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THE STABILITY OF EQUILIBRIUM

by

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1. INTRODUCTION

This paper presents an analysis of the stability of equilibrium for a broad class of models of intertemporal maximising behaviour that arise in dynamic economics. This class of models is capable of handling not only traditional problems in capital theory such as the adjustment-cost theory of the firm and many variants of the Ramsey problem, but also simple instances of intertemporal rational expectations equilibrium [19, 22, 32].

The analysis of the stability of equilibrium for this class of models has been the subject of extensive recent research,[2] much of which has been surveyed by Brock [4]. Until the work of Magill-Scheinkman [21], attention was focused exclusively on sufficient conditions for the stability of equilibrium. This paper is an attempt to extend the necessary and sufficient conditions derived by Magill-Scheinkman.

After introducing the basic class of intertemporal maximum problems (section 2), I reduce the problem in the neighborhood of an equilibrium point to a simple canonical form in which the forces that determine the stability of equilibrium stand out with especial clarity (section 4). I show that a concept which is basic to an understanding of the stability of equilibrium is the

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[2]See References [3]-[11], [15]-[23], [26]-[31].

distinction between a symmetric and an asymmetric equilibrium point (section 3). It follows from the work of Magill-Scheinkman [21] that in the neighborhood of a symmetric equilibrium point the local topological structure of the trajectories arising from the intertemporal maximum problem can be inferred from considerations based on a static maximum problem, namely the problem of maximising steady state profit. In the neighborhood of an asymmetric equilibrium point dynamic forces come into play which prevent the local topological structure of the trajectories from being inferred from purely static considerations. These dynamic forces are skew-symmetric (rotational) and when present with sufficient magnitude lead to cyclical motion about the equilibrium point.[3] Under simplifying assumptions on the way in which these skew-symmetric forces are present I show that in certain instances their presence induces a stabilising effect and in other instances a destabilising effect (section 4). These stability conditions are interpreted in section 5, where I also indicate a number of results that are likely to hold under more general assumptions.

The class of symmetric variational problems introduced by Magill-Scheinkman [21] have a number of remarkable properties, which arise in essence from the fact that a single function, the steady state profit function, characterises the equilibria and their stability properties. One of these properties concerns the results that may be obtained by an application of the Correspondence Principle, a method of far-reaching importance first explored by Samuelson in [30]. This Principle provides a natural way of generalising the method of comparative statics to the simplest class of dynamical systems namely those for which equilibria

[3]For an analysis of the way in which these skew-symmetric terms can give rise to cycling in a rational expectations equilibrium for a competitive industry see [19].

represent the important type of limiting behaviour (the ω-limit sets). The Principle rests on two ideas. First, only stable equilibria or motion in the neighborhood of such equilibria can expect to be observed for any reasonable length of time, motion in the neighborhood of an unstable equilibrium being a transient phenomenon. Second, the necessary conditions for the local stability of an equilibrium point can be used to obtain a qualitative restriction on the way in which the equilibrium point varies when certain underlying parameters in the model vary. Until the work of Magill-Scheinkman [21], the applicability of the Correspondence Principle for dynamical systems arising from maximising behaviour was limited by a failure to have a complete set of necessary conditions. It was shown in [21], using the necessary conditions for stability, that the term which appears in applying the Correspondence Principle to determine the way in which an equilibrium point varies with a given parameter consists of two components of which one is precisely the inverse of the Hessian matrix of the steady state profit function.

Using the necessary conditions derived in section 4, I show in section 6 the qualitative restrictions that can be obtained from the Correspondence Principle when the stable equilibria are allowed to be asymmetric. It is shown that the dynamic skew-symmetric forces which come into play in the neighborhood of an asymmetric equilibrium point no longer make it necessary for a certain local steady state profit function to attain a maximum at a stable equilibrium point but only a maximin, a condition that carries with it a correspondingly weaker condition on the inverse of the Hessian matrix of the local steady state profit function. I apply these results to the dynamic theory of the firm in a stationary environment and show that the basic result of the static theory, namely that the Jacobian matrix of the input demand function is negative definite, no longer holds in the dynamical case. Inputs in fact can exist for which an increase in the rental price leads to an increase in the steady state demand.

Mention should be made of the interesting work of Burmeister-Hammond [8] on the stability of equilibrium when the maximisation of an integral of discounted utility (profit) is replaced by an intertemporal version of Rawl's Maximin Criterion of Justice. It appears that the skew-symmetric terms which complicate the analysis under the conventional criterion may well be absent under Rawl's Maximin Criterion thereby leading to a potentially simplier theory of the stability of equilibrium.

A historical remark may be of interest. The analysis of the local stability of equilibrium for a conservative dynamical system was first given by Lagrange in the Mécanique Analytique [14, Part II, Section VI]. Routh [27; 28, Chapters III-VI] and Lord Kelvin and Tait [33, Section 345] were the first to analyse the stability of steady motion for a conservative dynamical system.[4] The equilibria of the first are symmetric in my terminology, the equilibria of the second are asymmetric when the standard Routhian function[5] is introduced. Lord Kelvin and Tait showed, in simple cases, that the skew-symmetric (centrifugal) forces arising from the steady motion can stabilise the unstable equilibrium of a conservative dynamical system. The spinning top provides the simplest classic example of this quite general phenomenon. With no spin, the verticle position, in which the potential energy is a maximum, is unstable, but with sufficiently rapid rotation the vertical position becomes stable.

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[4]See also [2, Ch. VIII] and [24, Chs. IX, X].

[5]See [24, p. 159].

2. THE INTERTEMPORAL MAXIMUM PROBLEM

Let $I = [0,\infty)$ denote the non-negative time-interval and let $\mathcal{K}$ denote the state space, where $\mathcal{K}$ is a convex set in R^n, $n \geqq 1$.

DEFINITION. For fixed $k_0 \varepsilon \mathcal{K}$, the class of absolutely continuous paths

(1) $$k(t) = k_0 + \int_0^t \dot{k}(\tau)d\tau : I \longrightarrow \mathring{\mathcal{K}}$$

for which $$\|k(t)\| \leqq \|k_0\| + \int_0^t \|\dot{k}(\tau)\| d\tau < \infty \qquad \text{for all } t \varepsilon I$$

where $\mathring{\mathcal{K}}$ denotes the interior of $\mathcal{K}$ and $\| \ \|$ denotes the standard Euclidean norm, is called the class of feasible paths and is denoted by $\mathcal{P}$. It is convenient to let $\{k,\dot{k}\}$ denote the path (1).

Let $\mathcal{A} \subseteq R^s$, $s \geqq 1$ denote the parameter space. We consider a vector of exogenous parameters $\alpha = (\beta,\delta) \varepsilon \mathcal{A} = \mathcal{A}_\beta \times \mathcal{A}_\delta$ and real valued instantaneous utility (profit) functions

(2) $$L(k,\dot{k};\beta)e^{-\delta t} : \mathring{\mathcal{K}} \times R^n \times \mathcal{A} \longrightarrow R$$

which satisfy the following[6]

[6]The dependence of $L(k,\dot{k};\beta)$ on the parameter β is sometimes omitted to simplify the notation.

ASSUMPTION 1 (Concavity, differentiability). L(·; β) is a C^r concave function in $(k,\dot{k})$ for all $(k,\dot{k}) \in \mathcal{K} \times R^n$, for all $\beta \in \mathcal{A}_\beta$ where $r \geqq 2$.

We consider feasible paths (1) induced by (2) through the following

VARIATIONAL PROBLEM. Find a feasible path $\{\bar{k},\dot{\bar{k}}\} \in \Phi$ such that

$$(\mathcal{V}) \qquad \underline{\lim}_{T\to\infty} \int_0^T \Big[L\big(\bar{k}(\tau),\dot{\bar{k}}(\tau)\big) - L\big(k(\tau),\dot{k}(\tau)\big)\Big] e^{-\delta\tau} d\tau \geqq 0$$

for all $\{k,\dot{k}\} \in \Phi$. The path $\{\bar{k},\dot{\bar{k}}\} \in \Phi$ is said to be optimal.

DEFINITION. Let Φ^* denote the class of absolutely continous price paths

$$(3) \qquad p(t) = p_0 + \int_0^t \dot{p}(\tau)d\tau : I \longrightarrow R^n$$

for which $\qquad \|p(t)\| \leqq \|p_0\| + \int_0^t \|\dot{p}(\tau)\|d\tau < \infty \qquad$ for all $t \in I$

It is convenient to let $\{\dot{p} - \delta p, p\}$ denote the path (3).

DEFINITION. A feasible path $\{\bar{k},\dot{\bar{k}}\} \in \Phi$ is competitive if there exists an absolutely continuous path of prices $\{\dot{\bar{p}} - \delta\bar{p}, \bar{p}\} \in \Phi^*$ such that

$$(4) \qquad L(\bar{k},\dot{\bar{k}}) + \bar{p}'\dot{\bar{k}} + (\dot{\bar{p}} - \delta\bar{p})'\bar{k} \geqq L(k,\dot{k}) + \bar{p}'\dot{k} + (\dot{\bar{p}} - \delta\bar{p})'k$$

for all $(k,\dot{k}) \in \mathcal{K} \times R^n$ for almost all $t \in I$.

Remark. Since $(1,\bar{p})$ is the vector of (imputed) output prices and $-(\dot{\bar{p}} - \delta\bar{p})$ is the vector of (imputed) rental costs

$$L + \bar{p}\dot{k} + (\dot{\bar{p}} - \delta\bar{p})k$$

is the (imputed) profit which is maximised at almost every instant by a competitive path.

The following result is an immediate consequence of (4).

LEMMA 1. If Assumption 1 holds then $\{k,\dot{k}\} \varepsilon \mathcal{P}$ is competitive if and only if

(5) $$(\dot{p} - \delta p, p) = -(L_k, L_{\dot{k}}) \quad \text{for almost all } t \varepsilon I$$

Remark. (5) is equivalent to the Euler-Lagrange equation

($\mathcal{L}$) $$L_k + \delta L_{\dot{k}} - \frac{d}{dt}(L_{\dot{k}}) = L_k + \delta L_{\dot{k}} - L_{\dot{k}\dot{k}}\ddot{k} - L_{\dot{k}k}\dot{k} = 0$$

Remark. Under a standard transversality condition a competitive path $\{k,\dot{k}\} \varepsilon \mathcal{P}$ is optimal [20, p. 177]. The converse is established by Benveniste-Scheinkman [3] under certain additional conditions. For our purposes it is sufficient to know that an optimal path is a solution of ($\mathcal{L}$).

DEFINITION. A path $\{k,\dot{k}\} \varepsilon \mathcal{P}$ which satisfies ($\mathcal{L}$) with $\ddot{k}(t) = \dot{k}(t) = 0$ for all $t \varepsilon I$ is called an equilibrium point (stationary state).

$$\mathcal{E} = \{(k^*,\alpha) \varepsilon \mathcal{K} \times \mathcal{A} \mid L_k(k^*,0;\beta) + \delta L_{\dot{k}}(k^*,0;\beta) = 0\}$$

is called the equilibrium set for the variational problem ($\mathcal{V}$).

DEFINITION. Let $(k^*,\alpha) \varepsilon \mathcal{E}$. The local coordinates around the equilibrium point $k^* = k^*(\alpha)$ are given by

$$x = k - k^*$$

Let $\mathcal{P}'$ denote the class of absolutely continuous paths $\{x,\dot{x}\}$ for which $\{k,\dot{k}\} \varepsilon \mathcal{P}$. The second variation problem about k^*

($\mathcal{V}'$) $$\inf_{\{x,\dot{x}\} \varepsilon \mathcal{P}'} -\frac{1}{2}\int_0^\infty L^0(x,\dot{x}) e^{-\delta t} dt$$

where

(8) $$L^0(x,\dot{x}) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}' \begin{bmatrix} L^*_{kk} & L^*_{k\dot{k}} \\ L^*_{\dot{k}k} & L^*_{\dot{k}\dot{k}} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

and where the asterisk signifies that the Hessians are evaluated at $(k^*,0)$, has associated with it the Euler-Lagrange equations

$$(\mathcal{L}') \qquad L^*_{\dot{k}\dot{k}}\ddot{x} + (L^*_{\dot{k}k} - L^*_{k\dot{k}} - \delta L^*_{\dot{k}\dot{k}})\dot{x} - (L^*_{kk} + \delta L^*_{\dot{k}k})x = 0$$

which are the linearised equations for $(\mathcal{L})$ about k^*.

DEFINITION. An equilibrium point $k^* = k^*(\alpha)$ is said to be regular (hyperbolic) if $\lambda_i \neq 0$ (Re $(\lambda_i) \neq 0$), $i = 1,\ldots,2n$ where $\lambda_i \in \mathbb{C}$ is a root of the characteristic polynomial

$$(9) \qquad D(\lambda_i) = |L^*_{\dot{k}\dot{k}}\lambda_i^2 + (L^*_{\dot{k}k} - L^*_{k\dot{k}} - \delta L^*_{\dot{k}\dot{k}})\lambda_i - (L^*_{kk} + \delta L^*_{\dot{k}k})| = 0$$

A parameter value $\alpha \in \mathcal{A}$ is regular (hyperbolic) if all the associated equilibria $k^*(\alpha)$ are regular (hyperbolic). We let $\mathcal{E}^r$ $(\mathcal{E}^h)$ denote the set of regular (hyperbolic) equilibria in $\mathcal{E}$. Similarily we let $\mathcal{A}^r$ $(\mathcal{A}^h)$ denote the set of regular (hyperbolic) parameter values in $\mathcal{A}$.

Remark. Let $(k^*,\alpha) \in \mathcal{E}$, then $k^* \in \mathcal{E}^r$ if and only if

$$\Delta = |L_{kk}(k^*,0;\beta) + \delta L_{\dot{k}k}(k^*,0;\beta)| \neq 0$$

Remark. Hyperbolic equilibria are of basic importance in the analysis that follows since it is only for these equilibria that the linearised equations $(\mathcal{L}')$ reveal the topological structure of the trajectories that are solutions of $(\mathcal{L})$ in a neighborhood of an equilibrium point. Hyperbolic equilibria are an important subset of the set of regular equilibria.

ASSUMPTION 2 (Profitability). There exist $\underline{k}_i$, $\overline{k}_i$, $i = 1,\ldots,n$ such that

$$\underline{\mathcal{K}} = \{k \in R^n \mid -\infty < \underline{k}_i < k_i < \overline{k}_i < \infty,\ i = 1,\ldots,n\} \subset \mathcal{K}$$

and for all $j = 1,\ldots,n$

$$L_{k_j}(k_1,\ldots,\underline{k}_j,\ldots,k_n,0,\ldots,0;\beta) + \delta L_{\dot{k}_j}(k_1,\ldots,\underline{k}_j,\ldots,k_n,0,\ldots,0;\beta) > 0$$

$$L_{k_j}(k_1,\ldots,\overline{k}_j,\ldots,k_n,0,\ldots,0;\beta) + \delta L_{\dot{k}_j}(k_1,\ldots,\overline{k}_j,\ldots,k_n,0,\ldots,0;\beta) < 0$$

for all $k_i \in (\underline{k}_i,\overline{k}_i)$, $i \neq j$.

Remark. The classical theorem of Kronecker-Poincaré [25, ch. XVIII] leads to the following result. If $\alpha \in \mathcal{A}^r$ and if Assumption 2 holds then there exists at least one regular equilibrium point $k^* \in \mathcal{K}$.

Assumption 2 is a natural economic condition to postulate: for each capital good j, the marginal revenue (L_{k_j}) from an additional unit of j must be greater (less) than its rental cost ($-\delta L_{\dot{k}_j}$) when the endowment of this capital good is sufficiently small (large), independent of the endowments of the other capital goods ($i \neq j$).

3. SYMMETRIC AND ASYMMETRIC EQUILIBRIA

DEFINITION. Let $(k^*,\alpha) \in \mathcal{E}$. k^* will be called a symmetric (asymmetric) equilibrium point if

$$L_{k\dot{k}}(k^*,0;\beta) - L_{\dot{k}k}(k^*,0;\beta) = 0 \ (\neq 0)$$

DEFINITION. The variational problem ($\mathcal{V}$) will be called symmetric (asymmetric) if

$$L_{k\dot{k}}(k,0;\beta) - L_{\dot{k}k}(k,0;\beta) = 0 \ (\neq 0) \quad \text{for all } k \in \mathcal{K}, \ \text{for all } \beta \in \mathcal{A}_\beta$$

Remark. Symmetric variational problems generate symmetric equilibria but the equilibria of an asymmetric variational problem need not be asymmetric.

DEFINITION. Let $k^* \in \mathcal{E}^h$. The solution of ($\mathcal{V}$) will be called locally cyclical (monotone) in a neighborhood of k^* if the characteristic polynomial $D(\lambda)$ has at least one (no) pair of complex conjugate roots.

Remark. Let $k^* \in \mathcal{E}^h$. If k^* is a symmetric equilibrium point and if $L^0(x,\dot{x})$ is negative definite, then the solution of $(\mathcal{V})$ is monotone in a neighborhood of k^*.[7]

Magill-Scheinkman [21] have given a complete characterisation of the local stability of regular symmetric equilibria. Their analysis is motivated by the following simple idea. Under Assumption 2 there are certain states in the region $\mathcal{K}$, namely the steady states or equilibria, which have the property that if the system starts in such a state it remains there permanently. Do these steady states have an economically interesting extremal property which might serve to characterise their stability properties? Is it possible that a certain function which depends only on the state of the system attains an extremum at such steady states, the extremum being a local maximum at locally stable regular equilibria and a local minimum at locally unstable regular equilibria. The analysis in [21] shows that this is indeed the case.

That the steady states of symmetric variational problems have an extremal property is a consequence of the following

LEMMA 2. If $L(\cdot;\beta) \in C^2$ then there exists a real valued function

$$\phi(k^*;\alpha) = \int_{\tilde{k}}^{k^*} \left(L_k(k,0;\beta) + \delta L_{\dot{k}}(k,0;\beta)\right)' dk \; : \; \mathcal{K} \times \mathcal{A} \longrightarrow R$$

where $\tilde{k}$ is an arbitrary fixed point in $\mathcal{K}$, such that

$$\phi_{k^*}(k^*;\alpha) = L_k(k^*,0;\beta) + \delta L_{\dot{k}}(k^*,0;\beta) \qquad \text{for all } (k^*,\alpha) \in \mathcal{K} \times \mathcal{A}$$

if and only if $(\mathcal{V})$ is a symmetric variational problem.

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[7] See Magill-Scheinkman [21, Lemma 3].

<u>Proof</u>. The symmetry of $L_{kk}(k,0;\beta)$ for all $k \in \mathcal{K}$, $\beta \in \mathcal{A}_\beta$ and the standard theorem for the existence of a potential function [1, pp. 293-297] yield the result. In view of (4) we are led to the following

DEFINITION. The function $\phi(k^*;\alpha)$ is called the <u>steady state profit function</u>.

This function characterises the steady states and their stability properties in the following way.

PROPOSITION 1 (Magill-Scheinkman). <u>If $(\mathcal{V})$ is a symmetric variational problem, then $k^* \in \mathcal{K}$ is a steady state if and only if the steady state profit function attains a local extremum at k^*. If $k^* \in \mathcal{E}^r$, then k^* is locally asymptotically stable (completely unstable) if and only if the steady state profit function attains a local maximum (minimum) at k^*.</u>

For a more complete statement of the results the reader is referred to [21]. The proof of the above result hinges on a relation which may be established between the eigenvalues of the linearised Euler-Lagrange equations $(\mathcal{L}')$ at k^* and the eigenvalues of the Hessian matrix $\phi_{k^*k^*}(k^*;\alpha)$ of the steady state profit function.

The results of Magill-Scheinkman lead us to ask the following question. <u>Is it possible to obtain a complete characterisation of the local stability properties of hyperbolic asymmetric equilibria</u>? The section that follows provides an answer to this question under certain simplifying assumptions.

4. STABILITY OF ASYMMETRIC EQUILIBRIA

To analyse the stability of hyperbolic asymmetric equilibria for ($\mathcal{V}$) and ($\mathcal{L}$) in terms of ($\mathcal{V}'$) and ($\mathcal{L}'$), I will reduce the problem ($\mathcal{V}'$) with associated Euler-Lagrange equation ($\mathcal{L}'$) to a simple <u>canonical form</u>. Assumption 1 implies $L^0(x,\dot{x})$ is <u>non-negative definite</u>. It is convenient to strengthen this property to

ASSUMPTION 1' (Strong concavity). <u>$L^0(x,\dot{x})$ is negative definite</u>.

I will introduce the notation

(10) $$A = -L^*_{kk}\,, \quad B = -L^*_{\dot{k}\dot{k}}\,, \quad N = -L^*_{k\dot{k}}$$

and consider the transformation

(11) $$x(t) = e^{\frac{\delta}{2}t}\,y(t)\quad .$$

which reduces ($\mathcal{V}'$) to

($\mathcal{V}''$) $$\inf_{\{y,\dot{y}\}\varepsilon\mathcal{P}''} \frac{1}{2}\int_0^\infty M^0(y,\dot{y})\,dt$$

where $\mathcal{P}''$ is defined in terms of $\mathcal{P}'$ through (11) and where

$$M^0(y,\dot{y}) = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' \begin{bmatrix} \bar{A} & \bar{N} \\ \bar{N}' & B \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

(12) $$\bar{A} = A + \frac{\delta}{2}(N+N') + \left(\frac{\delta}{2}\right)^2 B, \quad \bar{N} = N + \left(\frac{\delta}{2}\right)B$$

<u>Remark</u>. In view of Assumption 1', $M^0(y,\dot{y})$ is <u>positive definite</u>. Thus the

matrices $\bar{A}$ and B are both <u>positive definite</u>.

Under the transformation (11), ($\mathcal{L}'$) reduces to

$$(\mathcal{L}'') \qquad B\ddot{y} - C\dot{y} - \bar{A}y = 0, \quad C = N - N'$$

DEFINITION. $\bar{\alpha}_j \in \mathbb{C}$ will be called an eigenvalue of $\bar{A}$ in the metric of B and $\bar{w}^j \in \mathbb{C}^n$, $\bar{w}^j \neq 0$ is an associated eigenvector if B is positive definite and

$$(\bar{A} - \bar{\alpha}_j B)\bar{w}^j = 0$$

Remark. Since $\bar{A}$ is positive definite, symmetric, $\bar{A}$ has n real positive eigenvalues $(\bar{\alpha}_1, \ldots, \bar{\alpha}_n)$ and n real associated eigenvectors $(\bar{w}^1, \ldots, \bar{w}^n)$ in the metric of B. Furthermore the $n \times n$ matrix of eigenvectors $\bar{W} = [\bar{w}^1 \ \ldots \ \bar{w}^n]$ may be chosen in such a way that [12, pp. 310-319]

$$\bar{W}'B\bar{W} = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \qquad \bar{W}'\bar{A}\bar{W} = \bar{\mathcal{A}} = \begin{bmatrix} \bar{\alpha}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\alpha}_n \end{bmatrix}$$

Under the nonsingular transformation

$$(13) \qquad y = \bar{W}z$$

($\mathcal{V}''$) reduces to

$$(\mathcal{V}''') \qquad \inf_{\{z,\dot{z}\} \in \mathcal{P}'''} \frac{1}{2}\int_0^\infty Q^0(z,\dot{z})dt$$

where $\mathcal{P}'''$ is defined in terms of $\mathcal{P}''$ through (13) and where

$$Q^0(z,\dot{z}) = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}' \begin{bmatrix} \bar{\mathcal{A}} & \bar{\mathcal{N}} \\ \bar{\mathcal{N}}' & I \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix}, \qquad \bar{\mathcal{N}} = \bar{W}'\bar{N}\bar{W}$$

Remark. Since $M^0(y,\dot{y})$ is positive definite, $Q^0(z,\dot{z})$ is <u>positive definite</u>. Thus the matrix $\bar{\mathcal{A}} - \bar{\mathcal{N}}'\bar{\mathcal{N}}$ is <u>positive definite</u>.

Under the transformation (13), ($\mathcal{L}''$) reduces to

($\mathcal{L}'''$) $$\ddot{z} - \Gamma\dot{z} - \bar{A}z = 0$$

where

(14) $$\bar{W}'C\bar{W} = \Gamma = \begin{bmatrix} 0 & \gamma_{12} & \cdots & \gamma_{1n} \\ -\gamma_{12} & 0 & \cdots & \gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{1n} & -\gamma_{2n} & \cdots & 0 \end{bmatrix}$$

LEMMA 3. <u>If $(\pi_1,\ldots,\pi_n)$ are the eigenvalues of $A + \frac{\delta}{2}(N+N')$ in the metric of B then</u>

(15) $$\pi_j = \bar{\alpha}_j - \left(\frac{\delta}{2}\right)^2, \quad j = 1,\ldots,n$$

<u>Proof</u>. From the definitions of $\bar{\alpha}_j$, π_j, $j = 1,\ldots,n$

$$0 = (\bar{A} - \bar{\alpha}_j B)\bar{w}^j = \left[\left(A + \frac{\delta}{2}(N+N') + \left(\frac{\delta}{2}\right)^2 B\right) - \bar{\alpha}_j B\right]\bar{w}^j$$

$$= \left\{A + \frac{\delta}{2}(N+N') - \left[\alpha_j - \left(\frac{\delta}{2}\right)^2\right]B\right\}\bar{w}^j$$

$$= \left(A + \frac{\delta}{2}(N+N') - \pi_j B\right)\bar{w}^j \qquad \Delta$$

DEFINITION. Let $(k^*,\alpha) \in \mathcal{E}$. The quadratic form

$$\pi(x; k^*,\alpha) = x'\left(A + \frac{\delta}{2}(N+N')\right)x = -x'\left(L^*_{kk} + \frac{\delta}{2}(L^*_{k\dot{k}} + L^*_{\dot{k}k})\right)x : R^n \times \underline{\mathcal{K}} \times \mathcal{A} \longrightarrow R$$

will be called the <u>local steady state profit function</u> and the eigenvalues $(\pi_1,\ldots,\pi_n)$ will be called the <u>steady state profit rates</u>.

<u>Remark</u>. If we assume, without loss of generality, that the steady state profit rates are placed in order of decreasing magnitude

$$\pi_1 \geqq \pi_2 \geqq \cdots \geqq \pi_n$$

then they satisfy the well-known <u>maximum property</u> [12, pp. 317-320]

$$\pi_j = \max_{x \in \mathcal{B}_j} \pi(x; k^*, \alpha) = \pi(w^j; k^*, \alpha), \qquad j = 1, \ldots, n$$

$$\mathcal{B}_j = \{x \in R^n \mid x'Bx = 1, \quad x'Bw^i = 0, \quad i = 1, \ldots, j-1\}$$

I will give two precise characterisations of the local stability of hyperbolic asymmetric equilibria for ($\mathcal{V}$) using the canonical form ($\mathcal{L}'''$). In the first I impose a simplifying assumption on Γ, in the second I impose a simplifying assumption on $\bar{\mathcal{A}}$. To prove the first I use the following

LEMMA 4. Under Assumption 1' if there exists a reordering of the components of z such that

(16) $$\Gamma = \begin{bmatrix} \Gamma_1 & & 0 \\ & \ddots & \\ 0 & & \Gamma_{\left(\frac{n}{2}\right)} \end{bmatrix}, \quad \Gamma_j = \begin{bmatrix} 0 & \gamma_j \\ -\gamma_j & 0 \end{bmatrix}, \quad j = 1, \ldots, \left(\tfrac{n}{2}\right)$$

where $\left(\frac{n}{2}\right) = \frac{n}{2}$ when n is even, $\left(\frac{n}{2}\right) = \frac{n+1}{2}$ when n is odd and where $\Gamma_{\frac{n+1}{2}} = [\,0\,]$ when n is odd, then the solutions of ($\mathcal{L}'''$) are monotone (cyclical) if and only if

(17) $$|\sqrt{\bar{\alpha}_j} - \sqrt{\bar{\alpha}_{j'}}| \overset{>}{(<)} |\gamma_j|, \quad j = 1, \ldots, \left(\tfrac{n}{2}\right)$$

where $(\bar{\alpha}_j, \bar{\alpha}_{j'})$ are the components of $\bar{\mathcal{A}}$ associated with γ_j.

Proof. ($\mathcal{L}'''$) splits up into $\left(\frac{n}{2}\right)$ pairs of second order differential equations when n is even [$\left(\frac{n}{2}\right) - 1$ when n is odd]. For each such pair the characteristic polynomial is

$$D(\lambda_j) = \lambda_j^4 + (\bar{\gamma}_j^2 - \bar{\alpha}_j - \bar{\alpha}_{j'})\lambda_j^2 + \bar{\alpha}_j\bar{\alpha}_{j'} = 0, \quad j = 1,\ldots,(\tfrac{n}{2})$$

the roots of which are[8]

(18) $$\lambda_j = \tfrac{1}{2}\left(\pm\sqrt{-H_j} \pm \sqrt{-J_j}\right), \quad j = 1,\ldots,(\tfrac{n}{2})$$

where $$H_j = \gamma_j^2 - \left(\sqrt{\bar{\alpha}_j} + \sqrt{\bar{\alpha}_{j'}}\right)^2$$

$$J_j = \gamma_j^2 - \left(\sqrt{\bar{\alpha}_j} - \sqrt{\bar{\alpha}_{j'}}\right)^2$$

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[8](18) is derived as follows. Let $\xi_j = a_j + ib_j = \lambda_j^2$, then

$$a_j = \tfrac{1}{2}(\alpha_j + \alpha_{j'} - \gamma_j^2), \quad b_j = \tfrac{1}{2}\sqrt{(-H_j)J_j}$$

The relation $(a_j + ib_j) = (\mu_j + i\nu_j)^2$ implies

$$\mu_j = \frac{1}{\sqrt{2}}\left(\frac{b_j}{\theta_j}\right), \quad \nu_j = \frac{\theta_j}{\sqrt{2}}, \quad \theta_j = \sqrt{-a_j + \sqrt{a_j^2 + b_j^2}}$$

which in turn implies

$$\mu_j = \tfrac{1}{2}\sqrt{(-H_j)}, \quad \nu_j = \tfrac{1}{2}\sqrt{J_j}$$

Assumption 1' implies $\sqrt{\bar{\alpha}_j} > 0$, $\sqrt{\bar{\alpha}_{j'}} > 0$ from which it follows that the roots are complex if and only if $J_j > 0$. Δ

Remark. Let $\mathcal{N} = \{\nu_{ij}\}$ then Assumption 1' implies

(19) $\sqrt{\bar{\alpha}_j} + \sqrt{\bar{\alpha}_{j'}} > |\nu_{2j-1,2j}| + |\nu_{2j,2j-1}| \geqq |\nu_{2j-1,2j} - \nu_{2j,2j-1}| = |\gamma_j|, \quad j=1,\ldots,(\frac{n}{2})$

so that pure imaginary eigenvalues cannot arise in (18).

PROPOSITION 2. Let $k^* \varepsilon \mathcal{E}^h$. If Γ satisfies (16), then k^* is locally asymptotically stable if and only if

(20) $$\sqrt{(\sqrt{\bar{\alpha}_j} + \sqrt{\bar{\alpha}_{j'}})^2 - \gamma_j^2} - \sqrt{(\sqrt{\bar{\alpha}_j} - \sqrt{\bar{\alpha}_{j'}})^2 - \gamma_j^2} > \delta, \quad j = 1,\ldots,(\tfrac{n}{2})$$

when $(z_j, z_{j'})$ are monotone, if and only if

(21) $$\sqrt{\bar{\alpha}_j} + \sqrt{\bar{\alpha}_{j'}} > |\bar{\gamma}_j|, \quad j = 1,\ldots,(\tfrac{n}{2})$$

when $(z_j, z_{j'})$ are cyclical, where $\bar{\gamma}_j^2 = \gamma_j^2 + \delta^2$.

Proof. In view of (18) the eigenvalues of $(\mathcal{L}')$ are given by

$$\frac{1}{2}\left(\delta \pm \sqrt{-H_j} \pm \sqrt{-J_j}\right), \quad j = 1,\ldots,(\tfrac{n}{2})$$

Two of these eigenvalues are negative, for each j, if and only if (20) holds when $(z_j, z_{j'})$ are monotone. The real parts of a complex conjugate pair are negative, for each j, if and only if (21) holds when $(z_j, z_{j'})$ are cyclical. Δ

Remark. $(z_j, z_{j'})$ are completely unstable if and only if

(22) $$\sqrt{(\sqrt{\bar{\alpha}_j} + \sqrt{\bar{\alpha}_{j'}})^2 - \gamma_j^2} + \sqrt{(\sqrt{\bar{\alpha}_j} - \sqrt{\bar{\alpha}_{j'}})^2 - \gamma_j^2} < \delta$$

when $(z_j, z_{j'})$ are monotone, if and only if

(23) $$\sqrt{\bar{\alpha}_j} + \sqrt{\bar{\alpha}_{j'}} < |\bar{\gamma}_j|$$

when $(z_j, z_{j'})$ are cyclical.

Remark. If k^* is symmetric $\gamma_j = 0$ and by (17), $(z_j, z_{j'})$ are monotone $j = 1, \ldots, (\frac{n}{2})$. In view of (15), (20) is equivalent to the condition

$$\bar{\alpha}_j - \left(\frac{\delta}{2}\right)^2 = \pi_j > 0, \quad \bar{\alpha}_{j'} - \left(\frac{\delta}{2}\right)^2 = \pi_{j'} > 0, \quad j = 1, \ldots, (\tfrac{n}{2})$$

which is the result of Proposition 1. Equilibria are always symmetric when $n = 1$ so in this case z is monotone and the stability condition reduces to

$$\pi = -(L^*_{kk} + \delta L^*_{\dot{k}k}) > 0$$

Remark. Let $k^* \in \Xi^h$. If k^* is an asymmetric equilibrium point then a sufficient condition for Γ to have the form (16) is that the matrices in (10) have the block-diagonal form

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_{(\frac{n}{2})} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_{(\frac{n}{2})} \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_{(\frac{n}{2})} \end{bmatrix}.$$

$$A_j = \begin{bmatrix} a_{j1} & a_{j2} \\ a_{j'1} & a_{j'2} \end{bmatrix}, \quad B_j = \begin{bmatrix} b_{j1} & b_{j2} \\ b_{j'1} & b_{j'2} \end{bmatrix}, \quad N_j = \begin{bmatrix} n_{j1} & n_{j2} \\ n_{j'2} & n_{j'2} \end{bmatrix} \quad j = 1, \ldots, (\tfrac{n}{2})$$

the last of each of the block-diagonal elements $A_{(\frac{n}{2})}$, $B_{(\frac{n}{2})}$, and $N_{(\frac{n}{2})}$ reducing to scalars when n is odd.

Remark. In this case the stability conditions (20) and (21), stated in terms of the derived parameters $(\bar{\alpha}_j, \bar{\alpha}_{j'}; \gamma_j)$, are readily transformed into conditions on the original matrices $(A_j, B_j; N_j)$, by noting that the matrix of eigenvectors W is block-diagonal, $W = \begin{bmatrix} W_1 & & 0 \\ & \ddots & \\ 0 & & W_{(\frac{n}{2})} \end{bmatrix}$, so that $|W_j' B_j W_j| = |W_j|^2 |B_j| = 1$ implies

$|W_j| = \frac{1}{\sqrt{|B_j|}}$ and $\gamma_j = |W_j| g_j = \frac{g_j}{\sqrt{|B_j|}}$ where $g_j = (n_{j2} - n_{j'1})$. Furthermore, recalling (12),

$$\bar{\alpha}_j, \bar{\alpha}_{j'} = \frac{1}{|B_j|}\left\{\frac{D_j}{2} \pm \sqrt{\left(\frac{D_j}{2}\right)^2 - |\bar{A}_j||B_j|}\right\}, \quad D_j = \bar{a}_{j1}b_{j'2} + \bar{a}_{j'2}b_{j1} - 2\bar{a}_{j2}b_{j2}$$

PROPOSITION 3. Let $k^* \in \mathcal{E}^h$, k^* an asymmetric equilibrium point. If $\bar{\alpha}_1 = \bar{\alpha}_2 = \ldots = \bar{\alpha}_n = \bar{\alpha}^*$ then the solution of ($\mathcal{V}$) in a neighborhood of k^* is locally cyclical. Let[9]

$$\pm i\gamma_1, \ \ldots, \ \pm i\gamma_{(\frac{n}{2})}$$

denote the eigenvalues of Γ, where $\gamma_{\frac{n+1}{2}} = 0$ if n is odd and let $\pi^* = \bar{\alpha}^* - \left(\frac{\delta}{2}\right)^2$, then k^* is locally asymptotically stable if and only if

$$\pi^* > \left(\frac{\gamma_j}{2}\right)^2 \qquad j = 1, \ldots, (\tfrac{n}{2}) \tag{24}$$

Proof. In view of the assumption $\bar{\alpha}_1 = \bar{\alpha}_2 = \ldots = \bar{\alpha}_n = \bar{\alpha}^*$, $\bar{\mathcal{A}} = \bar{\alpha}^* I$ so that the eigenvalue problem for ($\mathcal{L}'''$)

$$(\lambda^2 I - \Gamma\lambda - \mathcal{A})v = 0 \tag{25}$$

reduces to an eigenvalue problem for Γ

$$\left[\Gamma - \left(\frac{\lambda^2 - \bar{\alpha}^*}{\lambda}\right) I\right] v = 0$$

Let $\lambda = \mu + i\nu$ then $\frac{\lambda^2 - \bar{\alpha}^*}{\lambda} = i\gamma_j$ implies $\mu = \pm\sqrt{\bar{\alpha}^* - \left(\frac{\gamma_j}{2}\right)^2}$, $\nu = \left(\frac{\gamma_j}{2}\right)$. Thus the eigenvalues of (25) are given by

$$\pm\sqrt{\bar{\alpha}^* - \left(\frac{\gamma_j}{2}\right)^2} \pm i\left(\frac{\gamma_j}{2}\right) \qquad j = 1, \ldots, (\tfrac{n}{2})$$

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[9] Recall that the eigenvalues of a skew-symmetric matrix are pure imaginary [12, p. 285].

and the eigenvalues of $(\mathcal{L}')$ are given by

$$\frac{\delta}{2} \pm \sqrt{\left(\frac{\delta}{2}\right)^2 + \pi^* - \left(\frac{\gamma_j}{2}\right)^2} \pm i\left(\frac{\gamma_j}{2}\right) \qquad j = 1,\ldots,(\tfrac{n}{2})$$

from which the result follows at once. Δ

5. INTERPRETATION OF STABILITY CONDITIONS

The stability conditions (20)-(24) may be given a simple geometric interpretation which brings to light more clearly the underlying economic conditions under which a hyperbolic equilibrium point is stable. To this end let

$$(26) \qquad v(t) = e^{\frac{\delta}{2}t} z(t) \quad \text{so that} \quad x(t) = \bar{W}v(t)$$

Thus the degree of stability of v(t) at an equilibrium point is the same as the degree of stability of x(t). Let $v = (v_1, v_{1'}, \ldots, v_{(\frac{n}{2})}, v_{(\frac{n}{2})'})$. We may construct a <u>stability diagram</u> (Figure 1) for the components $(v_j(t), v_{j'}(t))$ of v(t) in the non-negative orthant of the space $(\sqrt{\alpha_j}, \sqrt{\alpha_{j'}})$.

Consider Figure 1. VW and V'W' represent the lines $\sqrt{\alpha_j} - \sqrt{\alpha_{j'}} = \mp |\gamma_j|$. In the region between these lines $(v_j(t), v_{j'}(t))$ are cyclical (equation (17)). VV' represents the line $\sqrt{\alpha_j} + \sqrt{\alpha_{j'}} = |\gamma_j|$, the boundary of the feasible $(\sqrt{\alpha_j}, \sqrt{\alpha_{j'}})$ values implied by the strict concavity Assumption 1' (equation (19)). QQ' represents the line $\sqrt{\alpha_j} + \sqrt{\alpha_{j'}} = |\bar{\gamma}_j|$ which partitions the region in which $(v_j(t), v_{j'}(t))$ are <u>cyclical</u> into a <u>stable</u> and an <u>unstable</u> region (equations (21) and (23)). QS and QS', which represent equation (20) with an equality sign, partition the region in which $(v_j(t), v_{j'}(t))$ are <u>monotone</u> into a <u>stable</u> and an <u>unstable</u> region. QT and Q'T', which represent equation (22) with an equality sign, lead to the regions VQT and V'Q'T' in which $(v_j(t), v_{j'}(t))$ are <u>completely unstable</u>.

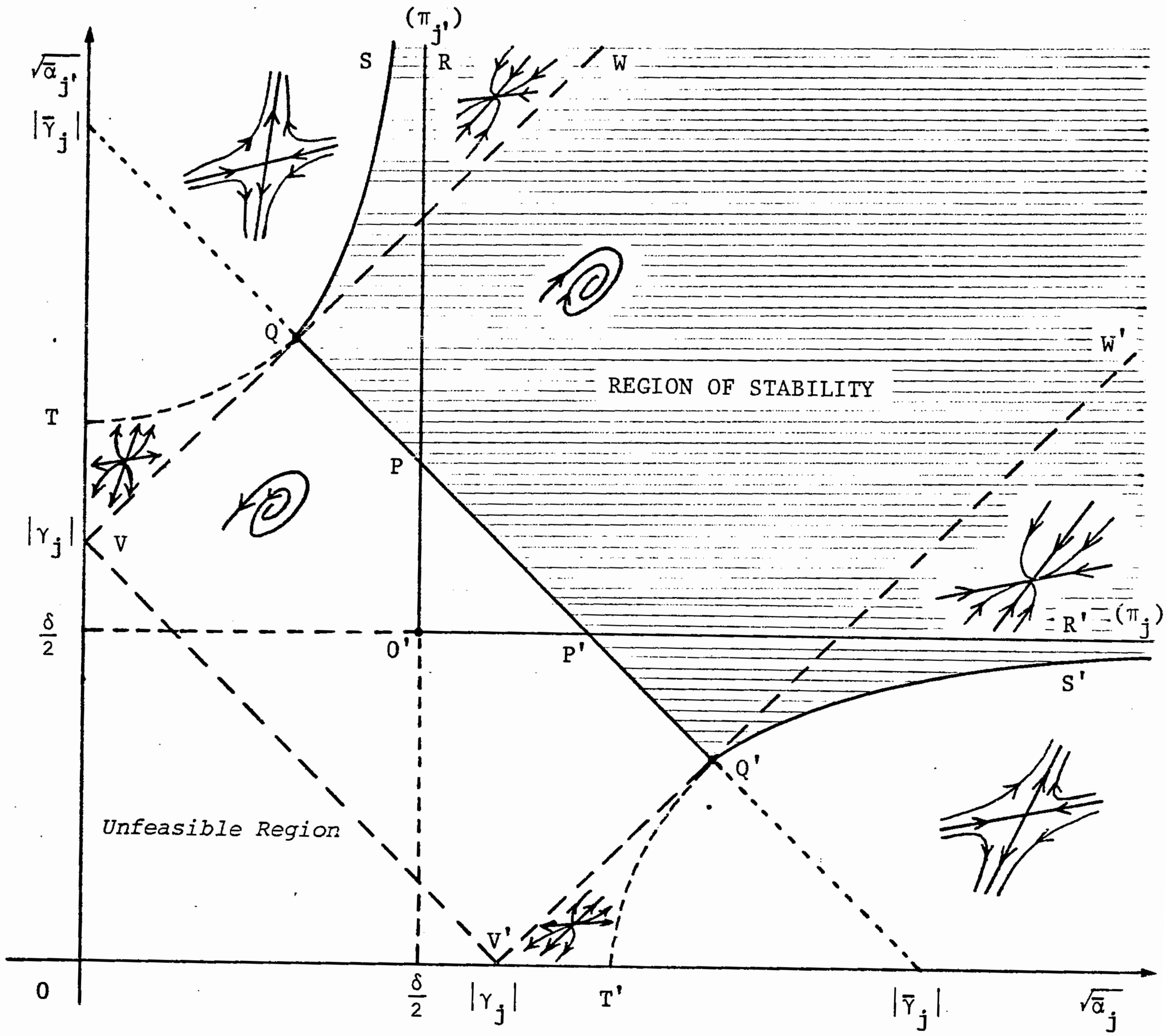
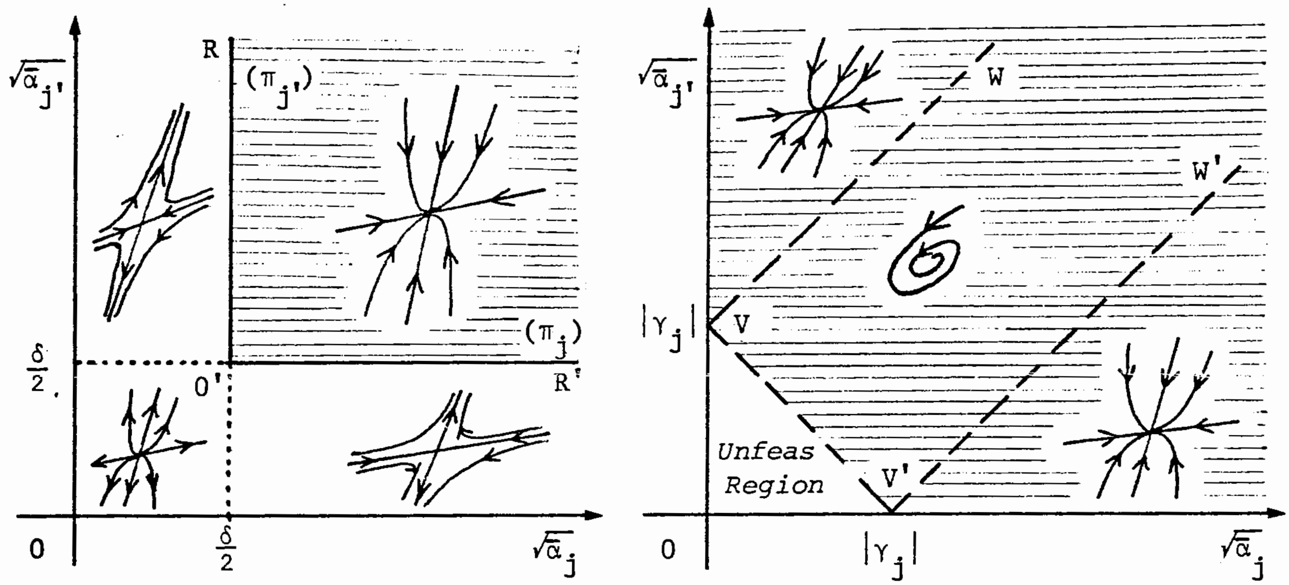


Figure 1. Stability diagram for $\{v_j(t), v_{j'}(t)\}$.

Two interesting limiting cases are shown in Figure 2 (a) and (b): the symmetric case $\gamma_j = 0$ and the undiscounted case $\delta = 0$.

As we know from the theorem of Magill-Scheinkman, stable symmetric equilibria are characterised by the fact that steady state profit rates $(\pi_j, \pi_{j'})$, $j = 1, \ldots, (\frac{n}{2})$ are positive: unstable symmetric equilibria are characterised by the fact that at least one steady state profit rate is negative. As we see



(a) $\gamma_j = 0$ (b) $\delta = 0$

Figure 2. The symmetric and undiscounted cases.

from the stability conditions (20)-(24) and in particular from Figure 1, asymmetric equilibria can be unstable even though the steady state profit rates are positive, when the solution of (V) is locally cyclical (region O'PP'). Asymmetric equilibria can be stable even though some steady state profit rate is negative (regions SQPR and S'Q'P'R').

Consider a symmetric equilibrium point. Suppose we introduce skew-symmetry, how does this affect the stability of equilibrium? Figure 3 provides an answer in the case where the skew-symmetry has the form (16). Let $\left(\sqrt{\bar{\alpha}_j^0}, \sqrt{\bar{\alpha}_{j'}^0}\right)$ denote the parameter values for a symmetric equilibrium ($\gamma_j = 0$). In view of (19) we may consider the impact of increasing $|\gamma_j|$, provided $|\gamma_j|$ is restricted to the interval

$$0 \leqq |\gamma_j| < |\gamma_j^0| = \sqrt{\bar{\alpha}_j^0} + \sqrt{\bar{\alpha}_{j'}^0}.$$

In Figure 3 (a) the curve FO'F' represents the locus of the points Q and Q' in Figure 1, as $|\gamma_j|$ is increased. $\left(v_j(t), v_{j'}(t)\right)$ are completely unstable, unstable of degree 1 and stable according as $\left(\sqrt{\bar{\alpha}_j^0}, \sqrt{\bar{\alpha}_{j'}^0}\right)$ lies in the regions

$00'Y'$, $R'0'Y'$, $Z0'R'$, where without loss of generality we restrict the analysis to the region below $00'Z$.

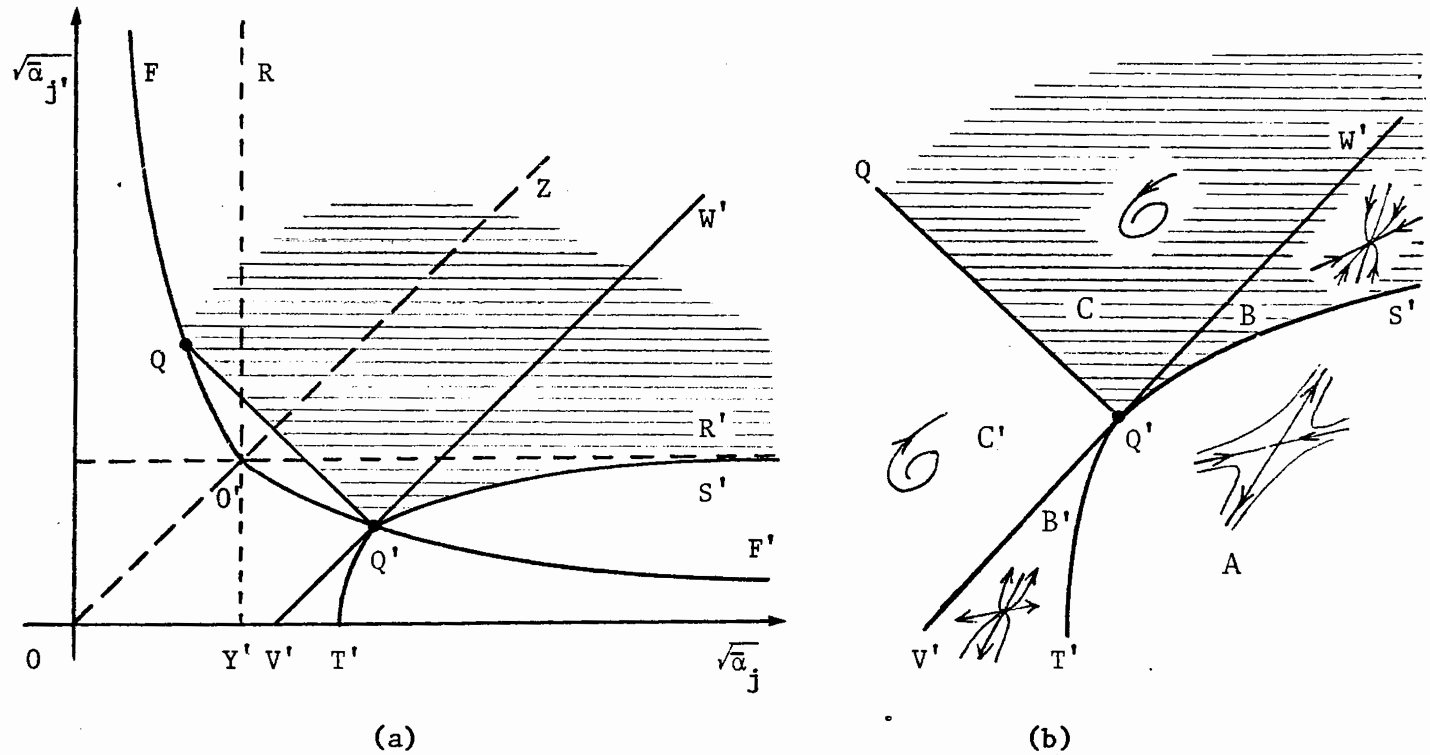


Figure 3. Analysis of effect of increase in skew-symmetry on stability of symmetric equilibrium.

Suppose the parameter value of the symmetric equilibrium lies in the region $00'Y'$. Then it starts in the region B' in Figure 3 (b) and passes into C' as the skew-symmetry $|\gamma_j|$ is increased. A parameter value which lies in $F'0'Y'$ starts in A, passes into B' and then into C'. If the parameter lies in $F'0'R'$, then it starts in A, passes into B, then into C and then into C'. Finally a parameter value which lies in $Z0'R'$ starts in B, passes into C and ends in C'.

Thus if the parameter value of a symmetric equilibrium lies in $F'0'R'$ then an increase in skew-symmetry $|\gamma_j|$ can lead to stability. However for every symmetric equilibrium a sufficient increase in skew-symmetry leads to instability (increasing $|\gamma_j|$ ultimately leads the parameter value into the region C').

Two further results follow from our analysis in terms of Figures 1 and 3. If the steady state profit rates are negative then an increase in skew-symmetry does not affect the complete instability of the symmetric equilibrium point. If the trajectory v(t) is monotone and if the steady state profit rates are positive then the equilibrium point is locally asymptotically stable. These results are likely to hold under quite general conditions.

It is clear from the equation ($\mathcal{L}'''$) that there are two forces at work in determining the stability of equilibrium. The first is summarised in $\bar{A}$ and the second in Γ. In view of (15) the information contained in $\bar{A}$ is summarised in the steady state profit rates $(\pi_1, \ldots, \pi_n)$. When $L^*_{k_i \dot{k}_j} \neq L^*_{k_j \dot{k}_i}$ so that the effect of investment in one capital good (j) on the marginal product of a second capital good (i) is asymmetric, a skew-symmetric matrix

$$(L^*_{\dot{k}k} - L^*_{k\dot{k}})' = -(L^*_{\dot{k}k} - L^*_{k\dot{k}})$$

$\{$since $L^*_{\dot{k}k} = (L^*_{k\dot{k}})'\}$ is induced leading to the matrix C in ($\mathcal{L}''$) and Γ in ($\mathcal{L}'''$). Γ thus summarises the asymmetries present in the capital-investment matrix $L^*_{k\dot{k}}$.

The forces induced by $\bar{A}$ in the equations of motion ($\mathcal{L}'''$) are symmetric. The forces induced by Γ are rotational. It seems that the rotational forces ultimately affect the stability of equilibrium as follows. When the eigenvalues of Γ are increased cyclical motion arises about the equilibrium point. This cyclical motion in turn slows down the rate at which z(t) converges to equilibrium. A sufficient increase in the magnitude of the eigenvalues of Γ slows down the rate at which z(t) converges to such an extent that v(t) and hence x(t) become unstable.

6. SAMUELSON'S CORRESPONDENCE PRINCIPLE

The basic idea that underlies Samuelson's Correspondence Principle [30] is that only stable equilibria can be observed. If we impose a similar condition on the parameter value then we are led to the following

DEFINITION. Let $(k^*,\alpha) \in \mathcal{E}$, then (k^*,α) will be called an observable equilibrium if $\alpha \in \mathcal{A}^h$ and k^* is locally asymptotically stable.

To simplify the analysis that follows I assume that there exists a set $\underline{\mathcal{A}} \subseteq \mathcal{A}$ such that $\alpha \in \underline{\mathcal{A}}$ and $(k^*,\alpha) \in \mathcal{E}$ implies $k^* \in \underline{\mathcal{K}}$. Consider a parameter value $\underline{\alpha} \in \mathring{\underline{\mathcal{A}}} \cap \mathcal{A}^r$. By the implicit function theorem there exist a neighborhood $\mathcal{N}_{\underline{\alpha}}$ of $\underline{\alpha}$ and m C^{r-1} functions, $m < \infty$

$$\psi^i(\alpha) : \mathcal{N}_{\underline{\alpha}} \longrightarrow \underline{\mathcal{K}}, \qquad i = 1,\ldots,m$$

such that

$$\mathcal{E}_{\underline{\alpha}} = \mathcal{E} \cap \underline{\mathcal{K}} \times \mathcal{N}_{\underline{\alpha}} = \{(\psi^i(\alpha), \alpha), \quad \alpha \in \mathcal{N}_{\underline{\alpha}}, \quad i = 1,\ldots,m\}$$

where $\mathcal{E}_{\underline{\alpha}} \neq \emptyset$ in view of Assumption 2 Let ψ^i be ordered so that

$$(\psi^1(\alpha), \ldots, \psi^\sigma(\alpha)) \quad \text{and} \quad (\psi^{\sigma+1}(\alpha), \ldots, \psi^m(\alpha))$$

denote the observable and the unobservable equilibria, respectively. If we let

$$g(k^*,\underline{\alpha}) = L_k(k^*,0; \underline{\beta}) + \underline{\delta} L_{\dot{k}}(k^*,0; \underline{\beta})$$

and let $\underline{k}^{*i} = \psi^i(\underline{\alpha})$, then if $\sigma \neq 0$, Samuelson's Correspondence Principle leads us to consider, by a second application of the implicit function theorem

$$(\mathfrak{S}) \qquad \psi^i_\alpha(\underline{\alpha}) = -\left[g_{k^*}(\underline{k}^{*i},\underline{\alpha})\right]^{-1} g_\alpha(\underline{k}^{*i},\underline{\alpha}), \qquad i = 1,\ldots,\sigma$$

where $-\left[g_{k^*}(\underline{k}^{*i},\underline{\alpha})\right]^{-1} = -\left[L_{kk}(\underline{k}^{*i},0; \underline{\beta}) + \underline{\delta} L_{\dot{k}k}(\underline{k}^{*i},0; \underline{\beta})\right]^{-1}, \qquad i = 1,\ldots,\sigma$

Remark. The number of positive (negative) eigenvalues of $-[g_{k^*}(\underline{k}^{*i},\underline{\alpha})]^{-1}$ is the same as the number of positive (negative) steady state profit rates

$$\pi_1(\underline{k}^{*i},\underline{\alpha}),\ \ldots,\ \pi_n(\underline{k}^{*i},\underline{\alpha})$$

of the local steady state profit function $\pi(x;k^*,\alpha)$ at $(\underline{k}^{*i},\underline{\alpha})$.[10]

Remark. If an observable equilibrium $(\underline{k}^{*i},\underline{\alpha})$ is known to be symmetric, then by the theorem of Magill-Scheinkman it is necessary for $-\pi(x;\underline{k}^{*i},\underline{\alpha})$ to attain a maximum at $\underline{k}^{*i}$: this forces the matrix $-[g_{k^*}(\underline{k}^{*i},\underline{\alpha})]^{-1}$ to be positive definite. If an observable equilibrium $(\underline{k}^{*i},\underline{\alpha})$ is asymmetric, as we must allow it to be in general, then Samuelson's Correspondence Principle ceases to yield such a precise qualitative result: the dynamic skew-symmetric forces no longer make it necessary for $-\pi(x;\underline{k}^{*i},\underline{\alpha})$ to attain a maximum at $\underline{k}^{*i}$, but rather only a maximin, a condition that carries with it a correspondingly weaker condition on the basic qualitative matrix $-[g_{k^*}(\underline{k}^{*i},\underline{\alpha})]^{-1}$.

In particular the results of sections 4 and 5 lead to the following

PROPOSITION 4. If $(\underline{k}^{*i},\underline{\alpha}) \in \mathcal{E}$, $i=1,\ldots,\sigma$ are observable equilibria then for each $i=1,\ldots,\sigma$

(i) there exist matrices Γ for which at least $\frac{n}{2}$ ($\frac{n-1}{2}$ if n is odd) of the eigenvalues of $-[g_{k^*}(\underline{k}^{*i},\underline{\alpha})]^{-1}$ are negative

(ii) if Γ satisfies (16), then at least $(\frac{n}{2})$ of the eigenvalues of $-[g_{k^*}(\underline{k}^{*i},\underline{\alpha})]^{-1}$ are positive.

I will consider but one application and refer the reader to Magill-Scheinkman [21] for further applications.

Example. In the dynamic theory of the firm in a stationary environment [16, 23], the firm is viewed as maximising the present value of the future stream of profit (in real terms)

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[10] See Proposition 2 in Magill-Scheinkman [21].

$$\int_0^\infty \left(f(k(\tau), \dot{k}(\tau)) - wk(\tau) - q\dot{k}(\tau)\right)e^{-\delta\tau}d\tau$$

where $$w = \left(\frac{W}{P}\right), \quad q = \left(\frac{Q}{P}\right), \quad k(0) = k_0 \in \mathring{R}^{n+}$$

The output price $P \in R^{1+}$, the rental and purchase prices of capital equipment $(W,Q) \in R^{n+} \times R^{n+}$ and the interest rate $\delta \in R^{1+}$ are taken as parameters determined on competitive markets independent of the actions of the firm, and the production function

$$f(k,\dot{k}) : R^{n+} \times R^{n} \longrightarrow R$$

is taken to satisfy Assumption 1. Let the parameter space be $\mathcal{A} = \{\omega \in R^{n+} \mid \omega = w + \delta q\}$ then

$$\mathcal{E} = \{(k^*,\omega) \in R^{n+} \times R^{n+} \mid f_k(k^*,0) + \delta f_{\dot{k}}(k^*,0) - \omega = 0\}$$

defines the firm's steady state demand correspondence for capital inputs. If we make Assumption 2 with $\underline{\mathcal{K}} \subset R^{n+}$, assume the existence of a subset $\underline{\mathcal{A}} \subseteq R^{n+}$ such that $\omega \in \underline{\mathcal{A}}$ and $(k^*,\omega) \in \mathcal{E}$ implies $k^* \in \underline{\mathcal{K}}$ and consider $\underline{\omega} \in \mathring{\underline{\mathcal{A}}} \cap \mathcal{A}^r$, then there exist a neighborhood $\mathcal{N}_{\underline{\omega}}$ and m C^{r-1} steady state input demand functions $\psi^i(\omega)$ such that

$$\mathcal{E}_{\underline{\omega}} = \{(\psi^i(\omega), \omega), \ \omega \in \mathcal{N}_{\underline{\omega}}, \ i = 1,\ldots,m\}$$

If $\sigma \neq 0$ then we may consider ($\mathcal{S}$) which becomes

$$\psi^i_\omega(\underline{\omega}) = \left[f_{kk}(\underline{k}^{*i},0) + \delta f_{\dot{k}k}(\underline{k}^{*i},0)\right]^{-1}, \quad i = 1,\ldots,\sigma$$

where $\underline{k}^{*i} = \psi^i(\underline{\omega})$ are observable equilibria.

Remark. If $(\underline{k}^{*i},\underline{\omega})$ is known a priori to be symmetric $\left(f_{k\dot{k}}(\underline{k}^{*i},0) = f_{\dot{k}k}(\underline{k}^{*i},0)\right)$, then the Jacobian matrix $\psi^i_\omega(\underline{\omega})$ is negative definite.

This is the well-known result of Mortensen [23] which extends the familiar result in the static case. However $(\underline{k}^{*i},\underline{\omega})$ is not in general symmetric, as

Mortensen recognised. Our stability analysis in conjunction with Samuelson's Correspondence Principle thus leads to the following conclusion. In the dynamic theory of the firm in a stationary environment the classical result of the static theory ceases to hold: the possible presence of asymmetric dynamic capital-investment interaction terms $f_{k\dot{k}}(\underline{k}^{*i},0)$ no longer makes it necessary for the Jacobian matrix $\psi^{i}_{\omega}(\underline{\omega})$ of the steady state input demand function $\psi^{i}(\omega)$ to be negative quasi-definite. Under quite general conditions inputs k^{*}_{j} can exist for which an increase in the rental price ω_{j} leads to an increase in the steady state demand.[11]

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[11]This was first observed by Mortensen [23] by means of a particular example. My object however has been to explain more generally the reason why the qualitative result in the symmetric case fails to carry over to the asymmetric case.

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