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STUDYING ECONOMIC EQUILIBRIA ON AFFINE NETWORKS VIA LEMKE'S ALGORITHM

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Studying Economic Equilibria on Affine Networks via Lemke's Algorithm[†]

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Abstract. The equilibria between supply and demand on certain affine, multicommodity, transshipment networks are characterized as the solutions to a standard linear complementarity problem to which Lemke's algorithm is applied.

Key words. Complementary pivot theory, economic equilibria, Lemke's algorithm, multicommodity networks, transshipment problem.

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†Essentially the same as the authors' previous report "Computing Economic Equilibria on Affine Networks with Lemke's Algorithm", but with more detailed explanations and derivations.

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1. Introduction. The economic equilibria on certain affine, multi-commodity, transshipment networks have previously been studied by the regional economists Takayama and Judge [4] via quadratic programming. However, utilizing the economic equilibrium conditions directly, without first passing to a quadratic programming problem, makes our approach more general. This approach also provides additional information and insight, as well as an alternative algorithm.

The economic equilibria on even more general (nonaffine) networks have previously been characterized in [3] as the solutions to a general (nonlinear) complementarity problem involving dual cones, but without the specific results obtained here. The study of such equilibria began at least as early as 1838 with the work of Cournot, and its history is summarized in both [3] and [4].

Section 2 of this paper develops the appropriate economic equilibrium conditions, whose solutions are subsequently characterized in Section 3 as the solutions to a standard linear complementarity problem. Section 4 then describes a rather general situation in which Lemke's algorithm either finds a solution or shows that no solution exists. Section 4 also describes two different situations in which a solution is always found. Finally, Section 5 proves the results described in Section 4.

2. The model. The problems studied here can be conveniently represented by a directed graph, with a finite number of nodes and links, on which a finite number of commodities can be transported.

Each node i represents the set of "producers" and/or "consumers" at a specific spatial location; and each link s represents a specific transport facility for transporting commodities from some node i to a different node j. (In particular then, there are no loops, i.e., no links connecting a given node to itself.) Each link is directed to coincide with the direction of a possible transport of commodities; so there are at least two links connecting those nodes between which there is a possible transport of commodities in either direction.

Both the nodes and the links are enumerated in any order, consecutively beginning with one -- as illustrated by the graph in Figure 1.

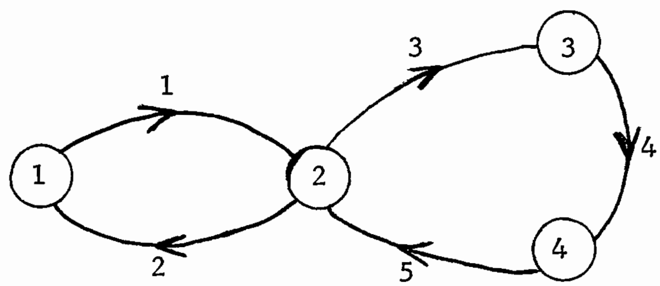

Figure 1

For clarity, the nodes are always indexed by the symbol i or j, and the links are always indexed by the symbol s or t.

To describe the topology of a general graph, suppose that

> the symbol $i\rightarrow$ denotes the set of all links s directed out of node i, while the symbol $\rightarrow i$ denotes the set of all links s directed into node i.

Needless to say, the absence of loops means that $i\rightarrow \cap \rightarrow i = \phi$ for each node i. Moreover, a given node i can be an "exporter" only if $i\rightarrow \neq \phi$, and an "importer" only if $\rightarrow i \neq \phi$. Consequently, a given node i for which $i\rightarrow = \rightarrow i = \phi$ can be neither an exporter nor an importer and is said to be "isolated".

In addition to the preceding notation, suppose that the symbol $)s$ denotes the "tail" of link s (i.e., the unique node i such that $s \in i\rightarrow)$, while the symbol $s)$ denotes the "head" of link s (i.e., the unique node i such that $s \in \rightarrow i)$.

In particular then,

$$s \in i\rightarrow \text{ if and only if })s = i$$

while

$$s \in \rightarrow i \text{ if and only if } s) = i.$$

Needless to say, the absence of loops means that $)s \neq s)$ for each link s.

For the graph in Figure 1, note that there are no loops, and note that none of the four nodes are isolated (in fact, each node might be either an exporter or an importer). Finally, to illustrate the preceding notation, note that $2\rightarrow = \{2, 3\}$ while $\rightarrow 2 = \{1, 5\}$; so $)2 =)3 = 2$ while $1) = 5) = 2$.

Each commodity c represents either a "raw material", an "intermediate product", or a "finished product"; and each might be produced and/or consumed by various nodes, as well as transported over various links. The commodities are enumerated in any order, consecutively beginning with one, and are always indexed by the symbol c.

We suppose that

the excess quantity of commodity c produced by node i is a variable q_{ic} (which is positive when node i produces more of commodity c than it consumes); and q_i denotes the vector variable with components q_{ic}.

Similarly, we suppose that

the unit price of commodity c for node i is a variable p_{ic}; and p_i denotes the vector variable with components p_{ic}.

We also suppose that

the quantity of commodity c transported via link s (in the direction of link s) is a variable $z^{sc} \geq 0$; and z^s denotes the vector variable with components z^{sc}.

Similarly, we suppose that

the unit price for transporting commodity c via link s is a variable p^{sc}; and p^s denotes the vector variable with components p^{sc}.

Notationally,

the symbol z denotes the vector variable with vector components z^s, and the symbol q denotes the vector variable with vector components q_i.

Moreover,

the symbol p denotes the vector variable with vector components p^s and p_i.

There are six conditions that collectively characterize those vectors (z, q; p) that place the network in a state of economic equilibrium. Two of the conditions involve only the quantity vector (z, q), and one of the conditions involves only the price vector p. The other three conditions involve both (z, q) and p.

The two conditions involving only (z, q) are the non-negativity condition

$$z^s \geq 0 \text{ for each link s} \tag{1a}$$

and the commodity conservation condition

$$q_i = \sum_{i\to} z^s - \sum_{\to i} z^s \text{ for each node } i. \tag{1b}$$

(Note that although z^{sc} is non-negative by virtue of our choice of link direction, q_{ic} might be either negative or positive, depending on whether node i is a net exporter or a net importer of commodity c. Needless to say, for any isolated node i, condition (1b) actually asserts that $q_i = 0$.)

The single condition involving only p is the price-stability condition

$$p_{)s} + p^s \geq p_{s)} \text{ for each link } s. \tag{1c}$$

(Note that condition (1c) just reflects the following circumstances. For any link s, if the unit purchase-price $p_{)sc}$ of a given commodity c for node)s plus the unit transport-price p^{sc} for commodity c on link s were less than the unit selling-price $p_{s)c}$ of commodity c for node s), some "entrepreneur" would obviously begin to purchase as much of commodity c as possible from node)s to be transported over link s for resale to node s) -- an economically unstable situation.)

One of the three conditions involving both (z, q) and p is the complementarity condition

$$\langle z^s, p_{)s} + p^s - p_{s)} \rangle = 0 \text{ for each link } s. \tag{1d}$$

(In view of conditions (1a) and (1c), note that condition (1d) just reflects the following circumstances. For any link s, if the quantity z^{sc} of commodity c being transported over link s were positive and if $p_{)sc}$ plus p^{sc} were greater than $p_{s)c}$, some entrepreneur would obviously be losing money and hence would begin to lower z^{sc} to zero -- another economically unstable situation.)

The remaining two conditions involving both (z, q) and p relate supply and demand to price. In particular, we assume the affine relations

$$p_i = A_i q_i + a_i \text{ for each node } i, \qquad \text{(ie)}$$

where the A_i and a_i are given constant matrices and vectors (that can arise from inverting the difference between given supply and demand quantities originally expressed as affine functions of price). We also assume the affine relations

$$p^s = A^s z^s + a^s \text{ for each link } s, \qquad \text{(1f)}$$

where the A^s and a^s are given constant matrices and vectors (that arise from describing the transport prices as functions of transport volumes).

Each solution (z, q; p) to the economic equilibrium conditions (1) is termed an <u>economic equilibrium</u>. For purposes of illustration, consider the 2-commodity network described by both the graph in Figure 1 and the data in Figure 2.

$$(A_1|a_1) = \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 2 & 1 & 2 \end{array}\right) \qquad (A_2|a_2) = \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & -1 \end{array}\right)$$

$$(A_3|a_3) = \left(\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 1 & -1 \end{array}\right) \qquad (A_4|a_4) = \left(\begin{array}{cc|c} 0 & -2 & 2 \\ 2 & 0 & 2 \end{array}\right)$$

$$(C_1|c_1) = \left(\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & 1 & -1 \end{array}\right) \quad (C_2|c_2) = \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -2 \end{array}\right) \quad (C_3|c_3) = \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 1 \end{array}\right)$$

$$(C_4|c_4) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \qquad (C_5|c_5) = \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 2 & 1 & 1 \end{array}\right)$$

Figure 2

An economic equilibrium for this example is computed via Lemke's algorithm at the end of Section 4.

3. A linear complementarity formulation. As previously noted, for any isolated node i, condition (1b) asserts that $q_i = 0$, which implies via condition (ie) that $p_i = a_i$. In particular then, equilibrium excess supplies and equilibrium prices are immediately determined for each isolated node. Consequently, we can assume (without loss of generality) that the network under study has no isolated nodes.

Condition (1b) now asserts that the excess supplies q_i depend (only) on the transport quantities z^s. Therefore, conditions (1e) and (1f) imply that the prices p_i and p^s also depend (only) on the transport quantities z^s; in particular, condition (1e) can be replaced by the condition

$$p_i = A_i\{\sum_{i\to} z^t - \sum_{\to i} z^t\} + a_i \text{ for each node } i. \tag{1e'}$$

Consequently, the "slack variables"

$$w^s \overset{\Delta}{=} p_{)s} + p^s - p_{s)} \text{ for each link } s \tag{2}$$

also depend (only) on the transport quantities z^s; in fact, with the constants

$$v^s \overset{\Delta}{=} a^s + a_{)s} - a_{s)} \text{ for each link } s, \tag{3}$$

condition (2) can be replaced by the condition

$$w^s = A^s z^s + A_{)s}\{\sum_{)s\to} z^t - \sum_{\to)s} z^t\} - A_{s)}\{\sum_{s)\to} z^t - \sum_{\to s)} z^t\} + v^s \quad \text{for each link } s. \tag{2'}$$

Needless to say, condition (1c) can be replaced by the condition

$$w^s \geq 0 \text{ for each link } s; \tag{1c'}$$

and condition (1d) can be replaced by the condition

$$\langle z^s, w^s \rangle = 0 \text{ for each link } s. \tag{1d'}$$

The net result is that the solutions (z, q; p) to the economic equilibrium conditions (1) can be described entirely in terms of the solutions (z, w) to the "linear complementarity conditions"

$$z \geq 0 \qquad \langle z, w \rangle = 0 \qquad w \geq 0 \tag{4a}$$

$$w \doteq Mz + v, \tag{4b}$$

where

M denotes the matrix with (matrix) elements M^{st} that make condition (2') equivalent to condition (4b).

In particular, <u>each solution</u> (z, w) <u>to the linear complementarity conditions</u> (4) <u>provides a solution</u> (z, q; p) <u>to the economic equilibrium conditions</u> (1) <u>via the equations</u>

$$p^s = A^s z^s + a^s \quad \underline{\text{for each link}}\ s \tag{5a}$$

$$q_i = \sum_{i\to} z^s - \sum_{\to i} z^s \quad \underline{\text{for each node}}\ i \tag{5b}$$

$$p_i = A_i \{\sum_{i\to} z^s - \sum_{\to i} z^s\} + a_i \quad \underline{\text{for each node}\ i}; \tag{5c}$$

<u>and all solutions</u> (z, q; p) <u>to conditions</u> (1) <u>can be obtained in this way</u>.

Now, elementary graph-theoretic considerations and an inspection of condition (2') show that the matrix elements

$$M^{st} = \begin{cases} A^s + (A_{)s} + A_{s)}) & \text{if } t = s \\ (A_{)s} + A_{s)}) & \text{if } t \in)s\!\to \cap \to\! s) \text{ but } t \neq s \\ A_{)s} & \text{if } t \in)s\!\to \text{ but } t \notin \to\! s) \\ A_{s)} & \text{if } t \in \to\! s) \text{ but } t \notin)s\!\to \\ -(A_{)s} + A_{s)}) & \text{if } t \in s)\!\to \cap \to\!)s \\ -A_{)s} & \text{if } t \in \to\!)s \text{ but } t \notin s)\!\to \\ -A_{s)} & \text{if } t \in s)\!\to \text{ but } t \notin \to\!)s \\ 0 & \text{if otherwise.} \end{cases}$$

For purposes of illustration, note that the graph in Figure 1 gives rise to the matrix and vector in Figure 3.

$$[M \,\|\, v] = \left[\begin{array}{ccccc|c} A_1+A_2+A^1 & -(A_1+A_2) & -A_2 & 0 & A_2 & v^1 \\ -(A_1+A_2) & A_1+A_2+A^2 & A_2 & 0 & -A_2 & v^2 \\ -A_2 & A_2 & A_2+A_3+A^3 & -A_3 & -A_2 & v^3 \\ 0 & 0 & -A_3 & A_3+A_4+A^4 & -A_4 & v^4 \\ A_2 & -A_2 & -A_2 & -A_4 & A_2+A_4+A^5 & v^5 \end{array}\right]$$

Figure 3

Hence, the data in Figure 2 gives rise to the linear complementarity conditions (4) with the data in Figure 4.

$$[M \,\|\, v] = \left[\begin{array}{rr|rr|rr|rr|rr||r}
4 & -1 & -2 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & -1 \\
2 & 3 & -2 & -2 & 0 & -1 & 0 & 0 & 0 & 1 & 2 \\
\hline
-2 & 1 & 3 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & -1 \\
-2 & -2 & 2 & 3 & 0 & 1 & 0 & 0 & 0 & -1 & -5 \\
\hline
-1 & 1 & 1 & -1 & 3 & -3 & -1 & 1 & -1 & 1 & -2 \\
0 & -1 & 0 & 1 & 2 & 3 & -1 & -1 & 0 & -1 & 1 \\
\hline
0 & 0 & 0 & 0 & -1 & 1 & 2 & -2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 4 & 2 & -2 & 0 & -3 \\
\hline
1 & -1 & -1 & 1 & -1 & 1 & 0 & 2 & 2 & -3 & 2 \\
0 & 1 & 0 & -1 & 0 & -1 & -2 & 0 & 4 & 2 & 4
\end{array}\right]$$

Figure 4

A solution for this example is computed via Lemke's algorithm at the end of Section 4.

4. <u>The main results</u>. To solve the linear complementarity conditions (4), we apply "Lemke's algorithm" to the modified conditions

$$z \geq 0 \qquad \langle z, w \rangle = 0 \qquad w \geq 0 \tag{6a}$$

$$w = Mz + ez^{+} + v \qquad z^{+} \geq 0, \tag{6b}$$

where

$$e_k \triangleq \begin{cases} 1 \text{ if } v_k \leq 0 \\ 0 \text{ if } v_k > 0 \end{cases}$$

and

$$z^+ \text{ is an additional scalar variable.}$$

Conditions (6) clearly have solutions $z = 0$ and $w = ez^+ + v$ for sufficiently large z^+. Starting with such a solution, Lemke's algorithm performs certain "pivots" that attempt to generate a solution (z, w) to the linear complementarity conditions (4) by driving z^+ to zero. The complete algorithm and a proof of the following theorem can be found in either [2] or [1].

<u>Theorem</u> 0 (Lemke). If the matrix M is "copositive plus", in that

(i) $\langle z, Mz \rangle \geq 0$ for all $z \geq 0$,

(ii) $\langle z, Mz \rangle = 0$ and $z \geq 0$ imply that $(M^t + M)z = 0$,

then Lemke's algorithm either generates a solution to conditions (6) with $z^+ = 0$ (in which event a solution to conditions (4) is obtained), or it generates a solution to the conditions

$$zM \leq 0 \qquad \langle z, Mz \rangle = 0 \qquad z \geq 0 \qquad \langle z, v \rangle < 0 \tag{7}$$

(in which event no solutions to conditions (4) exist).

The following three theorems are the main results of this paper.

<u>Theorem</u> 1. If the matrix A_i is positive semi-definite for each node i, and if the matrix A^s is copositive plus for each link s, then Lemke's algorithm either generates a solution to conditions (4) (and hence a solution to the economic equilibrium conditions (1)), or it demonstrates that no such solutions exist.

Theorem 2. If the matrix A_i is positive semi-definite for each node i, and if the matrix A^s is "strictly copositive" for each link s, in that

(i) $$\langle z^s, A^s z^s \rangle > 0 \text{ for all } z^s \geq 0 \text{ for which } z^s \neq 0,$$

then Lemke's algorithm generates a solution to conditions (4) (and hence a solution to the economic equilibrium conditions (1)).

Theorem 3. If the matrix A_i is positive definite for each node i, and if the conditions

$$z^s \geq 0 \qquad \langle z^s, A^s z^s \rangle = 0 \qquad \langle z^s, a^s \rangle < 0 \tag{8}$$

have no solution for each link s, then Lemke's algorithm generates a solution to conditions (4) (and hence a solution to the economic equilibrium conditions (1)).

For purposes of illustration, note that the data in Figure 2 satisfies the hypotheses of Theorem 2. Consequently, Lemke's algorithm generates a solution to conditions (4). In fact, computer calculations produce the solution vectors

$$z^1 = (.2353,.7059) \qquad z^2 = (0,2.2941) \qquad z^3 = (1.5294,0)$$

$$z^4 = (1.0098,.2451) \qquad z^5 = (0,0).$$

These vectors then generate, via relations (5), the additional solution vectors

$$p^1 = (-.5294,-.2941) \quad p^2 = (1.2941,.2941) \quad p^3 = (2.5294,2.5294)$$

$$p^4 = (1.2549,1.2549) \quad p^5 = (-1,1)$$

$q_1 = (.2353, -1.5882) \qquad q_2 = (1.2941, 1.5882)$

$q_3 = (-.5196, .2451) \qquad q_4 = (-1.0098, -.2451)$

$p_1 = (-.7647, .8824) \qquad p_2 = (-1.2941, .5882)$

$p_3 = (1.2353, -1.2745) \qquad p_4 = (2.4902, -.0196)$

Collectively, these vectors constitute an economic equilibrium (z, q; p) for the 2-commodity network described by both the graph in Figure 1 and the data in Figure 2.

5. Proofs. The following lemma is used repetitively and is of some interest in its own right.

Lemma. For the matrix M (defined immediately after conditions (4)),

$$\langle z, Mz \rangle \equiv \sum_s \langle z^s, A^s z^s \rangle + \sum_i \langle q_i, A_i q_i \rangle \text{ for all } z,$$

with the understanding that q is determined by z via the commodity conservation condition

$$q_i = \sum_{i\to} z^t - \sum_{\to i} z^t.$$

Proof. Conditions (4b) and (2'), together with elementary graph-theoretic considerations, show that

$$\begin{aligned}
\langle z, Mz\rangle &= \langle z, w - v\rangle = \sum_s \langle z^s, w^s - v^s\rangle \\
&= \sum_s \langle z^s, A^s z^s + A_{)s} q_{)s} - A_{s)} q_{s)}\rangle \\
&= \sum_s \langle z^s, A^s z^s\rangle + \sum_s \langle z^s, A_{)s} a_{)s}\rangle - \sum_s \langle z^s, A_{s)} a_{s)}\rangle \\
&= \sum_s \langle z^s, A^s z^s\rangle + \sum_i \langle \sum_{i\to} z^t, A_i q_i\rangle - \sum_i \langle \sum_{\to i} z^t, A_i q_i\rangle \\
&= \sum_s \langle z^s, A^s z^s\rangle + \sum_i \langle q_i, A_i q_i\rangle. \qquad \text{q.e.d.}
\end{aligned}$$

Proof of Theorem 1. According to Theorem 0, we need only show that M is copositive plus. Furthermore, since the sum of two copositive plus matrices is clearly copositive plus, we actually need only show that $A \overset{\Delta}{=} \text{diag}\ (A^1, A^2, \ldots)$ and $M - A$ are each copositive plus.

Now, A is copositive plus because A^s is assumed to be copositive plus for each s. Moreover, the Lemma shows that $M - A$ is positive semi-definite because A_i is assumed to be positive semi-definite for each i. This completes the proof because each positive semi-definite matrix is copositive plus (a fact that is an immediate consequence of the observation that $(N^t + N)z = 0$ characterizes the "critical points" for $\langle z, Nz\rangle$). q.e.d.

Proof of Theorem 2. First, note that A^s is copositive plus for each s because A^s is assumed to be strictly copositive for each s. Consequently, the hypotheses of Theorem 1 are satisfied.

If Lemke's algorithm demonstrates that no solution to conditions (4) exists, Theorem 0 asserts that a solution z to conditions (7) is generated. In that event, the second condition (7) along with the Lemma and the hypotheses of Theorem 2 show that $\langle z^s, A^s z^s\rangle = 0$ for each s. The third condition (7) and the strict copositivity of A^s for each s then imply that $z^s = 0$ for each s, which contradicts the fourth condition (7). q.e.d.

Proof of Theorem 3. First, note that the hypotheses of Theorem 1 are satisfied.

If Lemke's algorithm demonstrates that no solution to condition (4) exists, Theorem 0 asserts that a solution z to conditions (7) is generated. In that event, the second condition (7) and the Lemma imply that

$$0 = \sum_s \langle z^s, A^s z^s \rangle + \sum_i \langle q_i, A_i q_i \rangle.$$

From the third condition (7) and the hypotheses of Theorem 2, we now infer that both

$$\langle z^s, A^s z^s \rangle = 0 \text{ for each } s \tag{9}$$

and

$$\langle q_i, A_i q_i \rangle = 0 \text{ for each } i. \tag{10}$$

Moreover, the fourth condition (7) and the defining equation (3) along with elementary graph-theoretic considerations show that

$$\begin{aligned} 0 > \langle z, v \rangle &= \sum_s \langle z^s, a^s + a_{)s} - a_{s)} \rangle \\ &= \sum_s \langle z^s, a^s \rangle + \sum_s \langle z^s, a_{)s} \rangle - \sum_s \langle z^s, a_{s)} \rangle \\ &= \sum_s \langle z^s, a^s \rangle + \sum_i \langle \sum_{i\to} z^s, a_i \rangle - \sum_i \langle \sum_{\to i} z^s, a_i \rangle \\ &= \sum_s \langle z^s, a^s \rangle + \sum_i \langle q_i, a_i \rangle. \end{aligned}$$

Since the assumed positive definiteness of A_i for each i implies via relation (10) that $q_i = 0$ for each i, we infer that

$$0 > \sum_{s} \langle z^s, a^s \rangle.$$

Finally, the third condition (7) and relation (9) along with the preceding inequality clearly contradict the last hypothesis of Theorem 3. q.e.d.

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