DISCUSSION PAPER NO. 323

FUNCTIONS FOR MEASURING THE QUALITY OF
APPROXIMATE SOLUTIONS TO ZERO-ONE PROGRAMMING PROBLEMS

by

Eitan Zemel

April, 1978

Graduate School of Management
Northwestern University
Evanston, Illinois 60201

ABSTRACT

We point out the difficulties associated with measuring the quality of an approximate (heuristic) solution by "Percentage-Error" as is customary. We define a set of properties that are to be expected from a proper measure of solution quality and investigate the family of functions which satisfy those conditions. In particular, we show that for any class of 0-1 programming problems approximate functions do exist and that they are uniquely defined up to monotone transformations. We conclude with several examples of linear functions which are suitable for certain classes of problems.

I. Introduction

Consider the 0-1 integer program

(p)   max $\displaystyle\sum_{j=1}^{n} c_j x_j$

    s.t

$$\sum_{j=1}^{n} a_{ij} x_j \overset{\leq}{\underset{=}{or}} b_i \qquad i \in M = \{1\ldots m\}$$

$$x_j = 0 \text{ or } 1 \qquad j \in N = \{1\ldots n\}$$

We will denote by U the universe of problems of the type p. For $p \in U$ let

$$F(p) = \{x \in \{0,1\} \mid \sum_{j=1}^{n} a_{ij} x_j \overset{\leq}{\underset{=}{or}} b_i \quad i \in M\}$$

$$Z(p) = \max_{x \in F(p)} cx$$

$$O(p) = \{x \in F(p) \mid cx = Z(p)$$

be the feasible set, optimal value, and optimal set for p respectively.

Let now $P \subseteq U$ be a particular class of problems (say knapsack, travelling salesman, set covering, etc.) and let A be an exact algorithm for P, i.e. such that given any problem $p \in P$, A finds an optimal solution, $x \in O(p)$. The work of Cook [1], Karp [6], and others indicates that for many interesting classes of problems P (including the ones just mentioned), one is unlikely to find an exact algorithm A which will be able to solve every problem $p \in P$ without, on occasion, investing a computational effort which is essentially equivalent to enumerating a significant fraction of the vertices of $\{0,1\}^N$. It is also well known that practitioners of Integer and Combinational Programming have often been confronted with individual and classes of problems which defy solution in "reasonable" amounts of time. Thus, it is not surprising that

a considerable amount of effort has been recently devoted to the analysis of approximate (heuristic) algorithms, e.g., [3], [5], [7]. In contrast to an exact one, an approximate algorithm finds a feasible solution, $x \in F(p)$, but not necessarily an optimal one. The effectiveness of such an algorithm with respect to the problem p must then be evaluated along two attributes: the computational effort involved in finding x and the "quality" of x as an approximation for the optimal solution. It is this second attribute which is our domain of interest in this paper.

It will be more convenient to measure the "quality" of a given solution on a negative scale, i.e., we will be concerned with measuring the "distance", so to speak, of x from an optimal solution. We will call this "distance" the Error associates with x and any function which attempts to quantify it, the error evaluation of x relative to p, denoted $E_p(x,p)$. The main purpose of this paper is then to investigate and characterize functions which can serve as error evaluators. (It should be noted that this discussion has nothing to contribute towards the resolution, and is independent of, the deeper question associated with aggregating the characteristics of a given algorithm with regard to the individual problems $p \in P$, into a measure of effectiveness relevant to the class P as a whole. It is only after a proper measure is established for the individual problems that one needs decide whether averaging or worst case analysis is the meaningful, tractable, practical, etc. method of aggregation.)

The organization of the paper is as follows. In section II we present a set of properties to be expected from an error evaluating function and examine them vis-a-vis the traditional "percentage error" measure. In section III we show that functions which satisfy the conditions introduced in section II do indeed exist. We then discuss the possible uniqueness of such functions. Finally, in section IV, we conclude with several examples of error evaluating functions for particular classes of problems.

II. Basic Properties of Error Evaluating Functions

A given error evaluation function (E.E.F.) is not expected to perform properly unless it satisfies a set of reasonable properties. In this section, we attempt to propose such a set. To simplify notation, we will delete the parameter P from our notation for $E_P(x,p)$.

The first two properties, A and B, require hardly a comment. An E.E.F. should certainly be a monotone (at least weakly) function of the solution value cx. Also, the error associated with an optimal solution can be conveniently set to zero. Thus:

Condition A. Let $p \in P$, $x^1, x^2 \in F(p)$

$$cx^1 \geq cx^2 \Rightarrow E(x^1,p) \leq E(x^2,p).$$

Condition B. Let $p \in P$

$$x \in 0(p) \Rightarrow E(x,p) = 0$$

An E.F.F. often used by researchers is given by

$$E_1(x,p) = (cx^P - cx) \setminus (cx^P)$$

where $x^P \in 0(p)$ is an optimal solution for p.

If P is such that for every $p \in P$, $cx^P \neq 0$, then $E_1(x,p)$ is easily seen to satisfy conditions A and B. So, of course, do the functions

$$E_2(x,p) = cx^P - cx$$

$$E_3(x,p) = (cx^P - cx)/|c_1|$$

where c_1 is the first (index wise) non zero entry in the objective row.

$$E_4(x,p) = \begin{cases} 0 & \text{is } x \in 0(p) \\ 1 & \text{otherwise} \end{cases}$$

etc.

Let us examine $E_2(x,p)$ more carefully. One finds it hardly surprising

that this function is rarely proposed as an E.E.F. Intuitively, one feels

that the difference in the objective values is meaningless by itself and can

be gauged only relative to some other normalization factor. Put differently,

$E_2(x,p)$ is unacceptable since its value can be arbitrarily manipulated by changes

of units in the objective function. Likewise, $E_3(x,p)$ can be rejected on the

grounds that its value depends on the particular order in which the variables

are indexed. Several authors have observed that similar objections can be often

raised with respect to E_1. For instance, Cornuejols, Fisher and Nemhauser [2]

examined a set of problems of the plant location variety. Such problems are

known to be invariant under addition of certain row constants to the cost matrix.

Yet, as noted in [2], the value of the error for a given non-optimal solution,

as measured by $E_1(x,p)$, does change under such transformations. To overcome

this difficulty, an E.E.F. is proposed in [2] which is invariant under the

particular cost transformations in question. (An identical problem arises, of

course, when one applies $E_1(x,p)$ as an E.E.F. for transportation problems,

general (as opposed to Euclidean) travelling salesman problems and the like).

A related objection to $E_1(x,p)$ is brought forward by Korte in [7].
Consider say, the Binary Knapsack problem. This problem can be formulated as a

maximization or minimization problem, the two versions being virtually equivalent.

But the value $E_1(x,p)$ assigns to a given solution depends on the version chosen.

Even worse, one can show that for the maximization problem there exists a

fully polynomial, guaranteed performance (as measured by $E_1(x,p)$) approximation

scheme, [4]. Yet, the problem of solving the minimization version to a given

accuracy is NP complete [8]. As Korte suggests, this behavior casts some doubts

as to the appropriateness of measuring errors by $E_1(x,p)$ for this class of problems.

Condition C below is designed to take account of the ideas brought forward

in the foregoing discussion. Basically, it enforces equality of the values

assigned by an E.E.F. to corresponding solutions in problems which are obvious-
ly 'equivalent', but may appear different when viewed superficially. By equivalent
we mean here problems which are related to each other by permutations of the
rows or columns, complementing of variables, and trivial algebric manipulations of
the objective function and constraints. But first we must digress for some
notation and definitions.

For a problem $p \in P$, let us denote

$$c = c_j$$

$$A = a_{ij} \text{ with } a^j, (a_i) \text{ the jth column (ith } \rho \text{) of A.}$$

$$b = b_i$$

$$\rho = \rho_i \text{ is } "\underline{\leq}" \text{ or } "=" \text{ as the case may be.}$$

We then have the more compact representation:

$$(p) \quad \max \quad cx$$

$$\text{s.t.} \quad Ax\rho b$$

$$X \in \{0,1\}^N$$

On occasion, we will refer to p as (c, A, ρ, b).

For definitions 1 - 4 below, let $p = (c, A, \rho, b) \in U$, $x \in F(p)$.

<u>Definition 1</u> Let φ be a column permutation operator. By φp, we mean the
problem $p' = (\varphi c, \varphi A, \rho, b)$. $x' = \varphi x$ is defined accordingly.

<u>Definition 2</u> Let π be a row permutation operator. By πp we mean the problem
$p' = (c, \pi A, \pi \rho, \pi b)$. By πx we simply mean x.

<u>Definition 3</u> Let $S \subseteq N$. We will denote by D^S the operator which complements
the variables $x_j, j \in S$.
i.e.:

$$D^S(p) = p' = (c', A', \rho, b')$$

with

$$c'_j = \begin{cases} -c_j & j \in S \\ c_j & j \notin S \end{cases}$$

$$a'_j = \begin{cases} -a_j & j \in S \\ a_j & j \notin S \end{cases}$$

$$b'_i = b_i - \sum_{j \in S} a_{ij}$$

Also

$$D^S x = x' \qquad \text{is given by}$$

$$x'_j = \begin{cases} 1-x_j & j \in S \\ x_j & j \notin S \end{cases}$$

Unless specifically required, we will suppress the parameter S and refer to the operator as D.

<u>Definition 4</u> Let $M_p = \{ i \in M \mid \rho_i \text{ is } "=" \}$ be the index set for the equalities of p. Let also

$$\alpha^o, \ldots \alpha^M \in R_+ \quad \text{be any set of M+1 positive constants}$$

$$\beta^o, \ldots \beta^M \in R^{M_p} \quad \text{be any set of M+1 real, } M_p \text{ dimensional vectors such}$$

that the matrix β^i_j, $i \in M_p$, $j \in M_p$, is non singular. Then

$$G_{\alpha,\beta} p = p' = (c', A', \rho, b')$$

Where

$$c'_j = \alpha^o c_j + \sum_{i \in M_p} \beta^o_i a_{ij}$$

$$a'_{kj} = \alpha^k a_{kj} + \sum_{i \in M_p} \beta^k_i a_{ij}$$

$$b'_k = \alpha^k b_k + \sum_{i \in M_p} \beta^k_i b_i$$

Obviously,

$$G_{\alpha,\beta} x = x$$

Again we will usually suppress the parameter α, β and refer to the operator as G.

Definition 5 Let $p^1 \in U$, and let $p^2 \in U$ be a problem obtainable from p^1 by a sequence of π, φ, G and D operators applied in any order. We will then say that p^1 is reducible to p^2, denoted $p^1 \to p^2$. Also let $x^1 \in F(p^1)$, and let x^2 be obtained from x^1 by an identical sequence of operators. Then we say that x^2 corresponds to x_2, $x^1 \to x^2$, relative to $p^1 \to p^2$. One can easily verify that in this case, $x^2 \in F(p^2)$.

We are now ready to define the third property of E.E.F's:

Condition C. Let $p^1, p^2 \in P$ such that $p^1 \to p^2$. Further, let $x^1 \in F(p^1)$ and let x^2 correspond to x_1 relative to $p^1 \to p^2$.
Then:

$$E(x^1, p^1) = E(x^2, p^2).$$

III. Further Properties of Error Evaluating Functions

Condition C cuts significantly the number of available E.E.F.'s. For instance, $E_1(x,p)$ is excluded for several frequently occurring classes of problems P. Naturally, for a given P, the questions arrise as to the existance and uniqueness of an E.E.F. which satisfies conditions A-C. In this section, we address these questions. The first is settled in the affirmative (Theorem 1). The second in a qualified affirmative (Theorems 2 and 3). We open the discussion with some technical remarks concerning the relation $\rightarrow$.

Remark 1 It can be easily established that any of the row type operators, π and G, commutes with the column type operators, φ and D. On the other hand, π and G do not commute, nor do φ and D. However, the following weaker form of commutativity does hold:

(i) For every column permutation operator φ and every subset $S \subseteq N$, there exists $S' \subseteq N$ such that

$$(1) \qquad \varphi \; D^S = D^{S'} \varphi$$

The property goes the other way too, i.e., for every φ and every $S' \subseteq N$, there exists $S \subseteq N$ such that (1) holds.

(ii) For every row permutation operator π and every set of constants α, β as defined in condition C, there exists a set α', β' such that

$$(2) \qquad \pi G_{\alpha,\beta} = G_{\alpha'\beta'} \pi$$

Again, the statement is true in the reverse direction too, i.e., for every set $\alpha'\beta'$ there exists a set α, β such that (2) holds.

It also can be easily verified that any sequence of π type operators is in itself a π type operator. φ, G and D operators behaves likewise. Also, each of the four types of operators contains an identity operator. Thus, definition 5 can be stated equivelently:

<u>Definition 5'</u>: Let p^1, $p^2 \in U$. Then $p^1 \to p^2$ iff there exists π, G, φ, D such that

$$p^2 = \pi \varphi \ G \ D \ p^1$$

In such a case, we say that $x^1 \to x^2$ relative to $p^1 \to p^2$ if:

$$x^2 = \varphi \ D \ x^1$$

<u>Proposition 1</u>

The relation $\to$ among problems is

(i)   Symmetric, i.e., $p^1 \to p^2 \Rightarrow p^2 \to p^1$

(ii)   Transitive, i.e., $p^1 \to p^2$, $p^2 \to p^3 \Rightarrow p^1 \to p^3$

(iii)   Reflexive, i.e., $p \to p$

A corresponding set of statements can be made with respect to the relation $\to$ among the corresponding solutions x^1, x^2, x^3.

Proof:

(i)   follows from the fact that each operator has an inverse,

(iii)   follows from the fact that each operator type has an identity, and

(ii)   follows by the same reasoning used in the paragraph preceding definition 5'

<u>Proposition 2</u>

Let $P \subseteq U$ be arbitrary. Then P has a unique partition:

$$P = \bigcup_{\ell \in K} P_\ell \qquad \text{such that}$$

(i) $P_\ell \cap P_\ell{}' = \emptyset$ for $\ell \neq \ell'$

(ii)   Let $p^1 \in P_\ell$, $p^2 \in P_\ell{}'$ Then $p^1 \to p^2 <\Rightarrow \ell = \ell'$

<u>Proof</u>   immediate from proposition 1.

We will refer to the P_ℓ's as the <u>equivalence classes</u> within P. We will call two individual problems <u>equivalent</u> if they belong to the same equivalence class.

Remark 2

Let $p^i \in P$, $i = 1,2$ such that $p^1 \to p^2$. The transformations involved in this

relation are not necessarily unique. It follows then that there may be more

than one distinct $x^2 \in F(p^2)$ such that $x^1 \to x^2$ relative to $p^1 \to p^2$. For example, let

(p^1) max $x_1 + x_2$

       s.t.

$$x_1 + 2x_2 \leq 2$$

$$2x_1 + x_2 \leq 2$$

$$x_1, x_2 = o \text{ or } 1$$

and let $x^1 \in F(p^1)$ be $(1,0)$. Consider

(p^2) max $2x_1 + 2x_2$

       s.t.

$$x_1 + 2x_2 \leq 2$$

$$2x_1 + x_2 \leq 2$$

$$x_1, x_2 = 0 \text{ or } 1$$

One can observe that $p^2 = \pi \, \varphi \, G_{\alpha,\beta} \, D^S p^1$ where

$S = \emptyset$; $\alpha^o = 2$; $\alpha^i = 1$, $i = 1,2$; π and φ are the identity permutations.

The solution which corresponds to x^1 is then $x^2 = \varphi \, Dx^1 = (1,0)$.

But another set of possible operators which transforms p^1 to p^2 is given by

$$p^2 = \pi' \, \varphi' \, G_{\alpha,\beta} \, D^S p^1$$

where G and D are as before, but where φ' permutes the columns of p^1 and π'

permutes the rows. Under these operators, the solution which corresponds to

x^1 relative to $p^1 \to p^2$ is $y^2 = \varphi' x^1 = (0,1) \neq x^2$. However, there is a considerable

similarity between the feasible sets of equivalent problems as the following three

propositions indicate.

Proposition 3

For $i = 1, 2$ let

$$p^i = (c^i, A^i, \rho^i, b^i) \in P$$

be such that $p^1 \to p^2$. In particular, let

$$p^2 = \pi \, \varphi \, G \, D \, p^1$$

For an arbitrary $x \in F(p^1)$ let $y \in F(p^2)$ be given by

$$y = \varphi \, D \, x$$

Then,

$$c^2 y = k_1 c^1 x + k_2 \qquad \text{where } k_1 > o, \ k_2 \text{ depend on the transformations}$$

π, φ, G and D, but not on the particular solution x.

<u>Proof</u> Let us examine the effect of each of the transformations in question on

the value of the objective function of the corresponding solutions.

(i) Let $p = (c, A, \rho, b)$, $p' = D^S p$, $x \in F(p)$, $x' = D^S x$ as in definition 3.

$$c'x' = \sum_{j \in N} c'_j x'_j = \sum_{j \in N \setminus S} c_j x_j + \sum_{j \in S} (-c_j)(1-x_j)$$

$$= \sum_{j \in N} c_j x_j - \sum_{j \in S} c_j = cx + K_2$$

where $K_2 = -\sum_{j \in S} c_j$ and does not depend on x.

(ii)   Let $p = (C, A, \rho, b)$, $p' = G_{\alpha, \beta} p$, $x \in F(p)$, $x' = x$ as in definition 4.

$$c'x' = \sum_{j \in N} c'_j x'_j = \sum_{j \in N} (\alpha^o c_j + \sum_{i \in M_p} \beta_i^o a_{ij}) x_j$$

$$= \alpha^o \sum_{j \in N} c_j x_j + \sum_{i \in M_p} \beta_i^o \sum_{j \in N} a_{ij} x_j =$$

$$\alpha^o cx + \sum_{i \in M_p} \beta_i^o b_i = k_1 cx + k_2$$

where $k_1 = \alpha^o > o$, and $k_2 = \sum_{i \in M_p} \beta_i^o b_i$ do not depend on x.

(iii)   It can be easily verified that π and φ do not have any effect on the

objective value.   Q.E.D.

<u>Proposition 4</u>

Let $p^i \in P, i = 1, 2$ such that $p^1 \to p^2$. Let $Z_1^i > Z_2^i \dots > Z_{r(i)}^i$ be the distinct

values of $\{c^i x, | x \in F(p^i)\}$, with the corresponding solution sets N_j^i, $1 \le j \le r(i)$,

and multiplicities $n_j^i = |N_j^i|$. Then

(i) $r(1) = r(2) = r$

(ii) $n_j^1 = n_j^2$ for $j = 1, \ldots r$

__Proof__ Let $p^2 = \pi \varphi G D p^1$ be a particular transformation of p^1 to p^2. Let $x^1, y^1 \in F(p^1)$, $x^2 = \varphi D x^1$, $y^2 = \varphi D y^1$. It follows directly from proposition 3 that $c^1 x^1 > c^1 y^1 \Longleftrightarrow c^2 x^2 > c^2 y^2$. Thus, $x \in N_1^1 \Longleftrightarrow \varphi D x = y \in N_1^2$. Since φD is a one to one transformation of $F(p^1)$ onto $F(p^2)$, we have that $n_1^1 = n_1^2$. To complete the proof, we have to repeat the argument with respect to N_j^i, $j = 2, \ldots r(1)$.   Q.E.D.

__Proposition 5__

Let $p^1, p^2 \in U$ be such that $p^1 \to p^2$ and assume that there is more than one set of transformations which induce this relation . In particular,

$$p^2 = \pi \varphi G D p^1$$

but also

$$p^2 = \pi' \varphi' G' D' p^1$$

Let

$$x^1 \in F(p^1); \quad x^2 = \varphi D x^1; \quad y^2 = \varphi' D' x^1.$$

Then

$$c^2 y^2 = c^2 x^2$$

__Proof__ Using the notation of proposition 4, let $x^1 \in N_j^1$ for some $1 \leq j \leq r$. Then by the same logic applied in the proof of that proposition, we have that $x^2 \in N_j^2$ for the same index j. But then also $y^2 \in N_j^2$ by the same argument. Thus,

$$c^2 x^2 = c^2 y^2 = z_j^2$$ Q.E.D.

We are now ready to address the existance and uniqueness issues for those E.E.F.'s which satisfy conditions A-C for a given class P. We will call such E.E.F.'s __proper__ for P. It should be noted that this concept is defined relative to a given class P since condition C is in effect among problems within P only. Thus, it is entirely possible to have an E.E.F. which is proper for one class of problems P, but improper for another. Indeed, on occasion, one may wish to override the equivalence between problems implied by the relation $\to$. For instance, one may feel that a standard transportation problem, whose constraints have been manipulated by G and D type transformations, may not be easily recognized as

identically. The same distinction can be made between general and Euclidean cost traveling salesman problems, etc. Such cases can be conveniently handled by a proper choice of the class P. (One would choose P to be the class of standard format transportation problems in the first case, Euclidean traveling salesman problems in the second). Naturally, the larger is P, the fewer are the functions which are proper for it. One then may be concerned that if P is large enough, a proper E.E.F. may not exist. This possibility is ruled out by

Theorem 1   For every class $p \subseteq U$, there exists a proper E.E.F.

Proof   Let $p \in P$ and let $z_1 > z_2 \ldots > z_r$ be the distinct values of the objective function with the corresponding solution sets N_1, N_2, ..., N_r. Consider the E.E.F.

$$E^*(x,p) = j-1 \qquad \text{if} \qquad x \in N_j, \text{ i.e. if } cx = z_j$$

It can be easily seen that $E^*(x,p)$ satisfies conditions A and B. The fact that it satisfies condition C as well follows from propositions 4 and 5.   Q.E.D.

Can we expect uniqueness of E^* as a proper E.E.F. for P? The following Theorem shows that every proper E.E.F. yields in a natural way a vast number of new proper E.E.F.'s.

Theorem 2   Let $E(x,p)$ be a proper E.E.F. for P. Let $P_1, \ldots P_\ell$ be the partition of P to equivalence classes established in proposition 2. For each $1 \leq k \leq \ell$, let $b_k(\cdot)$ be any non-decreasing function of one argument such that $b_k(0) = 0$.

Then

$$E'(x,p) = b_k(E(x,p)) \qquad \text{for } p \in P_k$$

is also a proper E.E.F.

Proof   We have to show that $E'(x,p)$ satisfies conditions A, B and C. That it satisfies A follows from the facts that $E(x,p)$ satisfies A and that $b_k(\cdot)$ is

monotone. $E'(x,p)$ satisfies B since $E(x,p)$ does and since $b_k(0) = 0$. It remains to be seen that $E'(x,p)$ satisfies condition C. Let then p^1, $p^2 \in P$, such that $p^1 \to p^2$. It follows that $p^1, p^2 \in P_k$ for some $1 \le k \le \ell$. Hence, the same function, $b_k(\cdot)$, is used to define $E'(x,p)$ for both p^1 and p^2. Let $x^1 \to x^2$ relative to $p^1 \to p^2$. Since $E(x,p)$ is proper

$$E(x^1,p^1) = E(x^2,p^2)$$

but then

$$E'(x^1,p^1) = b_k(E(x^1,p^1)) = b_k(E(x^2,p^2)) = E'(x^2,p^2) \qquad \text{Q.E.D.}$$

In light of Theorem 2, the uniqueness of proper E.E.F. is ruled out. It is interesting to note, however, that every proper E.E.F. is accounted for by applying a monotone transformations of the type referred to in Theorem 2 to $E^*(x,p)$, the E.E.F. referred to in Theorem 1. Thus, as the following theorem indicates, we have uniqueness of $E^*(x,p)$ up to monotone transformations:

<u>Theorem 3</u> Let $P \subseteq U$ and let $E(x,p)$ be a proper E.E.F. for P. Then $E(x,p)$ can be represented as

$$E(x,p) = b_k(E^*(x,p))$$

where $E^*(x,p)$ is the E.E.F. referred to in Theorem 1 and $b_k(\cdot)$ are the monotone functions referred to in Theorem 2.

<u>Proof</u> The proof is by constructing the functions $b_k(\cdot)$, $1 \le k \le \ell$. For every $1 \le k \le \ell$, let us choose a representative problem $p^k \in P_k$ arbitrarily. Let $N_1^k, N_2^k \ldots N_{r(k)}^k$ be the partition of the feasible set of p^k according to the objective value as in the proof of Theorem 1. For each $1 \le j \le r(k)$, choose x^j arbitrarily such that $x^j \in N_j^k$. By definition

$$E^*(x^j,p^k) = j - 1$$

Let $\qquad\qquad E(x^j,p^k) = \alpha_{kj}$

Since $E(x,p)$ is proper, α_{kj} is well defined, i.e., its value depends on the class P_k and on the index j, but not on the particular representative problem $p^k \in P_k$ and solution $x^j \in N_j^{\;k}$ chosen. Since $E^*(x,p^k)$ ranges over the integers $0 - r(k)-1$ only, it suffices to define b_k on those values. Define

$$b_k(j-1) = \alpha_{jk} \text{ for } j = 1, r(k)$$

It then follows by construction that

$$E(x,p^k) = b_k(E^*(x,p^k)).$$

Since $p^k \in P_k$ was chosen arbitrarily, we have that

$$E(x,p) = b_k(E^*(x,p)) \text{ for } p \in P_k.$$

It remains to be seen that $b_k(\cdot)$ does have the properties asserted. Both follow the fact that $E(x,p)$ is proper. Thus, $b_k(\cdot)$, $k = 1...\ell$ is monotone since condition A implies that α_{kj} is monotone with j. Also, condition B implies that for $x \in 0(p^k)$, $E(x,p^k) = 0$. Thus, $\alpha_{1,k} = 0$ and hence $b_k(0) = 0$, $k = 1...\ell$ Q.E.D.

IV. Linear E.E.F.'s: Examples

The construction of Theorem 3 accounts for all the proper E.E.F.'s for a given class P. However, this construction is intractable computationally, since it calls for determination of the complete solution structure (i.e., the sequence of the z_i's), and for the identification of the appropriate monotone function, $b_k(\cdot)$, applicable to the problem in question. In this section, we examine some linear proper E.E.F.'s which can be calculated on the basis of considerably less information. One type of such functions uses the concept of a reference point, brought forward by Cornuejols, Fisher and Nemhauser [2]. Because of the vast richness of the family of proper linear E.E.F.'s, the section is intended to be a sketchy, rather than systematic, survey of some examples.

By a linear E.E.F., we mean a function of the form

$$E(x,p) = \beta_p - \alpha_p c^p x$$

where c^p is the cost vector for p, β_p and $\alpha_p \geq 0$ are constants which depend on p, but not on x. We will restrict our attention to the non-trivial case $\alpha_p > 0$. From condition B, we get an alternative expression for E

$$E(x,p) = \alpha_p(z(p) - c^p x)$$

denoting

$$\gamma_p = 1/\alpha_p$$

we have

$$E(x,p) = (z(p) - c^p x)/\gamma_p .$$

We refer to γ_p as the normalization factor. In order that $E(x,p)$ be proper, the γ_p's should be chosen in such a way that condition C is enforced. This can usually be done in several ways as the following proposition implies.

Proposition 7. Let $\alpha_i > 0$, w_i, $i = 1...k$ be arbitrary such that $\sum\limits_{i=1}^{k} w_i = 1$. Let also $E^i(x,p) = (z(p) - c^p x)/\gamma_p^i$, $i = 1...k$ be proper linear E.E.F.'s for P.

Then, so do
$$\widetilde{E}(x,p) = (z(p) - c^P x) / \sum_{i=1}^{k} \alpha_i \gamma_p^i$$

$$\overline{E}(x,p) = (z(p) - c^P x) / \prod_{i=1}^{k} \alpha_i (\gamma_p^i)^{w_i} .$$

and
$$\hat{E}(x,p) = (z(p) - c^P \cdot x) / \max_{i=1,k} \alpha_i \gamma_p^i$$

Proof. It is obvious that multiplying a proper E.E.F. by $1/\alpha$ for $\alpha > 0$,

leaves the E.E.F. proper (Theorem 2). Therefore, it is enough to consider

the case $\alpha_i = 1$, $i=1,\ldots,k$. Consider $\widetilde{E}$ first. It is enough to examine the

case k=2 since the general case follows by induction. Let then

$$\widetilde{E}(x,p) = (z(p) - c^P \cdot x) / (\gamma_p^1 + \gamma_p^2)$$

Let p, q $\in$ P, s.t. p $\rightarrow$ q and let x $\in$ F(p), x $\rightarrow$ y relative to p $\rightarrow$ q. We have

to show that

$$\widetilde{E}(x,p) = \widetilde{E}(y,q)$$

The statement is trivial if x $\in$ 0(p). Assume, then, that $z(p) - c^P \cdot x > 0$.

Since E^i are proper for i = 1,2 we know that

$$(z(p) - c^P \cdot x) / \gamma_p^i = (z(q) - c^q \cdot x) / \gamma_q^i \qquad i=1,2$$

since the numerator of the two expressions is non-zero

$$\gamma_p^i / (z(p) - c^P \cdot x) = \gamma_q^i / (z(q) - c^q \cdot x)$$

Adding the two equations for i=1,2, and flipping the numerator and denominator

back, we get the required result for $\widetilde{E}$.

Let us now examine $\overline{E}$ with p, q, x and y as before. Since E^i are proper:

$$(z(p) - c^P \cdot x) / \gamma_p^i = (z(q) - c^q \cdot x) / \gamma_q^i \qquad i=1,\ldots,k$$

$$z(p) - c^P \cdot x = (z(q) - c^q \cdot x) \cdot (\gamma_p^i / \gamma_q^i)$$

Raising each side to the power w^i, and multiplying the terms together, we get:

$$\prod_{i=1}^{k} (z(p) - c^P \cdot x)^{w_i} = \prod_{i=1}^{k} (z(q) - c^q \cdot x)^{w_i} \cdot \prod_{i=1}^{k} (\gamma_p^i \mid \gamma_q^i)^{w_i}$$

or since $\qquad \sum_{i=1}^{k} w_i = 1$

$$(z(p) - c^P \cdot x) = (z(q) - c^q \cdot x) / \prod_{i=1}^{k} (\gamma_p^i / \gamma_q^i)^{w_i}$$

Rearranging terms, we get the desired result.

Finally, consider $\hat{E}$. We have to show that

$$(z(p) - c^P \cdot x) / \max_{i=1,\ldots,k} \gamma_p^i = (z(q) - c^q \cdot y) / \max_{i=1,\ldots,k} \gamma_q^i$$

Let $\gamma_p^j = \max_{i=1,\ldots,k} \gamma_p^i$. It is enough to show that $\gamma_q^j = \max_{i=1,\ldots,k} \gamma_q^i$. Assume, on the contrary, that $\gamma_q^\ell > \gamma_q^j$ for some $1 \le \ell \le k$. It then follows that

$$(z(p) - c^P \cdot x)/\gamma_p^j \le (z(p) - c^P \cdot x)/\gamma_p^\ell = (z(q) - c^q \cdot y)/\gamma_q^\ell < (z(q) - c^q \cdot y)/\gamma_q^j$$

But then, since E^j is proper,

$$(z(p) - c^P \cdot x)/\gamma_p^j = (z(q) - c^q \cdot y)/\gamma_q^j, \text{ a contradiction.}$$

$$\text{Q.E.D.}$$

Proposition 7 suggests a strong relationship between normalization factors for Proper E.E.F.'s and homogeneous functions of degree 1. This relationship is further established in

Proposition 8. Let $P \subseteq U$. For $p \in P$, let c^P the cost vector of p. Let

$L^P \qquad$ be the row space spanned by the equalities rows, $i \in M_p$, of A

$c_L^P \qquad$ be the orthogranal projection of c^P on L^P

$d^P = c^P - c_L^P$

Then

$$E(x,p) = (z(p) - c^P \cdot x) \backslash f(d^P)$$

is a proper E.E.F. for p whenever $f(d^P)$ satisfies:

(i)   for every $i \in \{1,\ldots,N\}$

$$f(d_1{}^P \ldots d_i{}^P \ldots d_n{}^P) = f(d_1{}^P, \ldots -d_i{}^P \ldots d_p{}^n)$$

(ii)   for every $i, j \in \{1,\ldots,N\}$

$$f(d_1{}^P \ldots d_i{}^P \ldots d_j{}^P \ldots d_n{}^P) = f(d_1{}^P \ldots d_j{}^P \ldots d_i{}^P \ldots d_n{}^P)$$

(iii) $f(d)$ is homogeneous of degree one, i.e.,

$$f(\alpha d_1{}^P, \alpha d_2{}^P, \ldots, \alpha d_n{}^P) = \alpha f(d_1{}^P, \ldots, d_n{}^P)$$

Sketch of Proof and Discussion. Consider first the case $M_p = \emptyset$, i.e., there are no equality constraints in p. In this case, L^P and $c_L{}^P$ are both equal to $(0,\ldots 0)$ and $d^P = c^P$. If $M_p \neq 0$, then d^P is an invariant under G type transformations with $\alpha^0 = 1$, i.e., under additions of multiples of the equality constraints to the objective row. The case of $\alpha^0 \neq 1$, (i.e., rescaling of the objective row) is taken care of by requirement (iii) which assures that such scaling will affect the numerator and denominator proportionally. Condition (i) is designed to accomodate complementing variables (i.e., D type operators). It can be (fully or partially) relaxed if P itself excludes such operators. In other words, if P does not contain any two problems which are obtainable from each other by complementing a given column, then (i) can be relaxed with respect to this column. For instance, one can argue that for, say, travelling salesman problems, the operation of complementing variables is rather unnatural and therefore problems with complemented variables should be excluded from P. In such cases, there is no need to insist on the symetry imposed by (i). (Allowing for complementing variables, however, comes in very handy in the context of enumeration algorithms).

Condition (ii) ensures the invariance of E under permutation of columns (φ type operators). Again, it can be often relaxed in response to the structure of P. For example, if P is the set of, say, plant location problems, there is

a natural distinction between the variables which represent the on-off status of a plant and those which correspond to the distribution pattern. Clearly, there is no reason to impose condition (ii) between these two sets of variables. (Strictly speaking, in this case, f is not a function of d alone since specific entries in d are identified by their respective columns in A. Nevertheless, we will refer to the normalization factor as f(d)).

Examples.

(1) Let d be rearranged such that

$$|d_{\rho_1}| \geq |d_{\rho_2}| \ \cdots \ \geq |d_{\rho_n}| \ .$$

Let

$$f^k(d) = \sum_{i=1}^{k} |_{\rho_i}|$$

be the sum of k largest elements (in absolute value) in the (modified) objective row d. $f^k(d)$ satisfies (i), (ii) and (iii) and is then appropriate for a normalization factor. For instance, if P is the class of (binary) knapsack problems, then $f^1(d)$, which in this case, reduces to $|c_{max}| = \max|c_i|$, represents, in a way, a measure for what the algorithm is all about (since by rounding the optimal solution for the continuous relaxation of the problem, we get a solution which is at most $|c_{max}|$ units away from the optimum). Thus, $E(x,p) = (z(p) - c^P x)/|c_{max}|$ is, in my opinion, a meaningful measure for the quality of a solution to the knapsack problem. I am not aware of any approximate algorithm for this problem which can guarantee a performance better than the obvious level of 1 with respect to the measure.

(2) If the equalities defining the individual problems $p \in P$ are of the transportation type, then d can be obtained from c by simply adding row and column constants in such a way that the row and column sums of d are zeroes.

(This can be accomplished by a one pass scan of the rows and columns of c). If we exclude from P problems in which some of the variables are complemented, we can relax requirement (i) of Proposition 8. Let d be arranged such that

$$d_{\pi_1} \geq d_{\pi_2} \cdots \geq d_{\pi_n}$$

and consider the range of (modified) costs

$$f^R(d) = d_{\pi_1} - d_{\pi_n}$$

$f^R(d)$ measure the possible variability of the per unit cost over the different routes. Multiplied by the total amount shipped, it yields an upper bound on the difference in the objective function between any two feasible solutions to p, and is therefore meaningful as a normalization factor. This measure can be refined in several ways by considering the ranges of the individual rows and columns and aggregating them as in Proposition 7.

(3) We finally consider a class of normalization factors obtained by using a reference point, as suggested in [2]. For every $p \in P$, let $x^P \in [0,1]^N$ be a reference point (not necessarily feasible, not necessarily integer). Let the operators φ and D be extended in a natural way to the Unit Cube, B, so it will make sense to refer to the correspondence $x \to y$ relative to $p \to q$, for $x \in B$ even though $x \notin F(p)$. (In this case, of course, $y \notin F(q)$.)

Proposition 9. Let $P \subseteq U$. For each $p \in P$, let $x^P \in B$ be such that:

(i) $z(p) > c^P x^P$

(ii) $a_i^P x^P = b_i$, $i \in M_p$ i.e., x^P satisfies the equality constraints of p

(iii) $p \to q \Rightarrow x^P \to x^q$ relative to $p \to q$.

Then

$$E(x,p) = (z(p) - c^P x)/(z(p) - c^P x^P)$$

is a proper E.E.F. for P.

Proof. The proof resembles closely that of Proposition 8 (which was sketched only). The only comment in order is the fact that it is condition (ii), rather than feasibility or integrality of x, which enforces the invariance of E under G type transformations.   Q.E.D.

There are a number of ways to choose x^P to satisfy (i), (ii) and (iii). If $M_p = 0$, a generalization of the origin can be used by setting

$$x_j^P = \begin{cases} 0 & c_j \geq 0 \\ 1 & c_j < 0 \end{cases}$$

Similarly, if M_p is composed of a set of disjoint multiple choice constraints:

$$\sum_{j \in s_i} x_j = k_i \, ,$$

then x^P can be obtained by picking the worst (cost-wise) k_i variables from each set s_i.

Alternatively, x^P can be chosen as the solution of a given reference heuristic algorithm for P which is invariant under the transformations in question. For instance, for the knapsack problem, x^P can be chosen as the solution of the greedy (in the sense of the cost to weight ratios) algorithm, provided ties are broken properly. (E.g., breaking ties according to the size of the cost coefficients is o.k., however, breaking them according to the index of the variables in question is not.) Normalizing $E(x,p)$ by

$z(p) - c^P x^P$, where x^P is obtained in this way, is even more informative than normalizing by $\left| c_{max} \right|$ since the denominator is an ex-post rather than ex-ante bound on the relevant range of the objective function values. (The case $z(p) - c^P x^P = 0$ must then be disposed by some other means.)

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