,I,$$
$$(0,\ldots,0,\gamma_j D_{x_j}f^j(X^*,Y^*),0,\ldots,0) = (\rho^j,0) - \Sigma_{k\neq j}D\mu^{k*}(X^*,Y^*),\ j = I+1,\ldots,N.$$

From definition 3 (iii), we get

(5.2)
$$\Sigma_{n\leq N}D\mu^{n*}(X^*,Y^*) = (\rho,0).$$

Therefore,

$$(5.3)\quad (\rho^i,0) - \Sigma_{k\neq i} D\mu^{k*}(X^*,Y^*) = (\tau^i,0) + D\mu^{i*}(X^*,Y^*),\ i = 1,\ldots,N.$$

Suppose that $\tau^i \neq 0$, some i. Then, for some $r = 1,\ldots,L$, $\Sigma_{n\leq N} p_r^{n*} - Np_r^{i*} \neq 0$. Assume (without loss of generality) that $\Sigma_{n\leq N} p_r^{n*} - Np_r^{i*} < 0$. By hypothesis (1), we have $0 = \frac{\Sigma_{n\leq N} p_r^{n*} - Np_r^{i*}}{N(N-1)} + \mu^{i*}_{x_{hr}}(X^*,Y^*)$ for all $h \neq i$. Therefore $\mu^{i*}_{x_{hr}}(X^*,Y^*) > 0$ for all $h \neq i$. By hypothesis (3), it follows that $\mu^{j*}_{x_{hr}}(X^*,Y^*) \geq 0$ for all $h \neq j$, all $j = 1,\ldots,N$.

But, $\Sigma_{n\leq N} p^{n*} - Np^{i*} < 0$ implies that there exists $j \neq i$ such that $\Sigma_{n\leq N} p^{n*} - Np^{j*} > 0$. Since $0 = \frac{\Sigma_{n\leq N} p^{n*} - Np^{j*}}{N(N-1)} + \mu^{j*}_{x_{hr}}(X^*,Y^*)$ for all $h \neq j$, it follows that $\mu^{j*}_{x_{hr}}(X^*,Y^*) < 0$ for all $h \neq j$, and this is a contradiction.

Hence, $\Sigma_{n\leq N} p_r^{n*} - Np_r^{i*} = 0$ for all $i = 1,\ldots,N$, $r = 1,\ldots,L$, and therefore $p^{i*} = \Sigma_{n\leq N} p^{n*}/N$ for all $i = 1,\ldots,N$. This establishes (a).

(b) follows immediately from (5.1), (5.3) and (a). To establish (c), observe that the non-degeneracy condition is a trivial consequence of hypotheses (1) and (2), and that, by (b) and (5.2), summing on i yields

$$\Sigma_{i\leq N} D_{(X,Y)} \hat{C}^i(X^*,Y^*,m^{)i(*},P_2) = \Sigma_{i\leq N} D\mu^{i*}(X^*,Y^*) = (\rho,0).$$

Hence, Theorem 2 applies and $(X^*,Y^*)\in\theta$.||

Since the depletable commodities are pure private goods, the enforcement structure implicit in the specification of M admits a fairly large class of possible misrepresentations. If the government knows that depletable commodities are pure private goods, then, as Brock [3] suggests, it is unreasonable to bear the additional cost associated with processing messages m where $m^{i}_{x_{hr}}(X,Y) \neq 0$ for some $i \neq h$ and some $(X,Y) \in W$. A simple corollary to Theorem 3 is that changing the enforcement structure to reflect this extra information, i.e. to restrict allowable messages to the form $m^{i}_{x_{hr}} = 0$ for $i \neq h$, does not alter the conclusions. On the other hand, only conclusion (c) remains valid if hypotheses (1) and (3) are relaxed. (Observe that relaxing (1) means that depletable commodity externalities are being allowed.)

Theorem 4. Let $m^* = (\mu^*,p^*)$ be a Nash equilibrium for E relative to G', and suppose $(X^*,Y^*) = F(m^*,P_2)$. If E satisfies the non-degeneracy condition, then $(X^*,Y^*) \in \theta$.

Proof. By hypothesis, for $i = 1,\ldots,I$ and $j = I+1,\ldots,N$,

$$\gamma_i^{-1} Du^i(X^*,Y^*) = D_{(X,Y)}\hat{C}^i(X^*,Y^*,m^*)^{i}(,P_2) = (\tau^i,0) + D\mu^{i*}(X^*,Y^*)$$

and

$$\gamma_j Df^j(X^*,Y^*) = D_{(X,Y)}\hat{C}^j(X^*,Y^*,m)^{j}(^*,P_2) = (\tau^i,0) + D\mu^{j*}(X^*,Y^*)$$

where $\gamma_j Df^j(X^*,Y^*) = \Sigma_{s(j)}\gamma_{js(j)}Df^{js(j)}(X^*,Y^*)$. Summing on i and j gives

(5.4) $\Sigma_{i\leq I}\gamma_i^{-1}Du^i(X^*,Y^*) + \Sigma_{j>I}\gamma_j Df^j(X^*,Y^*)$

$= \Sigma_{n\leq N}D\mu^{n*}(X^*,Y^*)$ since $\Sigma_{n\leq N}\tau^n = 0$,

$= (\rho,0)$ by (5.2).

Since (5.4) is a special form of (3.9), Theorem 2 applies, and thus $(X^*,Y^*)\in\theta$.||

It is evident from the proof to Theorem 4 that two consumers can send _different_ equilibrium price messages, causing different evaluations of their initial endowments, and still the equilibrium allocation will be Pareto optimal. From this it follows that agents need not communicate their true marginal willingness to pay functions. However, any misrepresentation by one agent _must_ be cancelled by the aggregate of misrepresentations by other agents. Further, the amount of an agent's misrepresentation of his or her true marginal willingness to pay will equal the deviation of his or her price message from the average price communicated by the other agents.

Remark 4. The results of this section remain valid if any group of agents is excluded from the price making process provided two or more agents are required to send price messages.

Remark 5. Theorems 1 through 4 all give sufficient conditions for first-order Pareto optimality. To obtain the converses of these theorems, merely delete the non-degeneracy condition from each.

The proofs are simple consequences of Smale [9, Thm. A, p. 214].

Remark 6. We can also prove Theorem 4 (but not the incentive compatibility and equality of pieces of Theorem 3) by changing the allocation rule $F(m,p)$ to maximize $\Sigma_{i\leq N}\mu^{i}(X,Y)$ restricted to W_0. Then at equilibrium we get $D_{(X,Y)}\hat{C}^{i}(X^*,Y^*,m^{)i(},P_2)$ restricted to W_0 equals $D\mu^{i*}(X^*,Y^*)$ restricted to W_0. Summing, $\Sigma_{i\leq I}\gamma_i^{-1}Du^{i}(X^*,Y^*) + \Sigma_{j>I}\gamma_j Df^{j}(X^*,Y^*)$ restricted to W_0 equals zero, or, in the total space, equals some $(B,0)$ (a vector perpendicular to W_0).

FOOTNOTES

1. Brock assumes that each agent's message space is the allocation space, and that the allocation rule is the sum of the allocation vectors communicated by all agents.

2. This allows us to write the feasibility constraint as $\Sigma_{i \leq N} x_i = \Sigma_{i<I} \omega_i$.

3. $Du^i(\overline{X},\overline{Y})$ denotes the gradient of u^i with respect to (X,Y) evaluated at $(\overline{X},\overline{Y})$, and $D_X u^i(\overline{X},\overline{Y})$ denotes the gradient of u^i with respect to X evaluated at $(\overline{X},\overline{Y})$.

4. By the Local Submersion Theorem, this condition is equivalent to the assumption that for any allocation $(X',Y') = F(m',P)$ and any agent i, there is a differentiable function $\mu^i(X,Y) = m^i$ defined for (X,Y) near (X',Y') such that $F(m'/\mu^i(X,Y),P) = (X,Y)$. See [8, p. 20].

5. Multiplayer coalitions and wealth effects can nullify the Pareto efficiency properties of these mechanisms. See [2,6].

6. That is, all points in the tangent space TW can be realized.

7. $\gamma_i \neq 0$, $i = 1,\ldots,I$ follows from assumption 1. Observe, too, that if the non-degeneracy condition is satisfied, then $\gamma_j \neq 0$, all $j = I+1,\ldots,N$.

8. $\mathcal{C}^2(W,\mathbb{R})$ denotes the set of all $\mathcal{C}^2$ functions from W into $\mathbb{R}$.

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