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CONTINUITY RESULTS IN THE GAINS FROM TRADE WITH SIMILAR CONSUMERS*

by

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I

INTRODUCTION

The assumption that all consumers are identical has often been employed in economics. Use of this assumption usually simplifies the analysis considerably while allowing the most clear-cut, unambiguous conclusions to be drawn. Yet, the empirical significance of the case in which all agents are identical is negligible, and the general validity of welfare theorems and policy prescriptions derived under such an assumption must be questionable. Presumably, a major reason for the continued interest in this special case is an implicit hypothesis of continuity: the belief that the results obtained under the assumption of identical consumers would continue to hold even if the various consumers, while not strictly identical, were not too dissimilar. Although the idea that such continuity should obtain is intuitively very appealing, it has not, to my knowledge, been subject to rigorous examination.

Recently, the notion of similar consumers has been defined and studied in a series of very elegant papers. The first formalization of the concept of similarity was provided by Kannai (1970), who introduced a metric on consumer preferences. Using this metric, consumers could be said to be similar if their tastes and endowments were close. Debreu (1968) provided a significant generalization by introducing a topological space of agents' characteristics. The main published results using these concepts have been concerned with the relationship between core and competitive equilibrium allocations in large but finite economies: Kannai (1970) and W. Hildenbrand (1970b)

Rather, the best we can say is that free trade is potentially superior in that it is possible to make all consumers better off through redistribution. But if these redistributions are not actually executed, some ambiguity remains: some consumers may gain while others lose.

Intuitively, however, it might seem that if consumers do not differ "too much", then all would gain from trade. Presumably, the allocations to such similar consumers would be similar both under autarky and under free trade. Moreover, the way the different consumers rank their free trade and autarky consumption vectors would also be alike. Thus, if one consumer gained from trade, all would do so together, and the free trade allocation would be unambiguously superior to the autarky outcome.

In this paper, we establish that, under appropriate conditions, this argument can be made rigorous. We show that if all the consumers in a given economy benefit from the introduction of free trade, then the free trade solution(s) will also be Pareto-superior to the autarky outcome(s) in any economy where the consumers are sufficiently alike those in the given economy.

Establishing the result turns out to be a simple metter. In Proposition 1, we show that the quasi-equilibrium correspondence is upper hemi-continuous for production economies with a fixed (finite) number of agents.[3)] Given this result, the continuity result for the gains from trade (Proposition 2) follows immediately from the intersection of a finite collection of open sets being open. Thus, one's intuition is verified in this particular case and, while we cannot perhaps expect the argument to go so easily every case, a technique for checking presumed continuity is provided.

<u>Remark</u>: We do not claim that this approach via private production economies offers a better description than the more standard formulation, although certainly consumers do have production capabilities. However, it will be convenient to allow any finite collection of agents (from a certain class) to constitute an economy. This would cause difficulties if we had chosen the private ownership model, since there is no quaranty that the requirement $\sum_n^N \theta_k^n = 1$ will be met by an arbitrary choice of the N consumers. Moreover, as Hildenbrand (1970a) has noted, we can, given any private ownership economy, define a private production economy with the same preferences, aggregate production set, wealth distribution and feasible consumption vectors. (Essentially, if the production sets defining the K firms are denoted by Z^k, $k=1,\ldots,K$, we define the production set for consumer n by $Y^n = \sum_{k=1}^{K} \theta_k^n Z^k$.)

In order to formalize the notion of similar agents and, eventually, of similar economies, we employ the concept of a topological space of agents. Since an agent's consumption set and preferences are characterized by the graph P, we can identify an agent as a point $a \in \mathcal{A} \equiv [\mathcal{P} \times R^{(M)} \times \mathcal{F}]$, where $\mathcal{P}$ is the collection of closed, non-empty subsets of $R^{(M)} \times R^{(M)}$ and $\mathcal{F}$ is the collection of closed, non-empty subsets of $R^{(M)}$. We endow $\mathcal{P}$ and $\mathcal{F}$ with the (metric) topology of closed convergence, while we employ the usual metric topology on $R^{(M)}$. We then obtain a metric topology on $\mathcal{A}$ as the product topology. [For a discussion of the topology of closed convergence, see Hildenbrand (1970b) and the references given economy $A = (a_1, \ldots, a_N)$, we will occasionally write P_n, ω_n, Y_n, $\succsim_n$ and X_n for $P(a_n)$, $\omega(a_n)$, $Y(a_n)$, $\succsim(a_n)$ and $X(a_n)$ respectively.

For an economy $A = (a_1, \ldots, a_N)$, an allocation is a vector $(x_1, \ldots, x_N, z)$, x_n, $z \in R^{(M)}$, such that $x_n \in X(a_n)$, $n=1, \ldots, N$, and $z \in Z(A) \equiv \sum_{n=1}^{N} Y(a_n)$. An allocation $(x(a_1), \ldots, x(a_N), z)$ for $A = (a_1, \ldots, a_N)$ is feasible if

$$z = \sum_{n=1}^{N} x(a_n) - \omega(a_n).$$

A price system is a vector $p \in \Delta \equiv \{p \in R_+^{(M)} | \sum_{m=1}^{M} p_m = 1\}$. Given an economy A, a <u>quasi-equilibrium</u> for A is a vector $(p, x_1, \ldots, x_N, z)$ such that $(x_1, \ldots, x_N, z)$ is a feasible allocation for A, p is a price system,

1) $p \cdot z = \max p \cdot Z(A)$, and

2) $p \cdot x_n = w(a_n, p) \leqq p \cdot x$ for all $x \in X(a_n)$ such that $x \succsim_n x_n$, $n = 1, \ldots, N$,

where $w(a,p) \equiv p \cdot \omega(a) + \max p \cdot Y(a)$.

A <u>competitive equilibrium</u> for A is a vector $(p, x_1, \ldots, x_N, z)$ such that $(x_1, \ldots, x_N, z)$ is a feasible allocation for A, p is a price system,

1) $p \cdot z = \max p \cdot Z(A)$, and

2) $p \cdot x_n = w(a_n, p)$ and, if $p \cdot x \leqq p \cdot x_n$, then $x_n \succsim(a_n)\ x$.

Note that $w(a_n, p) > \min p \cdot X(a_n)$ for each n implies that the quasi-equilibrium $(p, x_1, \ldots, x_N, z)$ is a competitive equilibrium. [See Debreu (1959) p. 69].

We will limit our consideration to economies whose agents are drawn from a subset $\mathcal{C}$ of $\mathcal{A}$ such that

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Upper Hemi-Continuity

In order to show that all consumers in an economy will share in the gains from trade if the consumers are not too dissimilar (although not necessarily identical), we must first show that, given the set of equilibrium allocations for a particular economy, any equilibrium of any economy sufficiently close to the given one is similar to some equilibrium of the given economy. This result is Proposition 1, which establishes the upper hemi-continuity of the correspondence which assigns to each economy A in $\mathcal{C}^{(N)}$ its set of quasi-equilibria.[5)]

For each positive integer N and for each A in $\mathcal{C}^{(N)}$, let

$$Q_N(A) = \{(p,x_1,\ldots,x_N,z) \in \Delta \times R^M \times \ldots \times R^M \times R^M \mid (p,x_1,\ldots,x_N,z) \text{ is a quasi-equilibrium for } A\}.$$

Conditions sufficient for the non-emptiness of Q(A) are well-known. Essentially, we must limit consideration to those A such that $\sum_{a\in A} X(a)$ intersects the set $[Y(A) + \{\sum_{a\in A} \omega(a)\}]$ (so that feasible allocations exist) and we must also impose conditions on preferences or the technology (such as free disposal) sufficient to guaranty that prices can be taken to lie on the simplex Δ. Let $\mathcal{Q}_N$ denote the set of $A \in \mathcal{C}^{(N)}$ such that $Q(A) \neq \emptyset$. Then Q_N is a correspondence on $\mathcal{Q}_N$ for each positive integer N.

<u>Proposition 1</u>: The correspondence Q_N is upper hemi-continuous and compact-valued on $\mathcal{Q}_N$.

<u>Proof</u>: We follow the technique of proof given by Hildenbrand and Mertens (1972) for the upper hemi-continuity of the competitive equilibrium correspondence in exchange economies. However, the

Clearly, we can take the p^k to be bounded, since all belong to the compact set Δ. For each $A = (a_1,\ldots,a_N) \in \mathcal{C}^{(N)}$, let

$$C(A) = [\Pi_{n=1}^{N} X(a_n)] \times [\Pi_{n=1}^{N} Y(a_n)],$$

let

$$D(A) = \{(x_1,\ldots,x_N,y_1,\ldots,y_N) \in R^{(2NM)} | \sum_n x_n = \sum_n y_n + \sum_n \omega(a_n)\},$$

and define $E(A) = C(A) \cap D(A)$. Then $r^k \in E(A^k)$. For each $A \in \mathcal{C}^{(N)}$, Conditions II and IV imply that $C(A)$ is contained in the set

$$C^* = [\Pi_{n=1}^{N} (R_+^{(M)} - \{b_1\})] \times [\Pi_{n=1}^{N} Y^*],$$

while Condition III implies that, for each $A \in \mathcal{C}^{(N)}$,

$$D(A) \subset D^* \equiv \{(x_1,\ldots,x_N,y_1,\ldots,y_N) \in R^{(2NM)} | \sum_n x_n - \sum_n y_n \leqq Nb_2\}.$$

Then $E(A)$ is contained in $C^* \cap D^*$ for each A in $\mathcal{C}^{(N)}$. To establish the boundedness of the sequence (r^k) it is sufficient [see Debreu (1959)] to show that the intersection of the asymptotic cones of C^* and D^* is the set $\{0\}$.

Now,

$$\mathbf{A}(C^*) \subset [\Pi_{n=1}^{N} \mathbf{A}(R_+^{(M)} - \{b\})] \times [\Pi_{n=1}^{N} \mathbf{A}(Y^*)]$$

$$= [\Pi_n R_+^{(M)}] \times [\Pi_n Y^*],$$

while

$$\mathbf{A}(D^*) = \{(u_1,\ldots,u_N,v_1,\ldots,v_N) \in R^{(2NM)} | \sum u_n - \sum v_n \leqq 0\}.$$

If $(u_1,\ldots,u_N,v_1,\ldots,v_N) \in \mathbf{A}(C^*) \cap \mathbf{A}(D^*)$, then $\sum u_n \geqq 0$, and thus $\sum v_n \geqq 0$. But $v_n \in Y^*$, while $\sum_n Y^* = Y^*$ (since Y^* is a convex cone), so $\sum_n v_n \in Y^*$. But $Y^* \cap R_+^{(M)} = \{0\}$, so $\sum v_n = 0$. Thus $\sum_n u_n = 0$, and $u_n \in R_+^{(M)}$, so $u_n = 0$, while $Y^* \cap (-Y^*) = \{0\}$ implies that each v_n is zero. Thus, $(u_1,\ldots,u_N,v_1,\ldots,v_N) = 0$, so $\mathbf{A}(C^*) \cap \mathbf{A}(D^*) = \{0\}$.

Thus, the sequence $((p^k,x_1^k,\ldots,x_N^k,y_1^k,\ldots,y_N^k))$ is bounded, so there exists $\bar{p},\bar{x}_1,\ldots,\bar{x}_N$ and $\bar{y}_1,\ldots,\bar{y}_N$ such that $p^k \to \bar{p}$, $x_n^k \to \bar{x}_n$ and $y_n^k \to \bar{y}_n$, $n=1,\ldots,N$. Then too, $z^k = \sum_n y_n^k$ converges to $\bar{z} = \sum_n \bar{y}_n$.

condition I of the definition of $\mathcal{E}$, there exists $x'' \in X(a_n)$ with $x'' \succ(a_n)\ x'$. Further, the convexity assumption of condition I implies that $x_s = x' + \frac{1}{s}(x'' - x')$ is strictly preferred to x' for each positive integer s. For s sufficiently large, we have $\overline{p}\cdot x_s < \overline{p}\cdot\overline{x}_n$, and $x_s \succ(a_n)\ \overline{x}_n$, which we already saw led to a contradiction. Thus, $\overline{p}\cdot\overline{x}_n \leqq \overline{p}\cdot x$ for all x prefered or indifferent to $\overline{x}_n$ for a_n. Q.E.D.

An immediate corollary to Proposition I is the upper hemi-continuity of the correspondence Q_N^* on $\mathcal{Q}_N$, where

$$Q_N^*(A) = \{(x_1,\ldots,x_N) \in R^{(M)} \times \ldots \times R^{(M)} \mid \exists p \in \Delta, z \in R^M \ni (p,x_1,\ldots,x_N,z) \in Q_N(A)\}$$

i.e., $Q_N^*(A)$ is the set of quasi-equilibrium consumption vectors. As well, the correspondence P_N defined by

$$P_N(A) = \{p \in \Delta \mid \exists (x_1,\ldots,x_N,z) \ni (p,x_1,\ldots x_N,z) \in Q_N(A)\}$$

is also upper hemi-continuous on $\mathcal{Q}_N$.

The following corollary helps link our result on quasi-equilibria to competitive equilibria.

Corollary. Suppose $w(a,p) > \min p\cdot X(a)$ for all $a \in A$ and all $p \in P_N(A)$, where $A \in \mathcal{Q}_N$. Then there exists a neighborhood U of A in $\mathcal{Q}_N$ such that $w(a',p') > \min p'\cdot x(a')$ for all $a' \in A' \in U$ and all $p' \in P_N(A')$.

Proof: Suppose not, then for any neighborhood U^α of A there exists $A^\alpha \in U^\alpha$ with $w(a^\alpha,p^\alpha) = \min p^\alpha\cdot X(a^\alpha)$ for some $a^\alpha \in A$ and some $p^\alpha \in P_N(A^\alpha)$. Let $(p^\alpha,x_1^\alpha,\ldots,x_N^\alpha,z^\alpha)$ be a quasi-equilibrium associated with p^α, and suppose without loss of generality that $p^\alpha\cdot x_1^\alpha = \min p^\alpha\cdot X(a_1^\alpha)$ (i.e., the agent with minimal wealth is the first agent in each economy A^α). We can then construct a sequence (A^k) converging to A, with $(p^k,x_1^k,\ldots,x_N^k,z^k) \in Q_N(A^k)$. By Proposition I, there exists a subsequence (A^h) and a subsequence $(p^h,x_1^h,\ldots,x_N^h,z^h)$ converging to $(p,x_1,\ldots,x_N,z) \in Q_N(A)$. Thus, $p^h\cdot x_1^h = w(a_1^h,p^h) =$

IV

Continuity of the Gains from Trade

We are now in a position to state and prove our main result.

Given a collection of N agents $(\bar{a}_1,\ldots,\bar{a}_N)$, we can view these agents as comprising a national economy A in autarky. Alternatively, we may consider them as a nation in a world economy $W = (\bar{a}_1,\ldots,\bar{a}_N,\bar{a}_{N+1},\ldots,\bar{a}_S)$, where $(\bar{a}_{N+1},\ldots,\bar{a}_S)$ is the "rest of the world".

Suppose that A belongs to $\mathcal{L}_N$ and W to $\mathcal{L}_S$, so that quasi-equilibria exist both for the economy A under autarky and for the international economy with free trade. Further, suppose that every agent in A benefits from opening up trade with the rest of the world, i.e., every agent in A prefers the consumption bundle he receives under each quasi-equilibrium for the world economy to that which he gets under any autarky quasi-equilibrium. With these assumptions, we show that for any world economy $W' = (a'_1,\ldots,a'_S)$ comprised of agents sufficiently like those in W in their preferences, wealth and technology sets, all the agents in $A' = (a'_1,\ldots,a'_N)$ will also benefit from the opening up of free trade with the rest of the world $(a'_{N+1},\ldots,a'_S)$, (provided that A' belongs to $\mathcal{L}_N$ and W' to $\mathcal{L}_S$).

<u>Proposition 2</u>. Let $A = (a_1,\ldots,a_N) \in \mathcal{L}_N$ and let $W = (a_1,\ldots,a_N,a_{N+1},\ldots,a_S) \in \mathcal{L}_S$. Suppose for each n, n = 1,...,N, that $\bar{x}_n \succ(a_n)\ \bar{\bar{x}}_n$ for all $(\bar{\bar{x}}_1,\ldots,\bar{\bar{x}}_N) \in Q^*_N(A)$ and all $(\bar{x}_1,\ldots,\bar{x}_S) \in Q^*_S(W)$. Then there exists a neighborhood U of W in $\mathcal{L}_S$ such that if $W' = (a'_1,\ldots,a'_S)$ belongs to U, and $A' = (a'_1,\ldots,a'_N)$ belongs to $\mathcal{L}_N$, then $\bar{x}'_n \succ(a'_n)\ \bar{\bar{x}}'_n$ for each n = 1,...,N, each $(\bar{\bar{x}}'_1,\ldots,\bar{\bar{x}}'_N)$ in $Q^*_N(A')$ and each $(\bar{x}'_1,\ldots,\bar{x}'_S) \in Q^*_S(W')$.

of no member of an economy A' differ by more than d from a, the characteristics of the identical agents comprising A, then all will gain from trade. Thus, if we return to the more familiar private ownership model, if consumers do not differ too greatly in terms of their preferences, endowments and ownership claims, all will share in any gains from trade.

FOOTNOTES

1) Other significant contributions in this line are the papers by Bewley (1970), Delbaen (1971), K. Hildenbrand (1972) and Mertens (1970).

2) This follows from the fact that identical consumers receive bundles which are identical up to indifference under the competitive mechanism.

3) In an unpublished paper, Mertens (1970) has provided a very general treatment of the upper hemi-continuity of the competitive equilibrium correspondence in production economies where the number of agents is not required to be fixed or finite. The convergence concepts used by Mertens, which are based on the weak topology on the space of probability measures, would appear to imply convergence of the type considered here when consideration is restricted to the case of a fixed, finite number of agents. Thus, his results are of a much stronger nature than that given here, although somewhat different correspondences are considered. On the other hand, Mertens' arguments employ much more sophisticated mathematical techniques than do ours, and so may not be so easily accessible as is the proof of Proposition 1.

4) The relation $\succsim$ is convex if $x \succ x'$ and $t \in (0,1)$ implies $x_t \equiv x' + t(x-x') \succ x'$, where, as usual, we write $x \succ x'$ iff $x \succsim x'$ and not $x' \succsim x$. If foreach $x \in X$ there exists x' with $x' \succ x$, then $\succ$ is non-saturated.

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