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OPTIMAL CONSUMPTION AND EXPLORATION OF NONRENEWABLE RESOURCES UNDER UNCERTAINTY

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ABSTRACT

We consider the intertemporal problem of optimally consuming a natural resource and exploring for new sources of supply of that resource. Resource consumption yields social utility while the exploration effort controls the uncertainty in the timings of discoveries as well as their magnitudes. The objective is to choose an optimal consumption and exploration policy so as to maximize the expected discounted utility of consumption net of the exploration cost over an infinite planning horizon. We present a controlled storage process model of the problem and under reasonable conditions we characterize the existence and the properties of optimal policies and prices.

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1. INTRODUCTION:

Economic planning in the presence of a nonrenewable natural resource raises a number of special and interesting problems of intertemporal choice. Since Hotelling's [12] classic work, a considerable amount of recent literature that exists in this area is exemplified by the papers in the 1974 symposium of the Review of Economic Studies. Although it is becoming recognized that uncertainty plays a crucial role in the problems of dynamically managing a nonrenewable resource, only a few of the studies reported so far have explicitly incorporated uncertainty into their models. Gilbert [10] and Loury [15] have analyzed optimal consumption rates when the total resource stock is uncertain and learning about the true stock size takes place only through the extraction process. Assuming the total stock size to be known, Dasgupta and Heal [7] and Dasgupta and Stiglitz [8] determine optimal resource consumption patterns when there is uncertainty about the (exogenously determined) time at which a perfect, producible substitute becomes available. MacQueen [16,17] permits the resource stock on hand to change by random quantities at random times that are beyond the planner's control. For a fixed consumption rate, he then characterizes a strategy for choosing among these randomly arising gambles, so as to maximize the probability of survival or the expected time until the Doomsday.

We consider a socially managed economy with a natural resource (such as oil, mineral deposits or, more generally, energy) which is essential and which can be stored without depreciation over the planning horizon. Although the resource cannot be produced, the amount on hand may be increased by exploring and searching for new sources of supply of the resource. The exploration process involves uncertainty regarding the time until a successful discovery as well as regarding the magnitude of supply gained upon discovery. This uncertainty can be partially controlled by the amount of costly exploration effort (search intensity) selected by the central planner. A larger exploration effort costs more but is also more likely to result, on average, in a more prompt discovery of a larger supply.

In addition to the possibility of increasing the stock on hand through exploration, this stock may be depleted through consumption. A higher consumption rate yields a greater immediate social utility (although at a diminishing rate) but also leaves less quantity of the resource for future consumption. Thus, current consumption and exploration decisions affect not only the immediate utilities and costs but also the uncertain future stock and therefore all future decisions and payoffs. At each instant of time, given the amount of proven reserves of the resource on hand, the central planner's problem is then to determine optimal consumption and exploration rates so as to maximize the total expected discounted value of the utility of consumption less the exploration cost over the infinite planning horizon, taking into account the uncertainties and the intertemporal considerations involved.

Recently and independently of our work, Arrow [1] has reported some preliminary work on a similar problem. He has proposed a dynamic programming model and has used heuristic arguments to obtain some partial results. Our approach is to develop a Markov process model for the level of the proven reserves and then to apply Markov decision theory to obtain the optimal control results. In Section 2 we provide a precise formulation of our problem within the framework of controlled storage processes. Specifically, we model the resource stock on hand as a storage process that is controlled by a policy which specifies the consumption and the exploration rates, at each instant of time, as a function of the stock on hand. Storage processes, as Markovian models of dams, have been studied by Moran [20], Cinlar and Pinsky [4,5], and Harrison and Resnick [11], among others. The general theory of continuous-time Markov decision processes.has been developed by Howard [13], Miller [18], Kakumanu [14], Doshi [8] and others, while optimal control of storage processes has been the subject of Morais [19] and Pliska [21].

In Section 3, we employ Dynkin's [9] theory of weak infinitesimal generators of Markov processes to characterize the total expected discounted return from any consumption and exploration policy in terms of a functional equation. We also provide the dynamic programming functional equation and show that it has a unique, nonnegative, increasing, concave and differentiable solution, which turns out to be the maximum expected discounted return following an optimal policy. Since our storage process differs in several details from those studied in the literature, the probabilistic results of Section 3 are also of independent interest; however, due to the technical nature of the proofs, we provide them in the Appendices.

In Section 4, we prove the existence of an optimal consumption and exploration policy and we characterize its economially meaningful structure. Specifically, we define a policy that is determined through the dynamic programming functional equation of Section 2 and we show that this policy is admissible, that its associated return is the solution of this functional equation, and that it is in fact the maximum attainable return. In this process of proving the existence of an optimal policy, we also show that, as a function of the positive resource stock on hand, the optimal consumption rate is strictly positive and nondecreasing, while the optimal exploration rate is nonincreasing, in the level of proven reserves.

Section 5 studies the dynamics of the shadow prices. There we demonstrate the stochastic analog of the classical result that the shadow price of the resource rises at the social rate of discount.

Finally, Section 6 concludes with some additional remarks, interpretations and possibilities for future research.

2. MODEL FORMULATION

Let $X_t \geq 0$ denote the level of proven reserves of a natural resource at time $t \geq 0$. It represents the stock of the resource that is known to exist at time t and is measured in physical units (such as the number of barrels of oil or, more generally, the number of BTU's of energy). We do not distinguish between known reserves in the ground and extracted reserves held in inventory. We call X_t the state of the process at time t and $(\mathbb{R}_+, \mathcal{R}_+)$ the state space, where $\mathbb{R}_+ = [0,\infty)$ and $\mathcal{R}_+$ is the Borel σ-field on $\mathbb{R}_+$.

At each time $t \geq 0$, the central planner observes X_t and determines the consumption rate c_t (measured in physical units per unit time) at which the resource stock is consumed. (In particular, this allows for the case where the resource is extracted and consumed directly and not held in inventory, so that c_t also equals the extraction rate.) Let $\bar{c} < \infty$ be an upper bound on the rate at which the resource can be consumed. Thus we require $c_t \in C \equiv [0,\bar{c}]$, and we shall also stipulate that $c_t = 0$ is the only admissible decision whenever $X_t = 0$.

In addition to the consumption rate, the planner also determines the exploration rate e_t, which is the intensity of search effort he expends in order to discover additional sources of supply of the resource. The exploration effort can be measured in physical units (such as the geographical area searched per unit time, as in Arrow [1]) or in dollars per unit time. Let $\bar{e} < \infty$ be an upper bound on the rate at which exploration can be carried out, using the available technology. Thus, we require $e_t \in E \equiv [0,\bar{e}]$. We shall sometimes use the notation $d = (c,e)$ for the vector of the consumption and exploration decisions and $D = C \times E$ for the <u>decision space</u>.

These decisions are selected according to a policy which specifies the consumption and exploration rates as functions of the resource stock on hand. For technical reasons, we shall place some restrictions on the policies to be considered by making the following

<u>Definition 1</u>. An admissible policy is a Borel - measurable function $\pi(\cdot) = (c(\cdot),\ e(\cdot))$ mapping $\mathbb{R}_+$ into D such that (i) $c(0) = 0$, (ii) $\pi(\cdot)$ is left continuous and (iii) for any $x_0 > 0$, there exists some $\epsilon > 0$ such that $c(x) > \epsilon$ for all $x \geq x_0$.

Let A denote the set of all such admissible policies. Using $\pi \in A$ means that the decisions $\pi(x)$ are made whenever the process is in state x. The functions $c(\cdot)$ and $e(\cdot)$ will be called the consumption and the exploration policies, respectively. In Definition 1, the measurability and left continuity requirements are essential for technical reasons, while $c(0) = 0$ is a natural condition. The requirement (iii) has been made to rule out troublesome and uninteresting cases where the resource stock is not allowed to decrease below some positive level. Later on it will be seen that a policy that is optimal in A is also optimal in the larger class of policies in which (iii) is not required. To avoid confusion, we will write the consumption and exploration policies as $c(\cdot)$ and $e(\cdot)$ and the corresponding decisions as c and e, respectively.

Each $\pi \in A$ gives rise to a Markov process $\{X_t; t \geq 0\}$ which is governed by the storage equation

$$(2.1) \qquad X_t = X_0 + I_t - \int_0^t c(X_s)ds \qquad , \ t \geq 0,$$

where I_t is the total quantity of the resource that has been discovered during $[0,t]$. Equation (2.1) simply says that the resource on hand at any time is composed of the initial stock level and the additional amount discovered less the total amount consumed so far.

The consumption policy $c(\cdot)$ affects the resource level deterministically through the integral term in (2.1), while the exploration policy $e(\cdot)$ affects the discovery process $\{I_t; \ t \geq 0\}$ which involves uncertainty. An exploration rate $e \in E$ determines the probabilistic rate $\lambda(e)$ at which discoveries take place, as well as the distribution $G(e,\cdot)$ of the size of a discovery if it occurs. In precise terms, the discovery rate is specified by a continuous function $\lambda: E \to \mathbb{R}_+$, while the discovery size is specified by a Markov kernel $G: E \times \mathcal{R}_+ \rightarrow [0,1]$ such that (i) for every $e \in E$, $G(e,\cdot)$ is a probability measure on $\mathcal{R}_+$ and (ii) for every $B \in \mathcal{R}_+$, $G(\cdot,B)$ is a continuous function on E. Roughly speaking, if the exploration rate is $e \in E$ when a discovery takes place, then $G(e,B)$ is the probability that the size of the discovery is equal to some element of the Borel set B; in particular, $G(e,[0,y])$ is the distribution function of the discovery size evaluated at $y \geq 0$. With these specifications, the stochastic discovery process $\{I_t;\ t \geq 0\}$ has almost surely nondecreasing right-continuous paths of the pure jump type and only finitely many jumps in any finite time interval, where the jump rates and the jump size distributions are dependent upon the exploration rate decisions. Each jump corresponds to a discovery and the magnitude of a jump equals the size of the discovery.

It is reasonable to assume that a positive exploration rate is essential to make a discovery and that a higher exploration rate promotes quicker discoveries of larger supplies, in a probabilistic sense. To make this precise and for future convenience we introduce the notation

$$\beta(e,\cdot) = \lambda(e)\ G(e,\cdot),$$

so that $\beta(e,B)$ is the probabilistic rate, under $e \in E$, at which discoveries of a size equal to some element of $B \in \mathcal{R}_+$ occur. The stochastic dominance requirement is then our

<u>Assumption 1</u>. For each fixed $y \geq 0$, the continuous mapping $e \to \beta(e,[y,\infty))$ is nondecreasing with $\beta(0,[y,\infty)) = 0$.

This says that for any $y \geq 0$, the probabilistic rate of discoveries of size in excess of y is nondecreasing in the exploration rate e. Since the discovery rate $\lambda(e) = \beta(e,[0,\infty))$, an immediate consequence of Assumption 1 is that $\lambda(0) = 0$ and that $\lambda(e)$ is nondecreasing on E. An example of a measure β which satisfies Assumption 1 is $\beta(e,dy) = \lambda(e)\, F(dy)$, where $\lambda(\cdot)$ is continuous nondecreasing on E and $F(\cdot)$ is a fixed probability distribution. Here the exploration effort directly affects the discovery rate but not the size of a discovery. Another example is obtained by taking $\beta(e,dy) = \lambda F(e^{-1}dy)$ for $e > 0$, so that the discovery rate $\lambda \geq 0$ is unaffected by the exploration rate but increasing the exploration rate shifts the discovery size distribution to the right.

For any particular consumption and exploration policy $\pi \in A$, the resource stock level process $\{X_t;\ t \geq 0\}$ evolves in a simple manner. Following Çinlar [3], let, for $t \geq 0$, $x \geq 0$,

$$q(x,t) = \begin{cases} \inf\{y > 0 : \int_y^x [1/c(z)]dz \leq t\} & \text{if } x > 0 \\ 0 & \text{if } x = 0, \end{cases}$$

so that $q(x,t)$ is the resource level at time t if the initial level is x and no new discoveries occur in $[0,t]$. Let $T_1, T_2, \ldots$ be the successive (random) times at which discoveries occur with $Y_1, Y_2, \ldots$ the (random) magnitudes of the corresponding discoveries. Then

$$X_t = q(X_{T_n}, t - T_n) \qquad T_n \leq t < T_{n+1}$$

and

$$X_{T_{n+1}} = q(X_{T_n}, T_{n+1} - T_n) + Y_{n+1} \qquad n = 0,1,2,\ldots\ldots\ldots ,$$

where $T_0=0$ and X_0 is the initial resource level. The probability distribution of Y_n is governed by $G(e(X_{T_n}-), \cdot)$. As to the probability distributions of times between successive discoveries, we have, for $s \geq 0$ and $x \geq 0$,

$$P\{T_{n+1} - T_n \leq s \,|\, X_{T_n} = x\} = 1 - \exp[-\int_0^s \lambda(e(q(x,u)))du] .$$

Note that if the exploration policy $e(\cdot)$ or the exploration rate $\lambda(\cdot)$ is constant, the interdiscovery times will be exponentially distributed. In particular, the sojourn time in state 0 is exponentially distributed with parameter $\lambda(e(0))$.

To complete the description of the storage process (2.1) under the policy $\pi \in A$, we define

$$R(x) = \int_1^x [1/c(z)]dz \qquad 0 \leq x < \infty$$

so that, for $0 \leq y < x < \infty$, the quantity $R(x)-R(y)$ is the time required to deplete the resource level from x to y if no new discoveries occur. Note that the admissibility requirement (iii) of Definition 1 implies that $0 < R(x)-R(y) < \infty$ whenever $0 < y < x < \infty$, even though perhaps $R(0) = -\infty$. If $R(0) = -\infty$, then state 0 is <u>inaccessible</u>, i.e.,given any positive resource level, the Doomsday will, almost surely, never come following that policy. Otherwise $R(0) > -\infty$ and state 0 is accessible; we have to allow for both possibilities under policies $\pi \in A$. This and some other aspects distinguish our storage process from the ones studied in the existing literature. However, it is straightforward to show, as in Çinlar and Pinsky [4,5] and Morais [19], that, under any $\pi \in A$, the stochastic process $\{X_t; t \geq 0\}$ constructed above is a strong Markov process with stationary transition probabilities and with sample paths which are right-continuous, have left-hand limits at each $t \geq 0$ and have only a finite number of discontinuities in any finite interval, almost surely.

In order to compare different policies in A and to choose the best one, it remains to specify the utilities and costs associated with the consumption and exploration decisions. Let $u: C \to \mathbb{R}_+$ denote the consumption utility rate function and we assume it to be continuous, concave and nondecreasing with $u(0) = 0$ and with a finite positive derivative at $c = 0$. Whenever the consumption rate is c, the economy earns utility at a rate $u(c)$, measured in money units. Finally, we denote the exploration cost rate function as $h: E \to \mathbb{R}_+$ and assume it to be continuous nondecreasing with $h(0) = 0$; whenever the exploration rate e is selected, the cost is incurred at the rate $h(e)$. Let $\rho > 0$ be the social rate of discount. Note that, by our assumptions, the net benefit rate $u(c)-h(e)$ is bounded and continuous on D and that we have not made any differentiability assumptions other than that $u(\cdot)$ is differentiable at the origin. The assumption $u'(0) < \infty$ is not restrictive, for if $u'(0) = \infty$ (as, for example, in Dasgupta and Heal [6]), then by the concavity and continuity assumptions, u can be uniformly approximated by a sequence of concave, continuous functions $\{u_n\}$ with $u_n'(0) < \infty$.

We should also point out that we are not treating any extraction costs explicitly, because we are assuming such costs are independent of the resource level and amount of discoveries to date. Hence the extraction costs are incorporated in the utility rate function, so that $u(\cdot)$ is the utility rate of consumption net of extraction costs, as if the extraction rate equals c_t at each point in time. Note that our assumption that $u(\cdot)$ is nondecreasing on C is not restrictive, for if, alternatively, $u(\cdot)$ were maximized at c^*, say, in the interior of C, then it would never be optimal to choose a consumption rate in excess of c^*, so one could simply re-define $\bar{c} = c^*$.

Corresponding to each consumption and exploration policy $\pi \in A$, we are interested in the infinite horizon expected discounted return $V_\pi(x)$ as a function of initial resource level $X_0=x \geq 0$. Thus,

$$(2.2)\quad V_\pi(x) = E_\pi\left[\int_0^\infty \exp(-\rho t)[u(c(X_t))-h(e(X_t))]dt \mid X_0=x\right],\quad x \geq 0.$$

Let
$$V(x) = \sup_{\pi\in A} V_\pi(x), \qquad x \geq 0$$

be the maximum expected discounted return function. Define $\pi^* \in A$ to be an optimal policy if

$$(2.3)\quad V_{\pi^*}(x) = V(x) \text{ for all } x \geq 0 .$$

The balance of the paper is concerned with the study of existence, uniqueness and properties of V_π, V and π^*

3. THE EXPECTED RETURN FUNCTIONS:

We are first interested in characterizing the expected return function V_π for any policy $\pi \in A$. This is done by expressing V_π as the unique solution of the functional equation in Theorem 1 below. The proof of this theorem uses the weak infinitesimal generator of the underlying Markov process (see Dynkin [9] and Morais [19]) and we postpone it to Appendix A.

Theorem 1. For any $\pi \in A$ with $R(0) = -\infty$, V_π is the unique bounded function which is absolutely continuous on $(0,\infty)$, which has a left-continuous left-derivative V_π' on $(0,\infty)$ with $V_\pi'(\cdot)\, c(\cdot)$ bounded on $(0,\infty)$, and which satisfies

$$(3.1)\quad \rho V_\pi(x) = u(c(x)) - h(e(x)) + \int_0^\infty [V_\pi(x+y) - V_\pi^-(x)]\beta(e(x),dy) - V_\pi'(x)c(x),\ x > 0$$

and

$$(3.2)\quad \rho V_\pi^-(0) = -h(e(0)) + \int_0^\infty [V_\pi(y) - V_\pi(0)]\beta(e(0),dy).$$

For any $\pi \in A$ with $R(0) > -\infty$, V_π is the unique bounded function which satisfies these same conditions and is continuous at 0.

Proof: See Appendix A.

Our next major objective is to obtain a similar, functional equation characterization of the maximum expected discounted return function V. Following standard Markov decision theory, this is accomplished in three steps, the first of which is easy: we write down the dynamic programming functional equation (see (3.3) and

(3.4) below). The second step is considerably more difficult: we verify that there exists a solution to this dynamic programming functional equation, and we investigate the solution's properties. These results, presented in Theorem 2 below, are based upon the theory of functional differential equations and generalize the results obtained by Pliska [21]. The final step, deferred until the next section, is to confirm that this solution is indeed equal to the maximum expected discounted return function.

Theorem 2. There exists a unique bounded, continuous function v which is differentiable on $(0,\infty)$ and satisfies

$$v'(x) = \sup_{\substack{c\in C \\ e\in E}} \{c^{-1}[u(c)-h(e) + \int_0^\infty [v(x+y)-v(x)]\beta(e,dy) - \rho v(x)]\}, \quad x > 0, \tag{3.3}$$

$$\rho v(0) = \sup_{e\in E} \{-h(e) + \int_0^\infty [v(y)-v(0)]\beta(e,dy)\} . \tag{3.4}$$

Moreover, v is nonnegative, concave, and strictly increasing, $\lim_{x\to\infty} v(x) = u(\bar{c})/\rho$, and v is continuously differentiable on $\mathbb{R}_+$.

Proof. See Appendix B.

In the next section, we shall prove that the solution v of the functional equation (3.3) - (3.4) is in fact identical with the maximum expected discounted return function V, and we shall characterize an optimal consumption and exploration policy π^* which yields this maximum value.

4. OPTIMAL POLICIES:

We begin by stating and interpreting the main result of this paper on the existence and the economic properties of an optimal consumption and exploration policy.

Theorem 3. The maximum expected discounted return function V is the unique solution of (3.3) and (3.4). There exists an optimal consumption and exploration policy $\pi^*(\cdot) = (c^*(\cdot), e^*(\cdot))$ in A such that

(i) the consumption policy $c^*(\cdot)$ satisfies $c^*(0) = 0$, $c^*(x) > 0$ whenever $x > 0$, and $c^*(\cdot)$ is nondecreasing on $\mathbb{R}_+$; and

(ii) the exploration policy $e^*(\cdot)$ is nonincreasing on $\mathbb{R}_+$ with $\lim_{x\to\infty} e^*(x) = 0$.

Note that $V_{\pi^*} = V = v$, so that all of the conclusions about v of Theorem 2 apply to V. Also, by (i), it is optimal to consume the resource at a positive rate whenever possible, and the larger the resource stock on hand the greater is the optimal consumption rate, as to be expected. Similarly, the lower the resource level the greater should the exploration rate be in searching for new supplies, while exploration becomes unnecessary if the resource stock is large enough. $V_{\pi^*}(x)$ is the expected value (net of extraction and exploration costs) following an optimal policy as a function of the initial resource level on hand; this value is nonnegative (since $c(\cdot) \equiv e(\cdot) \equiv 0$ is an admissible policy yielding zero total benefit), and it cannot exceed the maximum possible discounted utility from consumption $u(\bar{c})/\rho$. Also, by Theorem 2 this expected value increases with the initial resource level but at a diminishing rate. Note that $V'_{\pi^*}(x) = V'(x) = v'(x)$ is the contribution of the marginal unit of proven reserves x to the maximum expected discounted return and is therefore the shadow rent on x. Thus, the shadow rents are decreasing in x (since V is concave) and their behavior is smooth although that of the utilities and the exploration costs may not be so. We will examine the behavior of shadow rents through time in the next section. Although Theorem 3 characterizes properties of π^* and V_{π^*}, their numerical computations from (3.1)-(3.4) may be, of course, formidable.

The balance of this section is devoted to the proof of Theorem 3; this is carried out in a sequence of lemmas. Briefly, we first define a consumption and exploration policy π^* through the functional equation (3.3)-(3.4). We then show by Lemmas 2 and 4 that this policy is in fact an admissible one in the sense of Definition 1, i.e., $\pi^* \in A$. In doing this, we show that π^* has the desired monotonicity properties. The next step is to show (Lemma 5) that the expected discounted return V_{π^*} following this policy is in fact identical with the unique solution v of the functional equation (3.3)-(3.4) asserted in Theorem 2. The final task is then to show that $V_{\pi^*} \geq V_\pi$ for all $\pi \in A$, thereby allowing us to conclude π^* is optimal and possesses the economically meaningful properties asserted in Theorem 3.

To begin this plan, consider the functional equation (3.3)-(3.4) possessing the unique solution v as in Theorem 2. For each $x > 0$ define $c^*(x) \in C$ and $e^*(x) \in E$ as the decisions which attain the supremum on the right hand side of (3.3); similarly, define $e^*(0) \in E$ as the decision attaining the supremum in (3.4), and define $c^*(0) = 0$. By continuity of u, h, β and v and by compactness of C and E, the suprema are attained. In case of ties, define $c^*(x)$ as the smallest and $e^*(x)$ as the largest such rates. Thus, $\pi^*(\cdot) = (c^*(\cdot), e^*(\cdot))$ is a well-defined function from $\mathbb{R}_+$ into $C \times E$.

To show that π^* thus defined is an admissible policy and that it has the desired properties, it is convenient to introduce the notation

$$(4.1) \quad L(x,e) = -h(e) + \int_0^\infty [v(x+y)-v(x)]\beta(e,dy) - \rho v(x), \qquad (x,e) \in \mathbb{R}_+ \times E$$

and

$$(4.2) \qquad K(x) = \sup_{e \in E} L(x,e), \qquad x \geq 0$$

so that, exploiting separability of equation (3.3), it may be rewritten in an equivalent form as

$$(4.3) \qquad -K(x) = \sup_{c \in C} \{u(c) - cv'(x)\}, \quad x > 0,$$

while (3.4) is equivalent to

$$(4.4) \qquad K(0) = 0.$$

Then $c^*(x)$ and $e^*(x)$ satisfy

$$(4.5)\qquad u(c^*(x))-c^*(x)v'(x) = -K(x), \qquad x > 0$$

and

$$(4.6)\qquad K(x) = L(x,e^*(x)) \qquad x \geq 0 .$$

We summarize some properties of the function K as

<u>Lemma 1</u>. The map $x \to K(x)$ is bounded, continuous, and strictly decreasing on $[0,\infty)$ with $K(0) = 0$.

<u>Proof</u>: By our previous results and assumptions, $L(x,e)$ is continuous on $[0,\infty) \times E$ and, therefore, by Theorems 1 and 2 in Berge [2,p.115-116], $K(\cdot)$ is continuous. Next, for each $e \in E$, concavity of v implies that $\int_0^\infty [v(x+y)-v(x)]\beta(e,dy)$ is nonincreasing, while $-\rho v(x)$ is strictly decreasing in x. Therefore $L(\cdot,e)$ is strictly decreasing on $\mathbb{R}_+$ and, hence, so is $K(\cdot)$. We have already remarked why $K(0) = 0$. Finally, by continuity and boundedness of v and L we have

$$\begin{aligned}\lim_{x\to\infty} K(x) &= \sup_{e\in E} \lim_{x\to\infty} L(x,e) \\ &= -\rho \lim_{x\to\infty} v(x) \geq -u(\bar{c}).\end{aligned}$$

<u>Lemma 2</u>. The consumption policy $c^*(\cdot)$ is left-continuous and nondecreasing on $\mathbb{R}_+$ with $c^*(0) = 0$ and $c^*(x) > 0$ for all $x > 0$. It is, therefore, also admissible in the sense of Definition 1.

<u>Proof</u>: Since $-K(x) > 0$ for all $x > 0$ and $u(0) = 0$, it is clear from (4.5) that $c^*(x) > 0$ for $x > 0$. Also, $v'(x)$ is nonincreasing in x, so it is straightforward to compare (4.3) and (4.5) and

check that $c^*(x)$ is nondecreasing in x. By a selection theorem in Berge [2,p.116], $c^*(\cdot)$ is lower semi-continuous, so together with its monotonicity, this implies that $c^*(\cdot)$ must be left-continuous, and this proof is completed.

Digressing for a moment in the proof of Theorem 3, if u is also differentiable, then, by concavity of $u(c)-cv'(x)$, it is necessary and sufficient for $c^*(x)$ to satisfy

$$(4.7) \qquad \begin{aligned} u'(c^*(x)) &= v'(x) \qquad \text{if } c^*(x) < \bar{c}, \text{ and} \\ u'(\bar{c}) &\geq v'(x) \qquad \text{if } c^*(x) = \bar{c} \end{aligned}$$

As will be shown later, $v'(x)$ is the shadow rent on proven reserves and $c^*(x)$ is the optimum consumption rate. Equation (4.7) is then the stochastic analog of the familiar condition that, under an optimal consumption policy, at each instant of time the shadow rent must equal the marginal utility of consumption (the competitive price when the consumers are on their demand curves).

In order to investigate properties of the exploration rate function $e^*(\cdot)$ we need the following.

<u>Lemma 3</u>. Let f be any nonnegative, nondecreasing, continuous function on $\mathbb{R}_+$ with $\int_0^\infty f(y)\beta(e,dy) < \infty$ for all $e \in E$. Then under Assumption 1 the mapping $e \to \int_0^\infty f(y)\beta(e,dy)$ is nondecreasing on E.

Proof: First consider a sequence of functions of the form

$$(4.8) \qquad g_n(y) = \begin{cases} k_n & y \geq y_n \\ 0 & 0 \leq y < y_n \end{cases}$$

for any $k_n > 0$, $y_n > 0$, $n=1,2,\ldots,$

Then, for each $n=1,2,\ldots$, $\int_0^\infty g_n(y)\beta(e,dy) = k_n\beta(e,[y,\infty])$, which is nondecreasing in e, by Assumption 1. It follows that if $\{g_1,\ldots,g_n\}$ is a collection of functions as in (4.8) and if $f_n = \sum_{i=1}^{n} g_i$, then the mapping $e \to \int_0^\infty f_n(y)\beta(e,dy)$ is also nondecreasing on E. By a suitable choice of the sequences $\{k_n\}$ and $\{y_n\}$, we may define $\{g_n\}$ and $\{f_n\}$ so that $f_n \uparrow f$. By the monotone convergence theorem, we then have

$$\int_0^\infty f(y)\beta(e,dy) = \lim_{n\to\infty} \int_0^\infty f_n(y)\beta(e,dy), \quad e \in E$$

and, since the limit of nondecreasing functions is also nondecreasing, the desired result follows.

Lemma 4. The exploration policy $e^*(\cdot)$ is left-continuous and nonincreasing on $\mathbb{R}_+$ with $\lim_{x\to\infty} e^*(x) = 0$.

Proof: For any $x_1 \geq x_2 \geq 0$, we have

$$L(x_2,e) - L(x_1,e) = \int_0^\infty f(y)\beta(e,dy) - \rho v(x_2) + \rho v(x_1)$$

where we have defined

$$f(y) = [v(x_2+y) - v(x_2)] - [v(x_1+y) - v(x_1)], \quad y \geq 0.$$

By continuity, concavity and monotonicity of v, f is nonnegative, nondecreasing and continuous on $\mathbb{R}_+$ and, since v is bounded and $\beta(e,\mathbb{R}_+) < \infty$, $e \in E$, f is also integrable. Therefore, by Lemma 3, the map $e \to \int_0^\infty f(y)\beta(e,dy)$ is nondecreasing on E and, hence, so is the map $e \to L(x_2,e) - L(x_1,e)$. However, this implies $e^*(x_1) \leq e^*(x_2)$, because, if not, we would have

$$L(x_2,e^*(x_1)) - L(x_1,e^*(x_1)) \geq L(x_2,e^*(x_2)) - L(x_1,e^*(x_2))$$

which would contradict the fact that $e^*(x_1)$ and $e^*(x_2)$ maximize $L(x_1,e)$ and $L(x_2,e)$, respectively, over $e \in E$. Hence, $e^*(\circ)$ is nondecreasing on $\mathbb{R}_+$. In addition, by a selection theorem in Berge [2,p.116], $e^*(\cdot)$ is upper semi-continuous on $(0,\infty)$, so $e^*(\cdot)$ must be left-continuous.

Finally, boundedness and concavity of v implies that, for any $y \geq 0$, $[v(x+y) - v(x)]$ is nonincreasing in x and tends to 0 as $x \to \infty$. An application of the monotone convergence theorem now yields

$$\lim_{x\to\infty} L(x,e) = -h(e) - \rho \lim_{x\to\infty} v(x).$$

Continuity of $K(\cdot)$ and $L(\cdot,e)$, the fact that $h(e) \geq 0$, and the definition of $e^*(x)$ in (4.2) now imply that $\lim_{x\to\infty} e^*(x) = 0$, thereby completing this proof.

In particular, if the expoloration cost rate function h(e) is convex and if the discovery rate measure $\beta(e,dy)$ satisfies a second order stochastic dominance condition (viz., $e \to \beta(e,[y,\infty))$

is nondecreasing and concave over E), then it is easy to show that $L(x,e)$ is concave in $e \in E$ for all $x \geq 0$. In addition, if the functions involved are differentiable, then $e^*(x)$ may be found by setting the partial derivative of $L(x,e)$ with respective to e to zero, so that $e^*(x)$, if in the interior of E, satisfies

$$h'(e^*(x)) = \int_0^\infty [v(x+y)-v(x)]\beta'(e^*(x),dy),$$

i.e., the marginal exploration cost balances the marginal expected improvement in the maximum attainable return.

To summarize where we stand in the proof of Theorem 3, the consumption and exploration policy $\pi^*(\cdot) = (c^*(\cdot), e^*(\cdot))$ defined by (4.5) and (4.6) is admissible and possesses meaningful properties. The next two results prove that $\pi^*(\cdot)$ is in fact an optimal policy.

Lemma 5. The expected discounted reward $V_{\pi^*} = v$, where v is as in Theorem 2.

Proof: By (4.5) and the definition of π^*, it is apparent that v satisfies (3.1) and (3.2). Note that v is bounded and continuous on $[0,\infty)$ and v' is continuous on $(0,\infty)$. In addition, $v'(\cdot)c^*(\cdot)$ is bounded on $(0,\infty)$, because (4.5) holds and u and K are bounded. Since V_{π^*} is the unique function which satisfies all these requirements by Theorem 1, it must be that $V_{\pi^*} = v$.

<u>Lemma 6</u>. For any policy $\pi \in A$, $V_\pi \le V_\pi{}^*$. Hence $V_\pi{}^* = V$ and π^* is an optimal policy.

<u>Proof</u>: Let $\pi \in A$ be arbitrary and subtract the equation (3.1) with $\pi^*(\cdot) = (c^*(\cdot), e^*(\cdot))$ from that with $\pi(\cdot) = (c(\cdot), e(\cdot))$ to yield

$$\begin{aligned}
\rho[V_\pi(x) - V_{\pi^*}(x)] &= u(c(x)) - h(e(x)) + \int_0^\infty [V_\pi(x+y) - V_\pi(x)]\beta(e(x),dy) - V_\pi'(x)c(x) \\
&\quad - u(c^*(x)) + h(e^*(x)) - \int_0^\infty [V_{\pi^*}(x+y) - V_{\pi^*}(x)]\beta(e^*(x),dy) + V_{\pi^*}'(x)c^*(x) \\
&= \{u(c(x)) - h(e(x)) + \int_0^\infty [V_\pi(x+y) - V_\pi(x)]\beta(e(x),dy) - V_\pi'(x)c(x)\} \\
&\quad - \{\int_0^\infty [V_{\pi^*}(x+y) - V_{\pi^*}(x)]\beta(e(x),dy) - V_{\pi^*}'(x)c(x)\} \\
&\quad + \{\int_0^\infty [V_{\pi^*}(x+y) - V_{\pi^*}(x)]\beta(e(x),dy) - V_{\pi^*}'(x)c(x)\} \\
&\quad - \{u(c^*(x)) - h(e^*(x)) + \int_0^\infty [V_{\pi^*}(x+y) - V_{\pi^*}(x)]\beta(e^*(x),dy) - V_{\pi^*}'(x)c^*(x)\}
\end{aligned}$$

where we have added and subtracted the same term.

Thus, we have

$$(4.9)\quad \rho[V_\pi(x) - V_{\pi^*}(x)] = g(x) + \int_0^\infty \{[V_\pi(x+y) - V_\pi{}^*(x+y)] - [V_\pi(x) - V_\pi{}^*(x)]\}\, \beta(e(x),dy) - [V_\pi'(x) - V_\pi'{}^*(x)]c(x), \qquad x > 0$$

where (recall $v = V_\pi{}^*$ by Lemma 5)

$$\begin{aligned}
(4.10)\quad g(x) \equiv & \int_0^\infty [v(x+y) - v(x)]\beta(e(x),dy) - v'(x)c(x) + u(c(x)) - h(e(x)) \\
& - \int_0^\infty [v(x+y) - v(x)]\beta(e^*(x),dy) + v'(x)c^*(x) - u(c^*(x)) + h(e^*(x)).
\end{aligned}$$

A similar computation for the boundary equation (3.2) yields

$$(4.11) \quad \rho(V_\pi(0)-V_{\pi^*}(0)) = g(0) + \int_0^\infty \{[V_\pi(y)-V_{\pi^*}(y)] - [V_\pi(0) - V_{\pi^*}(0)]\}\beta(e(0),dy)$$

where, with $V_{\pi^*} = v$, we have defined

$$(4.12) \quad g(0) = \int_0^\infty [v(y)-v(0)]\beta(e(0),dy)-h(e(0)) - \int_0^\infty [v(y)-v(0)]\beta(e^*(0),dy)+h(e^*(0)).$$

We note that g defined by (4.10) and (4.12) is bounded and left-continuous. Moreover, by the definitions of $c^*(x)$ and $e^*(x)$ as the functions attaining the maxima in (4.2) and (4.3), we conclude that $g(x) \leq 0$ for all $x \geq 0$.

Now, with $f = V_\pi - V_{\pi^*}$, we see that f is absolutely continuous with a left-continuous derivative on $(0,\infty)$. If $R(0) > -\infty$ under π, then V_π, and hence f, are continuous at 0. Since $V_\pi'(\cdot)c(\cdot)$ and $v'(\cdot)$ are bounded on $(0,\infty)$ (see Theorem 1, Theorem 2, and Lemma B.10), the same property holds for $f'(\cdot)c(\cdot)$. Comparing (4.9) and (4.11) with (3.1) and (3.2), respectively, we conclude by Theorem 1 that $V_\pi - V_{\pi^*}$ is the expected discounted return under policy $\pi \in A$ when the immediate return function is given to be g as defined by (4.10) and (4.12). Since $g \leq 0$, we must have $V_\pi \leq V_{\pi^*}$. Since $\pi \in A$ is arbitrary, π^* is an optimal policy and this proof is completed.

Combining Lemmas 2, 4, 5, and 6, we note that the proof of Theorem 3 is now complete.

5. DYNAMICS OF THE SHADOW PRICES

As described previously, c_t is the rate at time t at which the resource is consumed while $u(c_t)$ is the utility rate earned net of any extraction costs. Therefore $V(x)$ is the optimal present value of future consumption after deduction of extraction and exploration costs and $V'(x)$ is the shadow price (net of extraction and exploration costs) of the initial resource deposits of size x. In absence of any exploration possibilities and uncertainties, it is well known that the shadow price of the resource (net of any extration costs) rises at the social rate of discount; see Hotelling [12], Dasgupta and Heal [6], Solow [22]. Allowing for resource exploration and the associated uncertainties, the stochastic analog of this result in our model would be that the expected rate of increase of the shadow price V' equals the discount rate ρ, that is

$$(5.1) \qquad \lim_{t \downarrow 0} \frac{\bar{E}_{\pi^*}[V'(X_t)|X_0=x]-V'(x)}{t} = \rho V'(x), \quad x > 0 .$$

By Markov process theory, the limit in (5.1) exists if V' is in the domain of the infinitesimal generator of the Markov process under π^*, in which case this limit equals this infinitesimal generator evaluated at $V'(x)$. Hence this study of the dynamics of the shadow prices proceeds in two steps. First we provide in Lemma 7 some additional assumptions which ensure that V' is in the domain of $\mathcal{J}_{\pi^*}$, the weak infinitesimal generator of the Markov process under π^*. Then we show in Theorem 4 that $\mathcal{J}_{\pi^*}V'(x) = \rho V'(x)$ for all $x > 0$.

Lemma 7. Suppose $u(\cdot)$ (respectively, $h(\cdot)$) is continuously twice differentiable and strictly concave (convex) on $C(E)$. Furthermore, suppose that there exists a weakly continuous kernel $\beta'(e,dy)$ such that

(i) $\frac{d}{de}\int_0^\infty [V(x+y)-V(x)]\beta(e,dy) = \int_0^\infty [V(x+y)-V(x)]\beta'(e,dy)$, and

(ii) $\frac{d}{de}\int_0^\infty [V(x+y)-V(x)]\beta'(e,dy)$ is continuous in e and nonpositive.

Then V' is in the domain of $\mathcal{A}_{\pi^*}$. Moreover, $c^*(\cdot)$ and $e^*(\cdot)$ are continuously differentiable, except possibly at those points where their values change from the boundary to the interior of their range.

Proof: See Appendix C.

The assumptions in Lemma 7 are satisfied in a variety of simple situations. For example, suppose $\beta(e,dy) = \lambda(e)F(dy)$, where $\lambda''(\cdot)$ exists and is continuous and nonpositive. Then $\beta'(e,dy) = \lambda'(e)F(dy)$ and the desired properties are satisfied. For a second example, suppose $\beta(e,dy) = \lambda f(e,y)dy$, where λ is a positive constant and f is a nonnegative function which is, for each fixed e, a density function. If $\frac{\partial f}{\partial e}$ is continuous and integrable, then we can set $\beta'(e,dy) = \lambda\frac{\partial f}{\partial e}(e,y)dy$ and (i) holds. If, in addition, $\frac{\partial^2 f}{\partial e^2}$ is continuous, nonpositive, and integrable, then (ii) holds.

We are now ready to prove

<u>Theorem 4</u>. Under the assumptions of Lemma 7,

$$\lim_{t\downarrow 0} \frac{E_{\pi^*}[V'(X_t)|X_0=x]-V'(x)}{t} = \rho V'(x) \qquad x \geq 0.$$

<u>Proof</u>: Since V' is in the domain of $\mathscr{L}_{\pi^*}$ by Lemma 7, our objective is to show that $\mathscr{L}_{\pi^*}V'(x) = \rho V'(x)$. By Markov process theory and Appendix A, we know that

$$(5.2) \qquad \mathscr{L}_{\pi^*}V'(x) = \int_0^\infty [V'(x+y)-V'(x)]\beta(e^*(x),dy)-c^*(x)V''(x), \; x > 0.$$

Since $V = V_{\pi^*}$ and (3.1) holds, we also know that

$$(5.3) \qquad \rho V(x) = u(c^*(x))-c^*(x)\; V'(x)-h(e^*(x)) + \int_0^\infty [V(x+y)-V(x)]\beta(e^*(x),dy).$$

By the differentiability assumptions and Lemma 7, we may differentiate (5.3) with respect to x and rearrange the terms to yield, for $x > 0$,

$$(5.4) \quad \rho V'(x) = c^{*\prime}(x)[u'(c^*(x))-V'(x)]$$
$$+ e^{*\prime}(x)[-h'(e^*(x)) + \int_0^\infty [V(x+y)-V(x)]\beta'(e^*(x),dy)]$$
$$+ \left\{\int_0^\infty [V'(x+y)-V'(x)\beta(e^*(x),dy) - c^*(x)\; V''(x)\right\} .$$

We know $c^*(x) \in (0,\bar{c}]$. Let $x_0 = \inf\{x : c^*(x) = \bar{c}\} \leq \infty$. If $x < x_0$, then $u'(c^*(x)) - V'(x) = 0$, by the optimality condition which $c^*(x)$ must satisfy. On the other hand, if $x > x_0$ the montonicity of $c^*(\cdot)$ implies $c^*(x) = \bar{c}$, $c^{*\prime}(x) = 0$. Hence, in either case the first term on the right hand side of (5.4) is zero. By a similar argument, the second term is also zero. Consequently, upon comparing (5.2) with (5.4), it is apparent that we have the desired result.

Thus, the expected scarcity rent on proven reserves rises exponentially in time at rate ρ, as in Hotelling [12]. Of course, in contrast with deterministic models, the actual rents may decrease by random amounts at random times (whenever new resource deposits are discovered), which is an explanation of why the prices of natural resources have not always followed Hotelling's theory.

6. CONCLUSION AND EXTENSIONS

We have considered the problem of optimal social management of the stock of a natural resource when the resource stock on hand can be increased through exploration but the exploration process involves uncertainty. Under mild conditions, we have shown that there exists an optimal policy of consumption and exploration which maximizes the net expected discounted social benefit. As functions of the size of known deposits of the resource, the optimal consumption rate is shown to be strictly positive (whenever possible) and nondecreasing, while the optimal exploration effort rate is nonincreasing. We have also studied and characterized the properties of the optimal net present value and the shadow prices representing the scarcity rents on the resource deposits. It would have been interesting to show that, under an optimal policy, the Doomsday will almost surely never come, i.e., state 0 is inaccessible, but we were unable to do this.

Throughout this paper, our primary emphasis has been on presenting and analyzing a model which explicitly incorporates the possibilities of and the uncertainties involved in the exploration process of searching for new resource stocks. In order to focus on this aspect of economics of natural resources, we have made some simplifications. Specifically, we have taken the social benefit function to be separable in the consumption utility and the exploration costs (as in Arrow [1], Dasgupta and Stiglitz [8] and Gilbert [10]), which facilitates proving the structure of optimal policies. Similarly, we do not have any penalty for running out of the resource, since a requirement such as $u(0) = -\infty$ leads to technical difficulties which we are unable to resolve. We have also assumed that the various functions representing the consumption utility, extraction costs, exploration costs and the uncertainties in the exploration process remain time invariant, which is essential to obtain time invariant optimal policies. Most importantly, we have assumed that the discovery process depends only on the exploration effort, regardless of the total resource stock discovered so far (or of some other measure of the history of past successes). Consequently, if the total resource stock is finite, then we are ruling out any possibilities of learning about its true size and of revising the expectations of future exploratory gains on the basis of the amounts discovered thus far. Our feeling is that over an intermediate planning horizon this seems reasonable, whereas over longer planning horizons the discount factor will diminish the learning effects.

A more accurate model might involve a discovery rate measure which depends not only on the exploration effort rate e_t but also on the total amount I_t discovered so far. This would require the state of the process to be (X_t, I_t) with values in $\mathbb{R}_+^2$. Unfortunately, a theory of controlled storage processes of this type is not available at present, and we shall pursue this approach elsewhere. See also Arrow [1] for an attempt to incorporate learning in his model.

In conclusion, we have modelled and studied some natural resource management issues such as the consumption, the necessary exploratory activity, and the inevitable uncertainties involved. However, some aspects, such as exhaustibility of the resource and the learning involved in exploration, are not captured satisfactorily, so we hope that the methodology employed in this paper will be useful in analyzing more complete models of this problem.

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APPENDICES

APPENDIX A - PROOF OF THEOREM 1

The proof of Theorem 1 is based upon the theory of the weak infinitesimal generators of Markov processes (see Dynkin [9]). The case where $R(0) > -\infty$, that is, where state zero is accessible, has already been worked out by Morais [19]. Here we are concerned with an arbitrary policy $\pi \in A$ under which $R(0) = -\infty$.

Consider the Markov process $\{X_t; t \geq 0\}$ corresponding to π, and let $\mathcal{L}$ be the class of bounded, measurable functions f on $\mathbb{R}_+$ such that $E_x f(X_t) \to f(x)$ as $t \downarrow 0$ for all $x \in \mathbb{R}_+$, where $E_x f(X_t) \equiv E[f(X_t)|X_o = x]$. Define $\mathcal{D}$ to the set of $f \in \mathcal{L}$ such that $\{E_x f(X_t) - f(x)\}/t$ converges boundedly pointwise on $\mathbb{R}_+$ as $t \downarrow 0$ to a function (denoted as $\mathcal{G}f$) in $\mathcal{L}$. Then $\mathcal{G}$ is called the weak infinitesimal generator with domain $\mathcal{D}$ and range $\mathcal{L}$. The main step in the proof of Theorem 1 is to characterize $\mathcal{G}$, $\mathcal{D}$, and $\mathcal{L}$. We do this by following the general method used by Harrison and Resnick [11, Propositions 3 and 4], paying attention to just those points where the arguments differ due to our more general situation.

It is apparent that $\mathcal{L}$ consists of the bounded measurable functions which are left continuous. Moreover $f \in \mathcal{D}$ if and only if

$$\text{(A1)} \qquad f(x) = E_\pi[\int_o^\infty \exp(-\rho t) g(X_t) dt | X_o = x]$$

for some $g \in \mathcal{L}$ and $\rho > 0$, so to characterize $\mathcal{D}$ we shall analyze (A1) for arbitrary $g \in \mathcal{L}$ and $\rho > 0$. If we condition on T_1, the time of the first jump, and denote

$$U(x,t) = \int_o^t \exp(-\rho s) g(q(x,s)) ds$$

and

$$W(y) = \int_0^\infty f(y+z)G(e(y),dz),$$

then (A1) becomes, for $x > 0$,

$$\begin{aligned} f(x) &= E_\pi[U(x,T_1)|X_o=x] + E_\pi[\exp(-\rho T_1)W(q(x,T_1))|X_o=x] \\ &= \int_0^\infty \lambda(e(q(x,t)))\exp[-\int_0^t \lambda(e(q(x,s)))ds][U(x,t) + \\ &\qquad\qquad \exp(-\rho t)W(q(x,t))]dt \\ &= \int_0^x (c(y))^{-1}\lambda(e(y))\exp[-\int_0^{R(x)-R(y)} \lambda(e(q(x,s)))ds]m(x,y)dy, \end{aligned}$$

where

$$m(x,y) \equiv U(x,R(x)-R(y)) + \exp[-\rho(R(x)-R(y))]W(y)$$

and we have made the change of variable $y = q(x,t)$, so that $t = R(x)-R(y)$ and $-dt = dy/c(y)$. Now although the variable x appears in the integrand, the boundedness of m and the absolute continuity of R and U can be used to demonstrate the absolute continuity of f on $(0,\infty)$. Moreover, a straightforward computation demonstrates that the left-derivative f' of f exists and is left-continuous.

At this point, the steps in the proof of Proposition 4 in Harrison and Resnick [11] can be followed fairly closely, the primary difference being the more general density function associated with the time T_1 until the first jump. We thereby conclude that $\mathcal{D}$ consists of all bounded functions f which are absolutely continuous on $(0,\infty)$ and have left-continuous left-derivatives f' on $(0,\infty)$ such that $c(\cdot)f'(\cdot)$ is bounded on $(0,\infty)$. Furthermore, for $f \in \mathcal{D}$,

$$\mathcal{A}f(x) = \lambda(e(x)) \int_0^\infty [f(x+y) - f(x)]G(e(x),dy) - c(x)f'(x), \ x > 0,$$

$$\mathcal{A}f(0) = \lambda(e(0) \int_0^\infty [f(y) - f(0)]G(e(0),dy).$$

Having characterized $\mathcal{A}$, $\mathcal{D}$, and $\mathcal{L}$, the proof of Theorem 1 is virtually complete. The function $u(c(\cdot)) - h(e(\cdot))$ is bounded and left-continuous; therefore it is an element of $\mathcal{L}$. By Markov process theory, V_π is the unique element of $\mathcal{D}$ satisfying

$$\rho V_\pi = u(c) - h(e) + \mathcal{A}V_\pi \ .$$

Substituting our expression for $\mathcal{A}$, we see that we are done.

APPENDIX B - PROOF OF THEOREM 2

Theorem 2 is very similar to the main result in Pliska [21]. One might suppose, in fact, that his results can be applied to Theorem 2, but this is not true for two reasons. First, if the state is positive, then in Pliska [21] the release rate (using storage process terminology) as a function of the action is bounded away from zero, whereas here the release rate dictated by the dynamic programming functional equation can be any element of the interval $[0,\bar{c}]$. Second, in Pliska [21], but not here, the cost rate function is continuous on $\mathbb{R}_+ \times D$ at the state $x = 0$.

The basic outline of this proof is the same as in Pliska [21] and is presented in four lemmas. For each finite number $s > 0$, define

$$\beta_s(x,e,dy) = \begin{cases} \beta(e,dy), & y < s - x,\ x < s \\ \beta(e,[s-x,\infty)), & y = s - x,\ x < s \\ 0, & \text{otherwise} \end{cases}$$

This amounts to defining a new jump measure that depends explicitly on the state so that jumps which would have landed to the right of s are transformed into jumps to the point s. This measure is used in the following.

(B.1) LEMMA. For each pair of finite, positive numbers s and $\delta < \bar{c}$ there exists a unique, differentiable function v_s on $[0,s]$ which satisfies

$$\text{(B.2)}\quad v_s'(x) = \sup_{\substack{\delta \le c \le \bar{c} \\ e \in E}} \{c^{-1}[\int [v_s(x+y) - v_s(x)]\beta_s(x,e,dy) - \rho v_s(x) + u(c) - h(e)]\}, \quad x \in (0,s]$$

and

$$\text{(B.3)}\quad \rho v_s(0) = \sup_{e \in E} \{\int [v_s(y) - v_s(0)]\beta_s(0,e,dy) - h(e)\}.$$

Moreover, v_s is nonnegative concave increasing on $[0,s]$ with $v_\delta(s) \le u(\bar{c})/\rho$.

Proof. First we observe that by the argument immediately following Lemma 6 in Pliska [21] there exists a differentiable function v_s on $[0,s]$ satisfying (B.2) and (B.3) with $v_s(s) \le u(\bar{c})/\rho$. This solution is unique by the argument used for Theorem 1 in Pliska [21].

Theorem 3 in Pliska [21] is based on a contraction fixed point theorem, so to show v_s is concave increasing we shall use a successive approximations argument. In other words, if f is any concave increasing function on $[0,s]$ with $f(s) = v_s(s)$, then it suffices to show that the mapping

$$x \to \sup_{\substack{c\in[\delta,\bar{c}] \\ e\in E}} \{c^{-1}[\int[f(x+y) - f(x)]\, \beta_s(x,e,dy) - \rho f(x) + u(c) - h(e)\}$$

is nonnegative decreasing on $[0,s]$.

Now $f(x) \leq u(\bar{c})/\rho$, so taking $e = 0$ when $x = s$ we see the nonnegativity assertion follows from the decreasing assertion. Clearly $-\rho f$ is decreasing. To show the mapping

$$\text{(B.4)}\quad x \to \int [f(x+y) - f(x)]\, \beta_s(x,e,dy)$$

is decreasing, we shall define for each $x \geq 0$

$$g(x) = \begin{cases} f(x), & x \leq s \\ f(s), & x > s, \end{cases}$$

so that

$$\int[f(x+y) - f(x)]\beta_s(x,e,dy) = \int[g(x+y) - g(x)]\beta(e,dy)$$

for all $x\in[0,s]$. Hence $0 \leq x_1 < x_2 \leq s$ implies

$$g(x_1+y) - g(x_1) \geq g(x_2+y) - g(x_2)$$

for all $y \geq 0$, so the mapping in (B.4) is decreasing in which case v_δ is concave increasing. Finally, taking $e = 0$ in (B.3) we see that $v_s(0) \geq 0$, so $v_s \geq 0$ and this proof is completed.

(B.5) LEMMA. For each positive $\delta < \bar{c}$, there exists a unique bounded, continuous, real-valued function v_δ on $[0,\infty)$ satisfying

$$(B.6)\quad v_\delta'(x) = \sup_{\substack{\delta \le c \le \bar{c} \\ e \in E}} \{c^{-1}[\int [v_\delta(x+y) - v_\delta(x)]\beta(e,dy) - \rho v_\delta(x) + u(c) - h(e)]\}, \quad x > 0$$

and

$$(B.7)\quad \rho v_\delta(0) = \sup_{e \in E} \{\int [v_\delta(y) - v_\delta(0)]\beta(e,dy) - h(e)\}.$$

Moreover, v_δ has a continuous derivative on $(0,\infty)$, v_δ is nonnegative, concave, and nondecreasing, and $v_\delta(x) \le u(\bar{c})/\rho$ for all $x \ge 0$.

Proof. Although the boundary condition (B.7) is slightly different, we can proceed exactly as in Pliska [21,Section 4]. Briefly, for each $s > 0$ we define the function

$$\bar{v}_s(x) = \begin{cases} v_s(x), & x < s, \\ v_s(s), & x \ge s, \end{cases}$$

where v_s is from Lemma (B.1). Then by Pliska's arguments we conclude $\bar{v}_s$ converges uniformly as $s \to \infty$ and the limit v_δ is the unique solution of (B.6) and (B.7). The continuity properties of v_δ and its derivative are by-products of Pliska's arguments. The remaining conclusions of this lemma follow immediately from Lemma (B.1).

(B.8) LEMMA. For each $x \ge 0$, $\lim_{\delta \downarrow 0} v_\delta(x)$ exists. The function v on $[0,\infty)$ defined by $v(x) = \lim_{\delta \downarrow 0} v_\delta(x)$ is nonnegative, concave, and nondecreasing with $v(x) \le u(\bar{c})/\rho$ for all $x \ge 0$. Moreover, v has a continuous derivative on $(0,\infty)$ and satisfies (3.3) and (3.4).

Proof. Although the boundary condition (B.7) is slightly different, we may use the methods in Morais [19, Chapter IV] to conclude that v_δ may be interpreted as the maximum expected discounted reward for a problem that is identical to the one we formulated in Section 2 except that the consumption rates are restricted to being greater than or equal to δ for all values of $x > 0$. Hence, as we enlarge the set of admissible controls, the maximum expected discounted reward will increase, that is, v_δ is decreasing in δ. Since $v_\delta \leq u(\bar{c})/\rho$, $\lim_{\delta \downarrow 0} v_\delta$ exists. In view of Lemma (B.5), this limit v is nonnegative, concave, and nondecreasing with $v(x) \leq u(\bar{c})/\rho$ for all $x \geq 0$.

It remains to show v has a continuous derivative and satisfies (3.3) and (3.4). Let $\epsilon > 0$ be fixed and consider, for $t > \epsilon$, the equation

$$\text{(B.9)} \quad v_\delta(t) = v_\delta(\epsilon) + \int_\epsilon^t f_\delta(x)dx,$$

where $f_\delta(x)$ equals the right-hand side of (B.6). Now $f_\delta(x)$ converges to $f(x)$, which is defined to be equal to the right-hand side of (3.3), as $\delta \downarrow 0$. The sequence $\{f_\delta(\epsilon)\}$ is bounded (or else concavity and $0 \leq v_\delta \leq u(\bar{c})/\rho$ would yield a contradiction), so $\{f_\delta\}$ is bounded on $[\epsilon,t]$. Hence by the bounded convergence theorem we can let $\delta \downarrow 0$ in (B.9) and conclude

$$v(t) = v(\epsilon) + \int_\epsilon^t f(x)dx.$$

In other words, v satisfies (3.3) and has, since f is continuous, a continuous derivative on $(0,\infty)$. Letting $\delta \downarrow 0$ in (B.7) demonstrates (3.4), so this proof is completed.

We do not know yet whether v is continuous at $x = 0$ nor whether (3.3) and (3.4) have a unique solution. These matters are clarified in the following.

(B.10) LEMMA. The function v of Lemma (B.8) is continuously differentiable on $\mathbb{R}_+$ and is the unique solution of (3.3) and (3.4).

Proof. If we emulate (4.1)-(4.3) and Lemma 1 by defining

$$K_\delta(x) = \sup_{e\in E} \{\int [v_\delta(x+y) - v_\delta(x)]\beta(e,dy) - \rho v_\delta(x) - h(e)\}, \quad x > 0,$$

then we see that $K_\delta(x) \leq 0$ and (B.6) is the same as

$$v_\delta'(x) = \sup_{\delta\leq c\leq \bar{c}} \{ \frac{K_\delta(x)+u(c)}{c} \} .$$

Hence, $v_\delta'(x) \leq \sup_{\delta\leq c\leq \bar{c}} \{u(c)/c\} \leq u'(0)$ by the concavity of u, so $v_\delta'(0)$ exists and is bounded above by $u'(0)$ for all $\delta > 0$. Since $v_\delta \uparrow v$ as $\delta \downarrow 0$, it follows that v is continuous at $x = 0$. In addition, since $v'(\cdot)$ is nonincreasing, $\lim_{x\downarrow 0} v'(x)$ exists, so $v'(\cdot)$ must also be continuous at $x = 0$. Finally, uniqueness follows as in Pliska [21, Theorem 2].

Except for the strictly increasing property of v, all of Theorem 2 now follows from the preceding lemmas. To see that this remaining property holds, suppose not. Then there exists some $\bar{x} > 0$ with $v'(\bar{x}) = 0$ and $v(x) = v(\bar{x})$ for all $x > \bar{x}$. By functional differential equation theory, this implies $v(x) = v(\bar{x})$ for all $x > 0$. Hence, looking at (3.3), we see that v must satisfy

$$G(x) = \max_{e \in E} \{\int_0^\infty [V(x+y) - V(x)]\beta(e,dy) - h(e)\}.$$

We know $e^*(x)$, the value which attains this maximum, is nonincreasing with respect to x, so for some $0 \le x_1 \le x_2 \le \infty$ we have $e^*(x) = \bar{e}$ on $[0,x_1]$, $0 < e^*(x) < \bar{e}$ on (x_1,x_2), $0 \le e^*(x_2) < \bar{e}$, and $e^*(x) = 0$ on (x_2,∞). On $[0,x_1]$ we have

$$\text{(C.4)} \quad G'(x) = \int_0^\infty [V'(x+y) - V'(x)]\beta(e^*(x),dy),$$

so $G'(\cdot)$ is bounded and continuous on $[0,x_1)$ by property (C.1). The same result holds on (x_2,∞). On $(x_1,x_2]$, finally, we know by the hypotheses of Lemma 7 that $e^*(x)$ satisfies

$$\int_0^\infty [V(x+y) - V(x)]\beta'(e^*(x),dy) - h'(e^*(x)) = 0$$

and that the implicit function theorem can be applied to conclude $e^*(\cdot)$ is continuous with a continuous derivative and $G'(x)$ is given by (C.4). Hence, in summary, $e^*(\cdot)$ and $G'(\cdot)$ are bounded and continuous on all of $\mathbb{R}_+$, whereas the derivative of $e^*(\cdot)$ is continuous on $\mathbb{R}_+$ except, possibly, at x_1 and x_2.

The next step in the proof of Lemma 7 is to make a similar analysis of $c^*(\cdot)$. We know that $c^*(\cdot)$ is nondecreasing and $c^*(x)$ is the value which attains the maximum in

$$\text{(C.5)} \quad V'(x) = \max_{c \in C} \{ \frac{K(x)+u(c)}{c} \},$$

where $K(x) = G(x) - \rho v(x)$ was defined in (4.2). In this case there exists some $x_3 \le \infty$ such that $0 < c^*(x) < \bar{c}$ on $(0,x_3)$ and $c^*(x) = \bar{c}$ on $[x_3,\infty)$. On $[x_3,\infty)$ we must have

(C.6) $$V''(x) = \frac{K'(x)}{c^*(x)} \quad .$$

On the other hand, for $x \in (0, x_3)$ we know that $c^*(x)$ satisfies

(C.7) $$c^*(x)u'(c^*(x)) - K(x) - u(c^*(x)) = 0,$$

so applying the implicit function theorem we conclude $c^*(\cdot)$ is continuously differentiable on $(0, x_3)$ with $\lim_{x \to x_3} c^*(x) = \bar{c}$. Hence $c^*(\cdot)$ is continuous on $\mathbb{R}_+$ with a derivative that is continuous except, possibly, at x_3. Substituting $c^*(x)$ in (C.5), differentiating, and then using (C.7), we see that (C.6) must be satisfied on all of $\mathbb{R}_+$. Noting by the preceding paragraph that $K'(\cdot)$ is bounded and continuous, we conclude that V' satisfies properties (C.2) and (C.3), thereby completing the proof of Lemma 7.

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