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STABILITY OF AGGREGATION PROCEDURES ULTRAFILTERS AND SIMPLE GAMES(*)

by

Pierre Batteau[1],

Jean-Marie Blin[2],(+)

and Bernard Monjardet [3]

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(1) Institut d'Administration des Entreprises, Université de Droit, d'Economie et des Sciences d'Aix-Marseille, France.

(2) Northwestern University, Evanston, Illinois, U.S.A.

(3) Ecole des Hautes Etudes en Sciences Sociales, Centre de Mathématique Sociale, et Universite de Paris V,France

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I. Introduction

It is well known that aggregating arbitrary sets of individual preferences in some ethically desirable way leads to some unappealing procedures, namely *a priori* externally imposed solutions ("imposed rule"), or internally imposed solutions ("dictatorial rule") [1]. Equally discouraging is the finding that incentive compatibility for democratic choice procedures is generally an unattainable goal short of an imposed or dictatorial rule [11] [20]. In the former case it has been shown [12] [16] that Arrow's conditions on social welfare functions (SWF) lead to the construction of the social ordering by considering subsets of voters whose preferences dictate society's preferences. Such subsets of voters form particular "families of majorities" in a qualititative sense rather than a purely quantitative one as in majority rule. Their properties have been studied in [16] [17]. These properties make them identical to proper strong simple games. This establishes a direct connection between the general preference aggregation problem and n-person games, specifically simple games. The same connection is explicitly exploited by Wilson [24] [5] and Bloomfield [4]. It would seem that in the context of the preference revelation problem, an even more direct connection should exist with game theory. Indeed, as Schmeidler and Sonnenschein [21] point out, the Gibbard-Satterthwaite problem amounts to asking whether one can construct a voting procedure for which sincere preference revelation is a Nash equilibrium (NE) or a dominant strategy (DS) (in Gibbard D.S. is used instead of N.E.) of the associated voting game for any conceivable profile of individual preferences. Furthermore, recent work on the relationship between Arrow-type Social Welfare Functions and non-manipulable voting procedures specifies the intimate connection linking the two concepts* [14]. We submit that another way of looking at this connection is to characterize the types of families of majorities underlying any Nash-stable voting procedure. The potential for this line of research is further suggested by a number of recent contributions to the preference aggregation-social choice theory elaborating upon the structure of "families of majorities". For instance Fishburn [9], Kirman and Sondermann [15] and Hansson [13] all recognize the ultrafilter structure of the families of majorities implied by Arrow's conditions. And Brown uses the filter-ultrafilter notion to seek ways to attenuate Arrow's impossibility theorem; defining the notion of pre-filter, he characterizes aggregation rules which are both socially decisive and not strictly dictatorial à la Arrow [6], [7].

In view of (1) the fruitfulness of this approach as evidenced by all these contributions to the preference aggregation area, (2) the game-theoretic nature of the preference revelation problem, and (3) the relationship found between Arrow-type aggregation functions and Nash-stable voting procedures it seems useful to examine whether such procedures can be characterized by families of majorities--or, equivalently, proper strong simple games. This is indeed the case as we show in this paper. It follows, of course, that the Gibbard-Satterthwaite theorem is also directly established through this method. However, we feel that the interest of our approach lies beyond the derivation of a simple constructive proof of this theorem. Rather it shows that the essential similarity in the mathematical structure of the two impossibility theorems--Arrow's and Gibbard-Satterthwaite's--stems from the identity of structure imposed on the one

(*) This connection was, of course, implicitly recognized--at least technically--in Gibbard's proof [11] and, explicitly in Satterthwaite's paper [20].

hand upon the "winning sets" of voters ("families of majorities") by Arrow's conditions and on the other hand upon the "preventing sets" of voters by the non-manipulability condition. This finding further suggests that, in dealing with the preference revelation problem, the duality between the preference profiles approach and the families of majorities approach as thoroughly investigated and exploited in [16] [17], [18], should also be useful, for instance in dealing with restricted domains.

This paper is divided in five sections. Following this introduction, section two sets the basic definitions and notation. Section three focuses on the logic of the derivation of the main results leaving the complete statement of the proofs to section four. Finally, to stress the originality of this approach, we briefly outline an alternative--but longer--route from these families of majorities to an Arrow-type theorem and, further, to the Gibbard-Satterthwaite result, thus establishing another link with the original proofs.

II. Notation and Definitions[(1)]

We consider a set A of (m) alternatives--$m \geq 3$--$A = \{x,y,z,\dots,w,t\}$ and a set V of v voters $V = \{1,2,\dots,i,j,k,\dots,v\}$. S.T.W are subsets of V, i.e. they are elements of the power set of V, denoted 2^V. Families of subsets of V are denoted by $\mathcal{F}$ and indexed appropriately according to the context. The empty family is denoted Φ whereas $\emptyset$ denotes the empty set. Thus any $\mathcal{F} \subset 2^V$ whereas $\emptyset \in 2^V$.

Each voter i has a complete ordering on A denoted P_i (or $>_i$). $\mathcal{P}$ denotes the set of all total orders on A.

[(1)] For simplicity, only the main definitions needed for the problem statement are given here. More technical concepts and definitions are introduced in subsequent sections as needed.

<u>Definition 1</u>: A preference profile $\pi = (P_1 \ldots P_i \ldots P_v)$ is a v-tuple of total orders on A.

Thus $\pi \in \mathcal{P}^v$ the v-fold cartesian product of $\mathcal{P}$. For convenience we denote $\boxed{D = \mathcal{P}^v.}$

<u>Definition 2</u>: <u>A Collective Choice Procedure</u> (CCP) d is a mapping from D onto A.(*)

<u>Definition 3:</u> A <u>Social Welfare Function</u> (SWF) g is a mapping from D onto $\mathcal{P}$.

In the context of collective choice procedures the classical manipulability problem can be stated as the following v-person ordinal game. Each player (economic agent, voter) has $\mathcal{P}$ as its strategy space--i.e. he is free to state any ordering $P \in \mathcal{P}$. For each strategy profile $\pi \in D$ the final outcome is given by the mapping d. Finally each player i orders the outcomes according to his sincere preference ordering $P_i^* \in \mathcal{P}$. Of course player i may find it advantageous to choose a strategy $P_i \neq P_i^*$ given d and $\pi_{-i} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_v)$.

<u>Definition 4</u>: A voting game (VG) is a five-tuple $\{V, D, \pi^*, d, A\}$.

For V,A and hence D given, every pair (π^*, d) yields a different game. For each such game one may inquire about the existence of Nash equilibria in pure strategies--mixed strategies being ruled out given the ordinal nature of the game. Formally we recall:

(*) Note that we set the image of D under d as the full range, i.e. $d(D) = A$. This, of course, rules out imposed solutions, e.g. some alternative is never chosen, whatever profile we pick. Given this assumption, we will not mention the imposed solution case in the statement of the results. The alternative assumption would only complicate slightly the arguments without changing the nature of the result.

Definition 5: A profile $\bar{\pi}$ is a Nash equilibrium of the game (V,D,π^*,d,A) if $\forall i \in V$ $d(\bar{P}_1,\ldots,\bar{P}_i,\ldots,\bar{P}_v)$ P_i^* $d(\bar{P}_1,\ldots,P_i,\ldots,\bar{P}_v)$ $\forall$ $P_i \in \mathcal{P}$.

Thus we can say

Definition 6: A profile $\bar{\pi} \in D$ is stable for d if $\bar{\pi}$ is a Nash equilibrium point of the game $(V,D,\bar{\pi},d,A)$.

Definition 7: A CCP, d, is stable (non-manipulable, strategy-proof) if any profile π is stable under d.

Of course it should be noted that if we used a more general strategy space than the set of v-tuples of total orders (D), we could still apply this notion of stability to the strategies--e.g. some abstract messages sent by the agents--but the interpretation of individual manipulability of the outcome no longer applies.

As noted in [11] the stability notion--"straightforwardness" in Gibbard's words--can be used without defining what is meant by sincerity; for in a value-free sense there are as many notions of sincerity as there are mappings from individual preferences to strategy spaces. Furthermore, even if strategies are taken to be individual preferences, sincerity is not always obviously defined--as for instance when outcomes under d are subsets of A, not necessarily singletons. Thus, according to our definition, a CCP is stable if and only if the sincere strategy π is a Nash equilibrium of the game (π,d) for any $\pi \in D$. This can easily be shown to be equivalent to saying that, for each player, this sincere strategy is a dominant strategy in any case.[1] We should also note that non-stability of a CCP, d, can result from the existence of at least one non-stable profile or from the non-existence of a pure strategy Nash equilibrium in the game.

In the sequel we proceed as follows. First we examine how preference profiles should be mapped into an outcome to yield a stable CCP. Specifically we consider which partitions of the set 2^V, i.e. which 'families of majorities' are implied by such a mapping. Then we show how these families define a simple game whose family of winning sets forms an ultrafilter. The Gibbard-Satterthwaite impossibility theorem for a finite voter set follows at once. Its meaning for the case of an infinite voter set is also mentioned.

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[1] Formally we can state

Proposition: d is stable if and only if (iff) $\pi \in D \Rightarrow \pi$ is a dominant strategy of the game (π,d).

Proof: (1) Sufficiency: π is a dominant strategy of the game (π,d), i.e.

$$\forall i\ [P'_i \in \mathcal{P},\ \pi'_{-i} \in \mathcal{P}^{V-1}] \Rightarrow d(P_i,\pi'_{-i}) \underset{P_i}{\gtrsim} d(P'_i,\pi'_{-i})$$

Specifically, if $\pi'_{-i} = \pi_{-i}$

$$\forall i\ [P'_i \in \mathcal{P} \Rightarrow d(P_i,\pi_{-i}) \underset{P_i}{\gtrsim} d(P'_i,\pi_{-i})$$

which means that π is a Nash equilibrium for the game (π,d). And this argument holds $\forall\ \pi \in D$

(2) Necessity: If d is stable this means that π is a Nash equilibrium of (π,d) $\forall\ \pi \in D$. Hence

$$\forall i\ \{P'_i \in \mathcal{P} \Rightarrow d(\pi_{-i},P_i) \underset{P_i}{\gtrsim} d(\pi_{-i},P'_i)\}$$

But since this holds true $\forall\pi$, we can write

$$\forall i:\ \{P'_i \in \mathcal{P}\ \underline{\text{and}}\ \pi_{-i} \in \mathcal{P}^{V-1} \Rightarrow d(\pi_{-i},P_i) \underset{P_i}{\gtrsim} d(\pi_{-i},P'_i)$$

which means that P_i is a dominant strategy for i, $\forall i$. Hence π is a dominant strategy for (π,d).

III. Stable CCPs and "Preventing Families" of Voters

3.1. Characterization of stable CCP's

Let us consider a stable CCP with full range $d(D) = A$. First we state a fundamental property which must always hold for any stable d. Let $\pi = (P_1 \ldots P_i \ldots P_v)$ and $V_{xy}(\pi) = \{ i \in V \mid x P_i y \}$

Proposition 1. For a stable CCP, d,

$$[d(\pi) = x, \pi' \in D, V_{xy}(\pi') \supseteq V_{xy}(\pi), y \neq x] \Rightarrow d(\pi') \neq y$$

In other words, if x prevails socially under d for some profile π, for any other profile π' in which the set of voters preferring x to y--for any y--still includes the voters preferring x to y initially (in π), then y should never prevail socially for such profiles if d is to be stable. Proposition 1 has some important immediate implications:

(i) First, if we consider two profiles π and π' with $d(\pi) = x$ and $y \neq x$, $V_{xy}(\pi) = V_{xy}(\pi')$ from proposition 1, it follows that $d(\pi') \neq y \; \forall y \neq x$ and thus $d(\pi') = x$. This is nothing but the independence property adapted for a CCP.

(ii) Secondly, if we note that proposition 1 does not require that $V_{xy}(\pi) = V_{xy}(\pi')$ but only that $V_{xy}(\pi) \subseteq V_{xy}(\pi')$, we conclude that d exhibits a monotonicity property: adding individuals to the group favoring x over y cannot lead to a reversal of the social outcome away from x in favor of y. Roughly, no perverse vote counting is to be allowed for a stable d.

(iii) But now assume there exists some profile, π, where $V_{yx}(\pi) = \emptyset$. Then $d(\pi) = y$ cannot hold. For suppose it did; then any other profile π' is such that $V_{yx}(\pi') \supseteq V_{yx}(\pi) = \emptyset$ while from proposition 1 $d(\pi') \neq x$. But then $x \notin d(D)$ contradicting our assumption of a full range CCP--i.e. $d(D) = A$. This property is known as the Weak Pareto property. Formally stated, d is <u>weakly paretian</u> iff $\forall\ \pi \in D, \forall\ x \neq y\ V_{xy}(\pi) = V \Rightarrow d(\pi) \neq y$.

(iv) Finally the strong Pareto property, $\forall\ y \neq x\ V_{xy} = V \Rightarrow d(\pi) = x$, also holds for d since $d(\pi) \neq z\ \forall\ z \neq x$ in this case.

These four remarks can be formally summarized in the following proposition:

Proposition 2: A stable CCP is independent, monotonic, weakly paretian and paretian.

Turning back to Proposition 1 above, it should be noted that it singles out a particular set of voters (V_{xy}) as having a particular power, namely the <u>power to prevent y</u> from winning whenever they agree on ordering x above y. Thus we say:

<u>Definition 8</u>: A subset of voters $S \subseteq V$ is d-preventing for y by x [(+)] for a stable CCP, d, if $\forall\ \pi \in D$ with $V_{xy}(\pi) = S \quad d(\pi) \neq y$.

For any pair of distinct alternatives $(x \neq y)$ we denote by $\mathcal{F}_{xy} \subset 2^V$ the family of all subsets of V which are d-preventing for y by x.

[(+)] The term 'preventing set' was suggested by David Gale.

As we are going to show now these 'preventing families' are fundamental for the study of stable CCP's. Characterizing these families by their properties will provide us with a constructive procedure to exhibit stable CCP's. Specifically, our next theorem specifies the structure of these preventing families and shows that the pair $(\mathcal{F}_{xy}, \mathcal{F}_{yx})$ is a 'blocking system' as introduced by Edmonds and Fulkerson [8].

Following their terminology, they define a <u>clutter</u> on V as a family $\mathcal{R}$ of subsets of V such that no member of $\mathcal{R}$ is contained in another member of $\mathcal{R}$. A <u>blocking clutter</u> $\mathcal{Y}$ (blocker) of $\mathcal{R}$ is the clutter consisting of the minimal subsets of V that have non-empty intersection with every member of $\mathcal{R}$. The pair $(\mathcal{R}, \mathcal{Y})$ is said to form a <u>blocking system</u>. Edmonds and Fulkerson have shown that such a blocking system is characterized by the following property. "For any partition of V into two sets V_o and $V_1 (V_o \cap V_1 = \emptyset$ and $V_o \cup V_1 = V)$, either a member of $\mathcal{R}$ is contained in V_o or a member of $\mathcal{Y}$ is contained in V_1 but not both".

The pair of preventing families $(\mathcal{F}_{xy}, \mathcal{F}_{yx})$ is now characterized as follows:

<u>Theorem 1</u>: Let d be a stable CCP. For each (x,y) pair of alternatives $(x \neq y)$ there exists a non-empty family of subsets of V, $\mathcal{F}_{xy}$, such that

(1) $\mathcal{F}_{xy} \neq \Phi, \ \emptyset \notin \mathcal{F}_{xy}$

(2) $S \in \mathcal{F}_{xy} <=> \{\forall \pi \in D: V_{xy}(\pi) = S \Rightarrow d(\pi) \neq y\}$

(3) $S \in \mathcal{F}_{xy} <\Rightarrow \exists \ \pi \in D: V_{xy}(\pi) = S$ and $d(\pi) = x$

(4) $\mathcal{F}_{xy}$ is a hereditary family: $S \in \mathcal{F}_{xy}, T \supseteq S \Rightarrow T \in \mathcal{F}_{xy}$

(5) $\mathcal{F}_{yx} = \mathcal{F}^{d}_{xy} = \{T \subseteq V: \overline{T} \notin \mathcal{F}_{xy}\}$

$\mathcal{F}_{yx} = \mathcal{F}^{t}_{xy} = \{T \subseteq V: \forall \ S \in \mathcal{F}_{xy}, T \cap S \neq \emptyset\}$

The following points should be noted.

(i) For the pair $(\mathcal{F}_{xy}, \mathcal{F}_{yx})$ to be a blocking system as we have defined it above, it suffices to take the minimal elements of these hereditary families $\mathcal{F}_{xy}$ and $\mathcal{F}_{yx}$. If we now extend the notion of a blocking system to encompass the hereditary families, we can call the pair $(\mathcal{F}_{xy}, \mathcal{F}_{yx})$, itself a blocking system.

(ii) It is easily seen that property 4 ('heredity') is equivalent to $\mathcal{F}^d = \mathcal{F}^t$.

(iii) Property 4 is also known in game theory as the defining property of a general simple game - i.e. not necessarily proper nor strong. Specifically, we recall that a general simple game is such that any superset of a winning coalition is also winning. (Here, the coalitions would be the elements of a family, say $\mathcal{F}_{xy}$). It is proper if and only if S winning implies $\bar{S}$ not winning. And it is strong if and only the converse holds: S not winning implies $\bar{S}$ winning.

3.2 <u>The Case of Two Alternatives</u>

It is interesting to examine the case of two alternatives $A = \{x,y\}$, for in this case we find that there are uncommon non-dictatorial stable CCP's. First, we state:

<u>Corollary</u>: Let d be a CCP defined on a two alternative set; d is stable if and only if there exists a blocking system[(+)] $(\mathcal{F},\mathcal{G})$ such that

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[(+)] In the extensive sense allowing hereditary families as explained in (i) above.

(1) $d(\pi) = x \Leftrightarrow V_{xy}(\pi) \in \mathcal{F}$

(2) $d(\pi) = y \Leftrightarrow V_{yx}(\pi) \in \mathcal{G}$

(3) $\mathcal{F}$ and $\mathcal{G}$ are hereditary families

(4) $\mathcal{G} = \mathcal{F}^d = \mathcal{F}^t$

That this condition is necessary follows from Theorem 1. Its sufficiency is readily verified. For since $\mathcal{F} = \mathcal{G}^d$, either $V_{xy}(\pi) \in \mathcal{F}$ or $\bar{V}_{xy}(\pi) = V_{yx} \in \mathcal{F}$. Hence, either $d(\pi) = x$ or $d(\pi) = y$. Furthermore, property 3 ($\mathcal{F}$ and $\mathcal{G}$ are hereditary families) ensures that d is stable. For suppose that $d(\pi) = x$ and π is unstable; then $\exists\ \pi' \in D$ and $i \in V$ such that $d(\pi') = y$ and $V_{xy}(\pi') = V_{xy}(\pi) \cup \{i\}$ which contradicts property 3.

Example

The following example for a set of 3 voters $V = \{1,2,3\}$ illustrates how applying the above conditions enables us to construct a non-dictatorial stable CCP, which differs from the commonly known neutral and symmetric procedure.

We let

$$\mathcal{F}_{xy} = \{(2,3);\ (1,2,3)\}$$

$$\mathcal{F}_{yx} = \{(2);\ (3);\ (1,2);\ (1,3);\ (2,3);\ (1,2,3)\}$$

Now if we pick, say, $\pi = (xy;\ yx;\ yx)$ we obtain $d(\pi) = y$. This example shows that, in a sense, 2 is dictatorial against x and agreement between 2 and 3 suffices to determine the outcome irrespective of 1, who does not belong to any of the two families.

3.3 The Ultrafilter Structure of 'Preventing Families' for More Than Two Alternatives

We first define a notion of neutrality among alternatives for a CCP,d.

Definition 9: A CCP is neutral if and only if $\forall \pi, \pi' \in D$, $\forall (x,y) \neq (z,t)$ $[V_{xy}(\pi) = V_{zt}(\pi')] \Rightarrow [d(\pi) \neq y \Leftrightarrow d(\pi') \neq t]$. In the case of stable CCP's characterized by their preventing families, it is readily seen that neutrality can be equivalently stated:

$$\forall(x,y) \neq \forall(z,t) \qquad \mathcal{F}_{xy} = \mathcal{F}_{zt}$$

Thus a neutral CCP can be characterized by a <u>unique</u> preventing family for any pair of alternatives (x,y). As it turns out, neutrality holds for a stable CCP. Formally, we can state:

<u>Proposition 3</u>: For $|A| > 2$, a stable CCP is neutral.

Thus, in the case of a stable CCP, we can meaningfully speak of the preventing family of subsets of voters as it is a unique family, denoted $\mathcal{F}$, irrespective of the pair of alternatives considered.

Now, focusing on this family $\mathcal{F}$, we further specify its properties beyond those stated in Theorem 1.

<u>Theorem 2</u>: The preventing family $\mathcal{F}$ underlying a stable CCP is an ultrafilter in 2^V.

An immediate consequence of theorem 2 is:

<u>Theorem (Gibbard-Satterthwaite)</u>: A stable CCP (for $|A| \geq 3$) is dictatorial.

This follows at once from the well-known fact that an ultrafilter on a finite set has a singleton base. Such a singleton is a privileged voter whose most preferred alternative always becomes the social outcome for a stable CCP. The case of an infinite voter set must be treated with some caution since free ultrafilters do exist on infinite sets. Just as Fishburn [9], Hansson [13], and Kirman and Sondermann [15] noted in the case of Arrow's theorem, it suffices to pick a free ultrafilter on the set of voters to obtain a 'non-dictatorial' stable CCP.

IV. Proofs

For clarity of exposition, we first prove two short lemmas. The following definitions are also needed.

Definition 6: Let P_i be a complete ordering on A. A subset B of A is optimal for P_i iff $(x \in B \text{ and } y \notin B) \Rightarrow [x P_i y]$

Definition 7: A subset $B \subset A$ is optimal for $\pi_I = \{P_i \mid i \in I, I \subset V\}$ iff B is optimal for each P_i in π_I

Definition 8: A sequence $(x_1, x_2, \ldots, x_k)$, $k \leq |A|$ is optimal for P_i iff

(i) The subset $\{x_1, x_2, \ldots, x_k\}$ is optimal for P_i

(ii) $x P_i x_2 P_i, \ldots, P_i x_k$

For instance, to say that $\{x,y\}$ is optimal for P_i means that x is first and y second in P_i or vice versa; whereas to say that (x,y) is optimal means that x is first and y second. Now, considering a profile π, any pair of alternatives (x,y) and a voter i, we wish to examine certain subsets of profiles π' obtained by leaving all but the ith component of π unchanged. As to the ith component, x and y go to the top of P_i as first and second respectively if we had $x P_i y$ originally; or in either order (but still at the top) if we had $i \in V_{yx}(\pi)$ initially. This operation defines a correspondence from D into itself for each i and each (x,y) pair. Namely, $\varphi_i^{xy}(\pi) = \{\pi' \mid P'_j = P_j \ \forall j \neq i$ and $\{x,y\}$ optimal for P'_i and, in addition, (x, y) optimal for P'_i if $i \in V_{xy}(\pi)\}$

In our first lemma, we consider the image profiles of π under the correspondence φ.

Lemma 1: If d is stable, $d(\pi) = x$ and $\pi' \in \varphi_i^{xy}(\pi) \Rightarrow d(\pi') = x$

Proof: Note first that $d(\pi') \neq x$ or y cannot hold if π' is to be stable since $\{x,y\}$ is optimal for P'_i.

Now, either $i \in V_{xy}(\pi)$ and $d(\pi') = y$ in which case π' is unstable for d;

or $i \in V_{yx}(\pi)$ and $d(\pi') = y$ in which case π is unstable for d.
Therefore, if d is to be stable, $d(\pi') = x$ must hold.
Then we extend the correspondences φ_i^{xy} to φ^{xy} by defining:

$$\varphi^{xy} = \varphi_1^{xy} \times \varphi_2^{xy} \times \dots \times \varphi_v^{xy}$$

If D is unrestricted φ^{xy} defines a correspondence from D into D; that is it associates with every element of D a non-empty subset of D[(1)]. For instance, $\varphi^{xy} = \varphi_1^{xy} \times \varphi_2^{xy}$ is formed by the union of φ_2^{xy} for all $\pi' \in \varphi_1^{xy}(\pi)$. In other words, for each π and each (x,y) pair φ associates all profiles π' such that

(i) $\{x, y\}$ is optimal for π'

(ii) if $i \in V_{xy}(\pi)$, then (x,y) is optimal for P_i'

If we repeatedly use the proof of Lemma 1 for each i and $\pi' \in \varphi_i^{xy}$ we are led to:

<u>Lemma 2</u>: If d is stable, $d(\pi) = x$ and $\pi' \in \varphi^{xy}(\pi) \Rightarrow d(\pi') = x$

Now note that the φ_i and φ correspondences verify the following property

$$\text{Property } (*) \begin{cases} \text{(i) } V_{xy}(\pi_2) \supseteq V_{xy}(\pi_1) \Rightarrow \varphi_i^{xy}(\pi_2) \subseteq \varphi_i^{xy}(\pi_1) \\ \text{(ii) } V_{xy}(\pi_2) \supseteq V_{xy}(\pi_1) \Rightarrow \varphi^{xy}(\pi_2) \subseteq \varphi^{xy}(\pi_1) \end{cases}$$

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(1) It should be noted that the unrestricted domain condition is crucial if the φ correspondence is to exist. For, with a restricted domain D we may well have $\varphi_i^{xy} = \emptyset$. For instance, if $A = R^2$ and D is the set of all profiles such that $P_i \in \pi$ has no maximal element, then we have $\varphi_i^{xy} = \emptyset$.

Proposition 1: For any stable CCP, d

$$[d(\pi) = x,\ \pi' \in D,\ V_{xy}(\pi') \supseteq V_{xy}(\pi),\ y \neq x] \Rightarrow d(\pi') \neq y$$

Proof: By property (*) $\varphi^{xy}(\pi') \subseteq \varphi^{xy}(\pi)$ since we assume $V_{xy}(\pi') \supseteq V_{xy}(\pi)$. Now by lemma 2 if d is stable and $x = d(\pi)$ then $d[\{\varphi^{xy}(\pi)\}] = \{x\}$. Thus $d[\{\varphi^{xy}(\pi')\}] \neq y$.

Q.E.D.

Finally a third lemma can be readily proven for stable CCP's.

Lemma 3: For a stable d, if B is optimal for $\pi \Rightarrow d(\pi) \in B$.

Proof: It follows directly from the weak Pareto property (implication iii, proposition 1).

This lemma ensures that if a CCP is to be stable, the outcome set must respect unanimous negative opinions

We are now in a position to prove our Theorem 1 regarding the preventing families of subsets of voters characterizing a stable CCP.

Consider the family $\mathcal{F}_{xy}$ of subsets of voters preventing y by x. According to Theorem 1 this family $\mathcal{F}_{xy}$ must have the following properties.

Property 1. It is not the empty family: $\mathcal{F}_{xy} \neq \Phi$

Since $d(D) = A$, $\forall x \in A\ \exists \pi \in D$ such that $d(\pi) = x$. From proposition 1 (implications (i) and (ii)) any profile π' such that $V_{xy}(\pi') \supseteq V_{xy}(\pi)$ yields $d(\pi') \neq y$. Thus $V_{xy}(\pi) \in \mathcal{F}_{xy}$.

Property 2. $[S \subseteq V,\ S \in \mathcal{F}_{xy}] \Leftrightarrow [\exists \pi \in D : V_{xy}(\pi) = S \text{ and } d(\pi) = x]$.

We show necessity first. Let $S \in \mathcal{F}_{xy}$ and consider a profile $\pi \in D$ such that

(1) $\forall i \in S$ x is first and y is second in P_i ((x,y) is optimal in P_i)

(2) $\forall i \notin S$ y is first and x is second in P_i ((y,x) is optimal in P_i)

Thus $V_{xy}(\pi) = S$, which means $d(\pi) \neq y$. Further, $\forall z \neq y$ $V_{xz} = V$. Now the weak Pareto property ensures that $d(\pi) \neq z$. Thus $d(\pi) = x$. Sufficiency readily follows from Proposition 1.

Property 3. $\mathcal{F}_{xy}$ is a hereditary family: $S \in \mathcal{F}_{xy}$, $T \supseteq S \Rightarrow T \in \mathcal{F}_{xy}$

Let $S \in \mathcal{F}_{xy}$. By Property 2 $\exists \pi \in D$ such that $V_{xy}(\pi) = S$ and $d(\pi) = x$. Let $T \supseteq S$. Now Proposition 1 ensures that $\forall \pi'$ with $V_{xy}(\pi') = T \supseteq S = V_{xy}(\pi)$ we must have $d(\pi') \neq y$ if d is to be stable. Thus $T \in \mathcal{F}_{xy}$.

The conjunction of Property 1 and 3 implies that the Pareto property holds for preventing families: $V \in \mathcal{F}_{xy}$.

Property 4. $\mathcal{F}_{yx} = \mathcal{F}^{d}_{xy} = \{S \subseteq V : \overline{S} \notin \mathcal{F}_{xy}\}$

We must show $S \in \mathcal{F}_{yx} \Leftrightarrow \overline{S} \notin \mathcal{F}_{xy}$

(i) Necessity: Let $S \in \mathcal{F}_{yx}$. Consider a profile π such that $\{x,y\}$ is optimal for π, $V_{yx}(\pi) = S$ and $V_{xy}(\pi) = \overline{S}$; that is, (x,y) is optimal for $\pi_{\overline{S}}$ and (y,x) is optimal for π_S. Lemma 3 guarantees that $d(\pi) = x$ or y. $S \in \mathcal{F}_{yx} \Rightarrow d(\pi) \neq x$. Thus $d(\pi) = y$ which means $\overline{S} \notin \mathcal{F}_{xy}$.

(ii) Sufficiency: Suppose $\overline{S} \notin \mathcal{F}_{xy}$. Taking the same profile π as in (i), property 2 implies $d(\pi) \neq x$. Then $d(\pi) = y$ must hold and $S \in \mathcal{F}_{yx}$.

Property 5. The empty set cannot be preventing: $\emptyset \notin \mathcal{F}_{xy}$.

Assume otherwise: $\emptyset \in \mathcal{F}_{xy}$. From property 4 this would imply $\overline{\emptyset} = V \notin \mathcal{F}_{yx}$ and then $\mathcal{F}_{yx} = \Phi$ contradicting property 1.

We now further characterize preventing families underlying any stable CCP by showing that there is only one preventing family for a given stable CCP whatever the pair of alternative considered. This is what we referred to as the neutrality property.

Proposition 3. A stable CCP is neutral.

Proof: We want to show that $\mathcal{F}_{xy} = \mathcal{F}_{zt}$ $\forall(x,y) \neq (z,t)$ for a stable CCP.

(1) First we show that $\mathcal{F}_{xy} \subseteq \mathcal{F}_{xz}$. Let $S \in \mathcal{F}_{xy}$ $T \in \mathcal{F}_{yz}$ and pick a profile π such that: (a) $\{x,y,z\}$ is optimal for π, and

$$\text{(b)}\quad \begin{array}{ll} \forall i \in S - T & x \underset{i}{>} z \underset{i}{>} y \\ \forall i \in S \cap T & x \underset{i}{>} y \underset{i}{>} z \\ \forall i \in (T-S) & y \underset{i}{>} z \underset{i}{>} x \\ \forall i \in (\overline{T \cup S}) & z \underset{i}{>} y \underset{i}{>} x \end{array}$$

Lemma 3 ensures that $d(\pi) = x,y$ or z

$$S \in \mathcal{F}_{xy} \Rightarrow d(\pi) \neq y$$

$$T \in \mathcal{F}_{yz} \Rightarrow d(\pi) \neq z$$

$\Rightarrow d(\pi) = x$ and since $V_{xz}(\pi) = S$ it follows that $S \in \mathcal{F}_{xz}$ (by Property 2, Theorem 1 above).

(2) Permute z and y in (1) to show $\mathcal{F}_{xz} \subseteq \mathcal{F}_{xy}$.

(3) The conjunction of (1) and (2) $\Rightarrow \mathcal{F}_{xz} = \mathcal{F}_{xy}$ $\forall z \neq x$. The same reasoning shows $\mathcal{F}_{xy} = \mathcal{F}_{ty}$ $\forall t \neq y$.

Then $\mathcal{F}_{xy} = \mathcal{F}_{tz}$ $\forall z \neq x$ and $t \neq y$. And $\mathcal{F}_{yx} = \mathcal{F}_{vu}$ $\forall u \neq y$ and $v \neq x$.

Letting $z = u$ and $t = v$, we get $\mathcal{F}_{xy} = \mathcal{F}_{tz} = \mathcal{F}_{vu} = \mathcal{F}_{yz}$. Neutrality holds for a stable CCP. Q.E.D.

This means that there exists a single family $\mathcal{F}$ for a given stable CCP.

Thus $\mathcal{F} = \mathcal{F}_{xy} = \mathcal{F}_{yx} = \mathcal{F}^{d}_{xy} = \mathcal{F}^{d}$, which means $S \in \mathcal{F} \Longleftrightarrow \bar{S} \notin \mathcal{F}$.

In other words, $\mathcal{F}$ is a 'family of majorities' (à la Guilbaud-Monjardet); or, equivalently, $\mathcal{F}$ defines a strong and proper simple game.(1)

We can now summarize the properties of the preventing family $\mathcal{F}$ characterizing a stable CCP.

(1) $\emptyset \notin \mathcal{F}$

(2) $V \in \mathcal{F}$

(3) $S \in \mathcal{F},\ T \supset S \Rightarrow T \in \mathcal{F}$

(4) $S \in \mathcal{F} \Longleftrightarrow \bar{S} \notin \mathcal{F}$

Now it has been shown by Guilbaud [12] that for a family $\mathcal{F}$ exhibiting properties (3) and (4) to be an ultrafilter it is sufficient that it also exhibits property (5) $[S \in \mathcal{F},\ T \in \mathcal{F},\ W \in \mathcal{F}] \Rightarrow [S \cap T \cap W \neq \emptyset]$. Theorem 2 establishes that $\mathcal{F}$ is indeed an ultrafilter by establishing property (5).

Theorem 2: The preventing family $\mathcal{F}$ underlying a stable CCP is an ultrafilter in 2^V.

Proof: As explained above, we only need to show property (5). Assume, a contrario, that $S, T, W \in \mathcal{F}$ and $S \cap T \cap W = \emptyset$. Consider the following profile π:

(1) $\{x,y,z\}$ is optimal for π

$\forall i \in S : x P_i y$

(2) $\forall i \in T : y P_i z$

$\forall i \in W : z P_i x$

$\forall i \in (S \cap T) \cup (T \cap W) \cup (W \cap S)$, $\{x,y,z\}$ come first, second or third.

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(1) See 3.1 for a definition of a strong and proper simple game.

Thus $\forall i \in S \cap T : x \; P_i \; y \; P_i \; z$

$\forall i \in T \cap W : y \; P_i \; z \; P_i \; x$

$\forall i \in W \cap S : z \; P_i \; x \; P_i \; y$

Such a profile can be constructed for $S \cap T \cap W = \emptyset$. Now from Lemma 3 we have $d(\pi) = x,y$ or z. But now note that

$V_{xy}(\pi) \supseteq S \;\Rightarrow\; d(\pi) \neq y$

$V_{yz}(\pi) \supseteq T \;\Rightarrow\; d(\pi) \neq z$

$V_{zx}(\pi) \supseteq W \;\Rightarrow\; d(\pi) \neq x$

This contradicts the fact that the outcome must be x, y or z as noted earlier.

Q.E.D.

It follows that

$\exists i \in V$ such that $\mathcal{F} = \{S \subseteq V : i \in S\}$. That is, i is the singleton base of this ultrafilter. In other words,

<u>Theorem (Gibbard-Satterthwaite)</u>: A stable CCP (for $|A| \geq 3$) is dictatorial.

V. An Alternative Approach

In this section we sketch an alternative indirect route based on an Arrow-type theorem. In contrast to the direct method of the previous sections, this derivation will, we hope, further highlight the differences with the traditional approach. In short it will be seen that the common underlying concept is once again that of a 'family of majorities' as explained before. Thus this notion can be truly regarded as the fundamental basis for key results in axiomatic social choice theory.

As we recall, Theorem 1 characterizes the class of stable CCP's by the existence, for each pair of alternatives (x,y), of a non-empty family of subsets of V $\mathcal{F}_{xy}$ verifying five fundamental properties.

Briefly, the alternative approach consists in: (1) associating, with any stable CCP thus characterized, an aggregation function f mapping any profile of individual preferences into a binary complete and asymmetric relation--i.e. a tournament relation (thus not necessarily transitive); (2) showing that for any such aggregation function certain properties hold, the conjunction of which make it dictatorial.

First, some definitions.

Definition 9: An Aggregation Function (AF) f is a mapping from D onto $\mathcal{T}$--where $\mathcal{T}$ denotes the set of all tournament relations on A.

Definition 10: A Decisive Aggregation Function (DAF) is an aggregation function f such that $f(\pi) = T \in \mathcal{T}_{max}$--where $\mathcal{T}_{max}$ denotes the subset of tournament relations on A which have a maximal element.[1]

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[1] This type of binary relation (in $\mathcal{T}_{max}$) is, in a sense, 'intermediate' between fully transitive tournaments (as in an Arrow SWF for strict preferences) and general tournaments. Thus we have AF $\supset$ DAF $\supset$ Arrow SWF.

Given any stable CCP d (thus characterized by its families of preventing sets $\mathcal{F}_{xy}$) we define its associated AF, $f = \psi(d)$, by

$$[(x,y) \in f(\pi)] \Longleftrightarrow [V_{xy}(\pi) \in \mathcal{F}_{xy}]$$

In words we say that x precedes y in the tournament $f(\pi) \in \mathcal{T}$ if and only if the group of voters who placed x above y in that profile π, belong to the family of sets that are d--preventing for y by x. Theorem 3, below, characterizes the AF f thus associated with d.

<u>Theorem 3</u>: The AF $f = \psi(d)$ associated with a stable CCP d is:

(1) a DAF: $f(\pi) \in \mathcal{T}_{max}$, i.e. there always exists a maximal element in the image tournament. Thus $\forall x \in A,\ \forall \pi \in D\ [d(\pi) = x] \Longleftrightarrow [\forall y \neq x, (x,y) \in f(\pi)]$

(2) independent

(3) weakly paretian

<u>Proof</u>: (1) f is a DAF. We must show that $f(\pi) \in \mathcal{T}_{max}\ \forall \pi \in D$.

(i) That $f(\pi) \in \mathcal{T}$ (i.e. is, indeed, a tournament) follows from the fact that $\forall (x,y)$ either $V_{xy}(\pi) \in \mathcal{F}_{xy}$ or $V_{yx}(\pi) \in \mathcal{F}_{yx}$; thus $(x,y) \in f(\pi)$ or $(y,x) \in f(\pi)$.

(ii) Now let $d(\pi) = x$. Then x is a maximal element of $f(\pi)$. For

$d(\pi) = x \Longleftrightarrow \forall y \neq x,\ V_{yx}(\pi) \notin \mathcal{F}_{yx} \Longleftrightarrow \forall y \neq x$

$V_{xy}(\pi) \in \mathcal{F}_{xy} \Longleftrightarrow \forall y \neq x,\ (x,y) \in f(\pi)$

(2) To show that the independence property holds for f, consider $\pi, \pi' \in D$ such that $V_{xy}(\pi) = V_{xy}(\pi') = S$. Now note that $[(x,y) \in f(\pi)] \Longleftrightarrow [V_{xy}(\pi) = S \in \mathcal{F}_{xy}]$. But then also $[V_{xy}(\pi') = S \in \mathcal{F}_{xy}]$. Thus $[(x,y) \in f(\pi')]$. Hence

$[(x,y) \in f(\pi)] \Longleftrightarrow [(x,y) \in f(\pi')]$.

(3) The weak Pareto property is immediate. Consider π such that $V_{xy}(\pi) = V$. Now since $V \in \mathcal{F}_{xy}$ then $(x,y) \in f(\pi)$ and thus $d(\pi) \neq y$.[(1)] Q.E.D.

In this case of three alternatives in A, we can readily demonstrate the following corollary.

Corollary 1. A stable CCP d, on a 3-alternative set A is dictatorial.

Proof. It is well known that for a 3-element set a tournament on A is transitive iff the tournament has a maximal element. Thus the DAF f associated with d always yields a total order on A for $|A| = 3$. Now since f is independent and Paretian, Arrow's theorem ensures that it is dictatorial. Q.E.D.

In the case of more than three alternatives, ($|A| > 3$) this simple corollary is not applicable since we are dealing with decisive aggregation functions (DAF) rather than the more restricted Arrow-type SWF. Clearly, what is needed is a result similar to Arrow's theorem but applicable to DAF's. The following Lemma and theorem 4 generalize Arrow's theorem to DAF's.[(2)]

Consider some DAF f. Following standard terminology we call $S \subset V$ (x,y)-decisive for f and π if and only if $(x,y) \in f(\pi)$ and $V_{xy}(\pi) = S$. Now if f is independent and S is (x,y)-decisive for f and π, S is (x,y)-decisive for any π' such that $V_{xy}(\pi') = S$. Thus, in this case, we can characterize f by the families

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(1) Although this is not needed for our purpose here, it can easily be shown that such a DAF is also monotone, paretian and non-imposed.

(2) It should be noted that the result is quite independent of the issue of stability of a CCP. We are considering any DAF (see Definition 10) not one necessarily associated with some CCP.

of such (x,y)-decisive subsets of V. Denoting by $\mathcal{D}_{xy}$ such a family it can be shown that it verifies the following properties (as in Theorem 1).

- $\mathcal{D}_{xy}$ is a hereditary family
- $\mathcal{D}_{yx} = \mathcal{D}^d_{xy}$
- $(x,y) \in f(\pi) \Leftrightarrow [V_{xy}(\pi) \in \mathcal{D}_{xy}]$

Furthermore, it can be shown [16] that if f is non-imposed all the $\mathcal{D}_{xy}$ families are different from Φ and 2^V. It can also be noted that if f is weakly paretian it is paretian and, thus, non-imposed.

Now we must show that if f is independent and weakly paretian it is neutral, i.e. for any distinct pairs (x,y) and (z,t): $\mathcal{D}_{xy} = \mathcal{D}_{zt}$. This is the object of the next lemma.

Lemma: Let f be an independent and weakly paretian DAF. Then f is neutral.

Proof. The proof of this lemma is identical to that of Proposition 3 establishing the neutrality of a stable CCP.

For instance, we show that for any triple of distinct alternatives (x,y,z) $\mathcal{D}_{xy} \subseteq \mathcal{D}_{xz}$ by picking $S \in \mathcal{D}_{xy}$ and $T \in \mathcal{D}_{yz}$ and a profile π as in the proof of Proposition 3. Then as f is weakly paretian, we necessarily have the maximal element of the tournament under f, denoted $\overline{f}(\pi) = x$ or y or z. Now $S \in \mathcal{D}_{xz} \Rightarrow (x,y) \in f(\pi) \Rightarrow \overline{f}(\pi) \neq y$. Similarly $T \in \mathcal{D}_{yz} \Rightarrow (y,z) \in f(\pi) \Rightarrow \overline{f}(\pi) \neq z$. Hence $\overline{f}(\pi) = x \Rightarrow (x,z) \in f(\pi)$ and since $V_{xz}(\pi) = S \Rightarrow S \in \mathcal{D}_{xy}$. The rest of the proof follows as in Proposition 3. Q.E.D.

Hence we conclude that $\mathcal{D}$ is the unique family of decisive sets: $\mathcal{D} = \mathcal{D}_{xy} = \mathcal{D}_{yx} = \mathcal{D}^d_{xy} = \mathcal{D}^d$.

Thus $S \in \mathcal{D} \Leftrightarrow \overline{S} \notin \mathcal{D}$ which means that $\mathcal{D}$ is a 'family of majorities '(à la Guilbaud-Monjardet) just like $\mathcal{F}$ the 'preventing family'.(1)

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(1) See Section IV end of Proposition 3 above for the properties of such a family.

The next theorem generalizes Arrow's theorem to DAF's.

Theorem 4. Let f be an independent and weakly paretian DAF. Then f is dictatorial.

Proof. The previous lemma establishes that such an f is neutral hence characterized by its 'family of majorities' $\mathcal{D}$. The rest of the proof is identical to the proof of Theorem 3 above.

Theorem (Gibbard-Satterthwaite). For $|A| \geq 3$, a stable CCP is dictatorial.

Proof. Follows immediately from Theorems 3 and 4.

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