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AN EXTENDED SINGLE PEAK
CONDITION IN SOCIAL CHOICE

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An Extended Single Peak Condition in Social Choice

The purpose of this note is to show that the single peak condition (see Black [1948] and Arrow [1963]) is a special case of a simpler condition of inseparability. This inseparability condition characterizes a large family of domains that admit nondictatorial Arrow-type social welfare functions. It follows, by Maskin [1976a] or Kalai-Muller [1977], that they also admit Gibbard-Satterthwaite ([1973][1975]) nonmanipulable voting procedures.

Let A denote a set of alternatives with at least two elements, and let Σ denote the set of all transitive anti-symmetric total binary relations on A . Let Ω , a non-empty subset of Σ , denote the admissible preference relations in the society. For an integer n ($n \geq 2$) an n -person social welfare function (SWF) on Ω is a function $f: \Omega^n \rightarrow \Sigma$ which satisfies the following conditions.

1. Unanimity. For every $P \in \Omega^n$, if $P = (p_1, p_2, \dots, p_n)$, $x, y \in A$ and for $i=1, 2, \dots, n$, $x p_i y$ then $x f(P) y$.
2. Independence of irrelevant alternatives (IIA). For $x, y \in A$ and $P, Q \in \Omega^n$ if [$x p_i y$ if and only if $x q_i y$ for $i=1, 2, \dots, n$] then [$x f(P) y$ if and only if $x f(Q) y$].

f is dictatorial if there exists an i , $1 \leq i \leq n$, for which $f(P) = p_i$ for every $P \in \Omega^n$. f is nondictatorial if it is not dictatorial. Ω is nondictatorial if it admits a nondictatorial n -person SWF f . (This definition is independent of n by

Maskin [1976b] and Kalai-Muller [1977]).

Ω is said to contain an inseparable ordered pair (IOP) of alternatives if there are $s, t \in A$ such that

- i. For some $p, q \in \Omega$ $s p t$ and $t q s$, i.e. the pair (s, t) is not trivial, and
- ii. For no $p \in \Omega$ and $x \in A$ $s p x p t$, i.e. (s, t) is inseparable.

Theorem:

If Ω contains an inseparable ordered pair of alternatives then Ω is nondictatorial.

Proof:

Let (s, t) be an inseparable ordered pair in Ω . Let h be any SWF on the set $\{s, t\}$ in which person 1 is not decisive for the ordered pair (s, t) . For example use majority rule, or any dictator different from 1.

Define $f: \Omega^n \rightarrow \Sigma$ as follows.

For $x, y \in A$ and $P \in \Omega^n$, if $\{x, y\} = \{s, t\}$ and $s p_1 t$, then $x f(P) y$ if and only if $x h(P) y$. In any other case $x f(P) y$ if and only if $x p_1 y$. That f is well defined and obeys both unanimity and IIA is very easy to show. So it remains to prove that f is transitive. Suppose that for $x, y, z \in A$ ($x \neq y, y \neq z$) and $P \in \Omega^n$, the social outcome is $x f(P) y f(P) z$. We have to show that in all the following cases, this implies $x f(P) z$.

Case 1. $x \notin \{s, t\}, y \notin \{s, t\}$. Then, by definition, $xf(P)yf(P)z$ implies xp_1yp_1z , which in turn implies xp_1z and therefore $xf(P)z$.

Case 2. $s=x$.

- i) $y=t$. Then $xf(P)y$ implies sp_1t and $sh(P)t$ and therefore $z \neq s$. $yf(P)z$ implies tp_1z . Hence it is sp_1z which implies $xf(P)z$.
- ii) $y \neq t$. Then $xf(P)y$ implies sp_1y , which implies $z \neq t$ (otherwise it is sp_1yp_1t - contradicting the inseparability condition). Therefore it is xp_1z and $xf(P)z$.

Case 3. $x=t, y=s$.

- i) tp_1s . Then obviously, since everything is determined by 1, $xf(P)z$.
- ii) sp_1t . Then $xf(P)y$ implies $th(P)s$ which implies that $z \neq t$. Therefore $yf(P)z$ implies sp_1z which in turn implies tp_1z and therefore $xf(P)z$.

Case 4. $x=t, y \neq s$.

- i) sp_1t . Then $yf(P)z$ implies $z \neq s$ and hence $xf(P)z$.
- ii) tp_1s . Then since everything is determined by 1 obviously $xf(P)z$.

Case 5. $x \notin \{s, t\}, y=s$. If sp_1t then $xf(P)y$ implies both xp_1s and xp_1t . Therefore, for every z it is $xf(P)z$. On the other hand, if it is tp_1s then $z \neq t$ and xp_1sp_1z implies $xf(P)z$.

Case 6. $x \notin \{s, t\}$, $y=t$. If sp_1t then $xf(P)y$ implies xp_1t and by inseparability, xp_1s . Therefore, for every z it is xp_1z and therefore it is $xf(P)z$. If, on the other hand, it is tp_1s , then, since everything is determined by 1, $xf(P)z$.

Hence f is transitive and therefore f is SWF on Ω . f is non-dictatorial by the careful choice of h . Q.E.D.

The following examples show that domains containing inseparable pairs are frequent in the social choice literature.

Example 1. Single peaked preferences. Let $q \in \Sigma$, and define the set of single peaked preferences relative to the linear order q by $\Omega_q = \{p \in \Sigma: \text{for every three distinct alternatives } x, y, z \text{ if } xqyqz \text{ then it is not the case that } xpy \text{ and } zpy\}$.

If there are $x, y \in A$ such that for every $z \in A$, $xqyqz$, then the pair (x, y) is inseparable, i.e. there are two distinct alternatives x, y which are always top and 2nd top ranked by linear order q . Notice that this condition is met for every finite A and for some versions of infinite A .

Example 2. If A contains exactly 3 alternatives ($|A|=3$), then whenever Ω is obtained by deleting one allowed preference in Σ , then Ω contains an inseparable ordered pair. For example, if we delete the preference $xpyz$, then the pair (x, z) becomes an inseparable ordered pair.

Example 3. If $|A|=2$ then there is an obvious inseparable pair.

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