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NASH-COURNOT EQUILIBRIUM WITH ENTRY

by

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Despite the fact that the assumptions underlying perfect competition never actually hold, the use of the competitive model, as an idealization, is justified if the predictions of the model approximate the outcomes of situations it is used to represent. In partial equilibrium analysis, this justification is embodied in the "Folk Theorem" which states that if firms are small relative to the market, then the market outcome is approximately competitive. This paper provides a precise statement and proof of the "Folk Theorem" for competitive markets with a single homogeneous good, and free entry and exit. It is shown that if firms are small relative to the market then there is a Nash-Cournot equilibrium with entry; furthermore, the Nash-Cournot equilibrium is approximately competitive. More specifically, if we consider an appropriate sequence of markets in which firms become arbitrarily small relative to the market, then there is a Nash-Cournot equilibrium with entry for all markets in the tail of the sequence, and aggregate equilibrium output converges to perfectly competitive output. If firms have strictly U-shaped average cost curves then individual firm behavior converges to competitive behavior. The treatment of free entry distinguishes this paper from other papers dealing with the "Folk Theorem," where either the number of firms is exogenous, ruling out free entry, or free entry is treated as being equivalent to a zero profit condition, ignoring the integer problem that arises when the number of firms is finite but unspecified.

The paper is organized as follows: section 1 contains the perfectly competitive model and its assumptions, section 2 contains the assumptions and definitions for the imperfectly competitive model with small firms, section 3 contains an example contrasting the usual treatment of the "Folk Theorem" and the present approach, section 4 contains the proofs of the main results, and section 5 contains remarks on the results and indicates how some of the assumptions that are used can be weakened.

Section 1

The classical long run perfectly competitive model of a market for a single homogeneous good, where factor prices are constant, can be found in most textbooks which survey microeconomics at any level. All firms have identical technology, and in the long run firms can choose plant size, and enter or leave the market. The long run perfectly competitive market result is aggregate output Y^* and price $F(Y^*)$ (F is inverse demand) such that $F(Y^*) = \text{minimum long run average cost} = \text{LRAC}(y^*)$, with each producing firm operating the optimally efficient size plant at output y^* , and earning zero profit. With constant input factor prices, long run supply is a horizontal line at price = $\text{LRAC}(y^*)$.

There are two reasons why firms must be infinitesimal in the perfectly competitive model: first, if firms produce significant output then they have an effect on price; second, if y^* is significant, then long run supply at price = $\text{LRAC}(y^*)$ is the discrete set of points which are integer multiples of y^* , not a horizontal

line, so if Y^* is not an integer multiple of y^* , long run perfectly competitive equilibrium does not exist. Both of these problems vanish when firms are infinitesimal.¹

The assumptions used in the long run perfectly competitive model are:

- (1) long run average cost is strictly U-shaped with minimum attained at $y^* \neq 0$ (nonzero in the scale of the firm);
- (2) there exists $Y^* \in (0, \infty)$ such that inverse demand $F(Y) \geq \text{LRAC}(y^*)$ as $Y \leq Y^*$;
- (3) (a) firms are identical and infinitesimal with respect to Y^* ,
 (b) firms choose quantity (and as a result of being infinitesimal, need not consider their own effect on price, since it is zero), maximizing profit, viewing price (and hence the aggregate output of other firms) as fixed,
 (c) firms are free to enter and leave the market.

The long run average cost and inverse demand functions are also commonly assumed to satisfy differentiability conditions.

The long run perfectly competitive equilibrium is characterized by:

- (4) each producing firm's output is a profit maximizing response to price (and hence to the aggregate output of all other firms);
- (5) each producing firm has nonnegative profit:

- (6) no potential entrant can earn strictly positive profit by entry, assuming price (and hence the aggregate output of all other firms) is fixed.

When firms are small not infinitesimal (y^* is small but significant) they have an effect on price, and we assume they recognize this effect, but no other substantial changes are made in the assumptions or the equilibrium properties. In this context, the "Folk Theorem" says that if y^* is significant but small relative to Y^* , then market equilibrium output exists and is approximately Y^* .

Section 2²

Let AC and C be long run average cost and long run cost functions respectively, and let F be the inverse demand function. The first two assumptions for imperfectly competitive markets correspond to assumptions (1) and (2) for the perfectly competitive model.

- (C1) There exists $y^* \in (0, \infty)$ such that $AC(y) \geq AC(y^*) \in (0, \infty)$ for all $y \in (0, \infty)$, and $AC(y^*) < \liminf_{y \rightarrow 0^+} AC(y)$.

- (F1) There exists $Y^* \in (0, \infty)$ such that $F(Y) \underset{Y \leq Y^*}{\geq} AC(y^*)$ as $Y \underset{Y \geq Y^*}{\leq} Y^*$.

Assumption (C1) does not require average cost to be strictly U-shaped, but does guarantee the existence of $\epsilon, \delta > 0$ such that $AC(y^*) + \epsilon < AC(y)$ for all $y \in (0, \delta)$, so very small outputs have average cost bounded away from minimum average cost. Uniqueness of y^* is not required, so a continuum of efficient outputs,

as in the case of a flat bottomed U-shaped average cost curve, is allowed.

The measure of firm size is a natural one that is based on technology: the smallest output at which minimum average cost is attained, minimum efficient scale. In our partial equilibrium framework, each firm is completely described by its average cost function, so for convenience we identify the firm with its average cost function, and speak of the firm, AC.

Definition 1: Let AC be an average cost function satisfying (C1).

Then the size of a firm, AC, is

$$\text{MES}(\text{AC}) := \inf\{y^* \in (0, \infty) \mid \text{AC}(y^*) \leq \text{AC}(y) \quad \forall y \in (0, \infty)\} .$$

By (C1), MES is well defined and strictly positive. If AC is continuous at $\text{MES}(\text{AC})$ then $\text{MES}(\text{AC})$ is the smallest output at which minimum average cost is attained.

In order to generate a family of average cost functions (indexed by α), a transformation which changes the scale of measurement of output is used.

Definition 2: Let AC be an average cost function satisfying (C1).

For each $\alpha \in (0, \infty)$, the α -size firm corresponding to AC

is the firm AC_α defined by $\text{AC}_\alpha(y) := \text{AC}(y/\alpha)$.

This transformation changes the scale of measurement of output by a factor of α ($\text{AC}(y) = \text{AC}_\alpha(\alpha y)$). With this transformation, we can assume that any basic average cost function under consideration has minimum efficient scale equal to one, and use α

to generate the other average cost functions with nonunity minimum efficient scales. This assumption that minimum efficient scale equals one for basic cost functions entails no loss of generality.

$$(C2) \quad \text{MES}(AC) = 1 .$$

If AC satisfies (C1) and (C2) then an α -size firm relative to AC has $\text{MES}(AC_\alpha) = \alpha$, and the use of size in Definitions 1 and 2 is consistent. Denote the cost function corresponding to AC_α by C_α , so $C_\alpha(y) = \alpha C(y/\alpha)$.

The next two assumptions correspond to the continuity and differentiability assumptions commonly made in the perfectly competitive model.³

- (C3) a) $C(0) = 0$ and C is a continuous, strictly positive, monotonically increasing function of y on $I \setminus \{0\}$, where I is the interval on which C is finite valued. If C is bounded on I then I is closed.
- b) C is twice continuously differentiable on $I \setminus \{0\}$ with $C' > 0$, and there exist $s \in (0,1)$ and $t \in (1,\infty)$ such that $C'' \geq 0$ for all $y \in [s,t] \cap I$.

This is a standard assumption, weakened to allow capacity constraints ($I \neq [0,\infty)$), but requiring that either I is closed on the right (the constraint can be attained) or cost is arbitrarily large for outputs near the constraint. The existence of s and t corresponds to the usual assumption about the shape of the marginal cost curve near the minimum of the average cost curve. Notice that C is not required to be continuous at 0, so $\lim_{y \rightarrow 0^+} C(y) > 0$ is possible.

(F2) For all $Y \in (0, \infty)$, $F(Y)$ is twice continuously differentiable with $F'(Y) < 0$ whenever $F(Y) > 0$.

This is also a standard assumption and allows unbounded price near zero output as well as strictly positive price for all finite outputs. The market under consideration is completely described by α , C , and F , so we denote the market by (α, C, F) . Note that in any (α, C, F) market all firms, including potential entrants, have the same cost function C_α .

The equilibrium concept used here has properties similar to those listed for the long run perfectly competitive equilibrium:

- (i) The output of the producing firms yields a Nash equilibrium with strategic variable quantity,
- (ii) all firms make nonnegative profit, and
- (iii) there is no profit incentive for additional firms to enter the market.

Definition 3: Given a cost function C an inverse demand function F , and an $\alpha \in (0, \infty)$, an (α, C, F) market equilibrium with free entry

is an integer n and a set $\{y_1, \dots, y_n\}$ of positive outputs

such that, for $Y_i = \sum_{j=1, j \neq i}^n y_j$ and $Y_T = \sum_{i=1}^n y_i$,

(a) $\forall i \in \{1, \dots, n\}$, $F(Y_i + y_i) y_i - C_\alpha(y_i) \geq F(Y_i + Y) Y - C_\alpha(Y)$
 $\forall Y \in [0, \infty]$; and

(b) $F(Y_T + Y) Y - C_\alpha(Y) \leq 0 \quad \forall Y \in [0, \infty]$.

The set of all (α, C, F) market equilibria with free entry is denoted by $E(\alpha, C, F)$.

Condition (a) is the Nash condition for producing firms. When firms are infinitesimal, y is infinitesimal with respect to (the integral) $\sum_{i=1}^n y_i$, so $F(\sum_{i=1}^n y_i + y)$ is a fixed price and (a) becomes the profit maximizing condition for a competitive firm. If $C(0) = 0$, condition (a) implies that all firms make nonnegative profit since $C'_\alpha(0) = 0$. Condition (b), the free entry condition, requires that no potential entrant, acting alone, can achieve positive profit by entry. When firms are infinitesimal this reduces to the competitive entry condition. Notice that the n used in the definition is endogenous, not prespecified, and given C , F , and α , $E(\alpha, C, F)$ may contain several equilibria, with different values of n .

The main results of the paper are Theorems 1 and 2.

Theorem 1: Given a cost function satisfying (C1) and (C2), with $C(0) = 0$, an inverse demand function F satisfying (F1), and an $\alpha \in (0, \infty)$, if n , $\{y_1, \dots, y_n\}$ is an element of $E(\alpha, C, F)$ then

$$Y_T = \sum_{i=1}^n y_i \in [Y^* - \alpha, Y^*].$$

Hence, if α is small relative to Y^* , then any equilibrium in $E(\alpha, C, F)$ (if one exists) yields a market output which is approximately the perfectly competitive output.

Theorem 2: Given a cost function C satisfying (C1), (C2), and (C3), and an inverse demand function F satisfying (F1) and (F2), there exists $\alpha^* > 0$ such that for all $\alpha \in (0, \alpha^*]$, $E(\alpha, C, F) \neq \emptyset$.

Theorems 1 and 2 provide a precise statement and proof of the "Folk Theorem": if firms are small relative to the market then there is a market equilibrium, and the market output is approximately competitive.

Section 3

In this section, for a simple example with U-shaped average cost, we contrast the usual treatment of the "Folk Theorem" with the approach adopted in Section 2. The usual treatment fixes cost and demand functions, fixes the number of firms, n , and finds an n -firm Cournot equilibrium. Then n is exogenously increased, so each firm's output (the measure of size) becomes arbitrarily small. In this context, the "Folk Theorem" is valid if and only if the aggregate output of the n -firm Cournot equilibrium converges to perfectly competitive output as n becomes arbitrarily large.⁴ When average cost is U-shaped, if each n -firm equilibrium is viable (all n firms producing positive output earn nonnegative profit) then the "Folk Theorem" must invariably fail, since price converges to $AC(0+)$ - the limit of average cost as output converges to zero - in order to maintain nonnegative profit for the firms whose output is becoming arbitrarily small.⁵

In contrast to the n -firm Cournot technique, the approach of Section 2 measures firm size by technology and treats the number of firms as an endogenous variable. In this context, the "Folk Theorem" is valid under very general assumptions.

Example A shows the failure of the "Folk Theorem" using the n-firm Cournot technique. Example B, using the same basic cost and demand functions, demonstrates the validity of the "Folk Theorem" using the approach of Section 2. For both examples, the nondifferentiability of the cost function serves to simplify the reaction correspondences, and does not affect the nature of the results.

Example A: The cost function is

$$c(y) = \begin{cases} \frac{3}{2}y - \frac{1}{2}y^2 & y \in [0,1] \\ \frac{1}{2}y + \frac{1}{2}y^2 & y \in (1,\infty) \end{cases}$$

and the inverse demand function is $F(Y) = 3 - 2Y$.

Let $\{y_1, \dots, y_n\}$ be an n-firm Cournot equilibrium, and let $Y_{i(\cdot)} := \sum_{\substack{j=1 \\ j \neq i}}^n y_j$ for each i . The reaction function for each firm is

$$y_i(Y_{i(\cdot)}) = \begin{cases} \frac{2}{3}(\frac{3}{4} - Y_{i(\cdot)}) & Y_{i(\cdot)} \in [0, \frac{3}{4}] \\ 0 & Y_{i(\cdot)} \in (\frac{3}{4}, \infty) \end{cases}$$

so summing over i , recognizing that $\sum_{i=1}^n Y_{i(\cdot)} = (n-1) \sum_{i=1}^n y_i$,

and rearranging, we get

$$\sum_{i=1}^n y_i = \frac{3}{4} \left(\frac{n}{n+1/2} \right) < \frac{3}{4}.$$

Then $Y_{i(\cdot)} < \frac{3}{4}$ so $y_i > 0$ for all i , and $y_i = \frac{3}{4} \left(\frac{1}{n+1/2} \right)$ for all $i = 1, \dots, n$ is an equilibrium, and it is the only n-firm equilibrium (to see that $y_i = y_j$ for all i, j , fix the output of

any $n-2$ firms and notice that the only equilibrium for the two remaining firms, facing the residual demand as a Cournot duopoly, is $y_i = y_j$). As n is exogenously increased, the output of each individual firm converges to 0, while aggregate output converges to $\frac{3}{4}$, not to competitive output 1. Price converges to $\frac{3}{2} = AC(0+) = C'(0+)$. Notice that for any finite n , price is greater than $\frac{3}{2}$, so if entry is allowed, it will take place. Also, as n increases the behavior of the firms does not converge to price taking behavior (the second order condition for a price taking profit maximizer is violated at y_i , and the price taking response to $Y_{-i} = \frac{3}{4}(\frac{n-1}{n+1/2})$ is always greater than 1, while y_i converges to 0).

Example B: The basic inverse demand and cost functions are the same as in Example A. For $\alpha \in (0, \infty)$,

$$C_\alpha(y) = \begin{cases} \frac{3}{2}y - \frac{1}{2\alpha}y^2 & y \in [0, \alpha] \\ \frac{1}{2}y + \frac{1}{2\alpha}y^2 & y \in (\alpha, \infty) \end{cases} .$$

We show that for $\alpha \in (0, 1/5]$, $E(\alpha, C, F) \neq \emptyset$.⁶ Fix $\alpha \in (0, 1/5]$.

The reaction correspondence of a firm with cost function C_α , reacting to output Y by other firms, is

$$y(Y|\alpha) = \begin{cases} (\frac{\alpha}{4\alpha+1})(\frac{5}{2}-2Y) & Y \in [0, \frac{3}{4}-2\alpha) \\ \alpha & Y \in [\frac{3}{4}-2\alpha, 1-\alpha) \\ \{0, \alpha\} & Y = 1 - \alpha, \\ 0 & Y = (1-\alpha, \infty) \end{cases} .$$

An instructive method for finding a symmetric equilibrium is to consider the cumulative reaction graph, CRG, which plots the points $(Y, Y+y(Y|\alpha))$ (see McManus [4]). If there is an n -firm Cournot symmetric equilibrium (without free entry), there is a point $(Y_0, Y_0+y(Y_0|\alpha))$ which lies on both the CRG and the line $L_n: Y+y = \left(\frac{n}{n-1}\right)Y$. In order for this Cournot equilibrium to satisfy the additional free entry condition, $0 \in y(Y_0+y(Y_0|\alpha)|\alpha)$ is required, so an optimal response of any potential entrant is to maintain zero output. The reaction correspondences for $\alpha \in (0, 1/5]$ have discontinuities at $Y = 1 - \alpha$, where they are not convex valued, so n -firm Cournot symmetric equilibrium exists if and only if $n \leq n(\alpha)$ where $n(\alpha)$ is the greatest integer less than or equal to $1/\alpha$ (see figure 1).

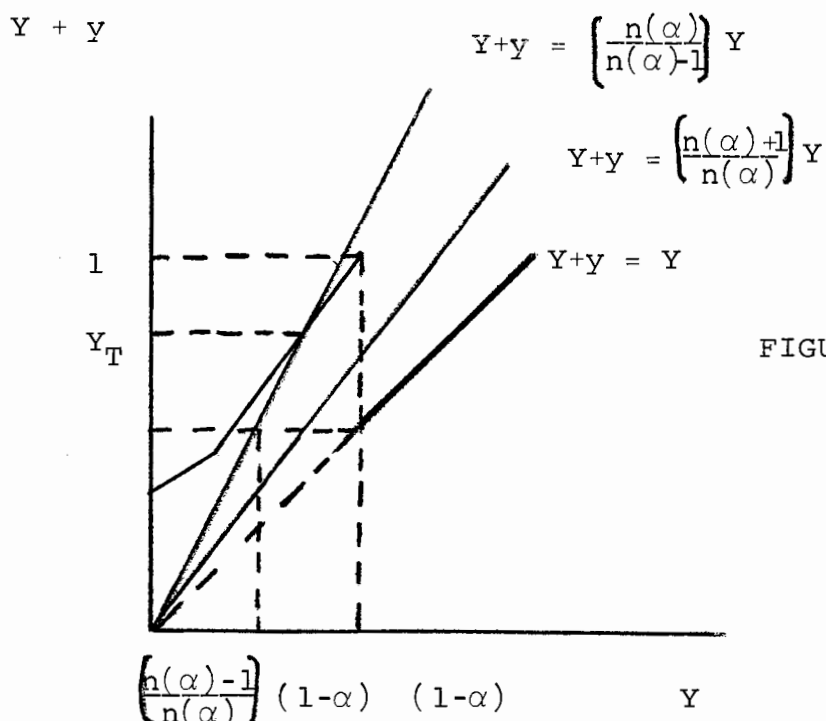


FIGURE 1

All that remains is to show that aggregate output Y_T in the $n(\alpha)$ firm Cournot equilibrium is at least $1-\alpha$, so $0 \in y(Y_T|\alpha)$, and no entry takes place when free entry is allowed. For $\alpha \in (0, 1/5]$, $n(\alpha)$, $\{y_1, \dots, y_{n(\alpha)}\}$, where $y_i = \alpha$ for all i , is an element of $E(\alpha, C, F)$ since

- (a) $Y_{i(\alpha)} = (n(\alpha)-1)\alpha \in (1-2\alpha, 1-\alpha) \subset [\frac{3}{4} - 2\alpha, 1-\alpha]$, so $y_i = \alpha$ is an optimal response to $Y_{i(\alpha)}$, and
- (b) no entry occurs since $Y_T = n(\alpha)\alpha \in (1-\alpha, 1]$ so $y(Y_T|\alpha) = 0$.

Notice that $Y_T \in (1-\alpha, 1]$, and Y_T converges to the perfectly competitive output as α converges to zero. Also, profit is strictly positive unless $n(\alpha) = \frac{1}{\alpha}$.

The optimal response to $Y_{i(\alpha)} = (n(\alpha)-1)\alpha$ for a firm that assumes price is fixed is approximately α (in fact, for $\alpha \in (0, \frac{1}{7})$, the price taking response in any element of $E(\alpha, C, F)$ is α , but this exact equality of price taking and non-price-taking responses is a consequence of the nondifferentiability of the cost function at α). Multiple equilibria are possible. For $\alpha = \frac{1}{n}$, where n is an integer greater than or equal to five, n , $\{y_1, \dots, y_n\}$ where $y_i = \frac{1}{n}$ for all i , and $n-1$, $\{y_1, \dots, y_{n-1}\}$ where $y_i = \frac{1}{n}$ for all i , are both elements of $E(\frac{1}{n}, C, F)$.

Section 4

The results of Example B are generalized in Theorems 1 and 2, the first of which shows that if α is small relative to Y^* , then every element of $E(\alpha, C, F)$ yields a market output which is approximately the perfectly competitive output.

Theorem 1: Given a cost function C satisfying (C1) and (C2), with $C(0) = 0$, an inverse demand function F satisfying (F1) and an $\alpha \in (0, \infty)$, if n , $\{y_1, \dots, y_n\}$ is an element of $E(\alpha, C, F)$ then $Y_T = \sum_{i=1}^n y_i \in [Y^* - \alpha, Y^*]$.⁷

Proof: If $Y_T > Y^*$ then $y_i > 0$ for some i , and market price $F(Y_T)$ is less than minimum average cost by (F1), so the firm producing $y_i > 0$ has strictly negative profit, contrary to the Nash condition and $C_\alpha(0) = 0$.

If $Y_T < Y^* - \alpha$, there exists $y \in [\alpha, Y^* - Y_T)$ such that $AC_\alpha(y)$ equals minimum average cost, by definition of MES and the fact that $MES(AC_\alpha)$ equals α (if AC_α is continuous at α there is no need to consider y values other than α). By producing y , an entrant changes price to $F(Y_T + y) > AC(y)$ since $Y_T + y < Y^*$, and earns profit $(F(Y_T + y) - AC_\alpha(y))y > 0$, contrary to the free entry condition. Q.E.D.

In order for this result to be meaningful, $E(\alpha, C, F)$ must be nonempty for α small relative to Y^* . There are two ways in which $E(\alpha, C, F)$ can be empty. First, as in Example B for $\alpha \in (\frac{1}{2}, \infty)$, no finite number of firms may be enough to remove the incentive for additional firms to enter, while n -firm Cournot equilibria without free entry exist for all n . Second, n -firm Cournot equilibrium may not exist for all but a finite number of n values, with the free entry condition failing at those Cournot equilibria that do exist.

The second way in which $E(\alpha, C, F)$ may be empty illustrates the integer problem that arises when free entry is allowed with non-infinitesimal firms. Most of the time, free entry has been treated as equivalent to a zero profit condition, and when firms are noninfinitesimal, the number of firms is treated as a continuous variable in order to get zero profit, after which some statement is made about rounding off the number of firms to an integer. Using that approach, equilibrium with free entry may fail to exist when the number of firms is rounded to an integer.⁸

Theorem 2 shows that if α is small enough relative to Y^* , then both types of nonexistence are overcome, and $E(\alpha, C, F)$ is not empty.

Theorem 2: Given a cost function C satisfying (C1), (C2), and (C3), and an inverse demand function F satisfying (F1) and (F2), there exists $\alpha^* > 0$ such that for all $\alpha \in (0, \alpha^*]$, $E(\alpha, C, F) \neq \emptyset$.

In order to prove the Theorem, we show that for α sufficiently small, the reaction correspondence is similar to those of Example B, at least in the interval that corresponds to $[1-2\alpha, 1-\alpha]$. In particular, three properties are vital for the method of proof used:

- (i) at the point corresponding to $1-\alpha$, $Y(\alpha)$, the reaction correspondence has two values, 0 and $y(\alpha) > 0$.
- (ii) the reaction correspondence is nonincreasing on $[Y^*-2\alpha, Y(\alpha)]$ (the exact meaning of this is explained below); and
- (iii) the reaction correspondence is continuous on $[Y^*-2\alpha, Y(\alpha))$.

These three properties are proved as follows.

Step 1: As α decreases, the average cost function AC_α becomes more sharply U-shaped, so for α sufficiently small, if we overlay a graph of AC_α on a graph of F , and shift the AC_α axes horizontally to the right until AC_α is just tangent to F , we obtain $Y(\alpha)$, and $y(\alpha) > 0$, where 0 and $y(\alpha)$ are both optimal responses to $Y(\alpha)$, and (i) holds (see figure 2).

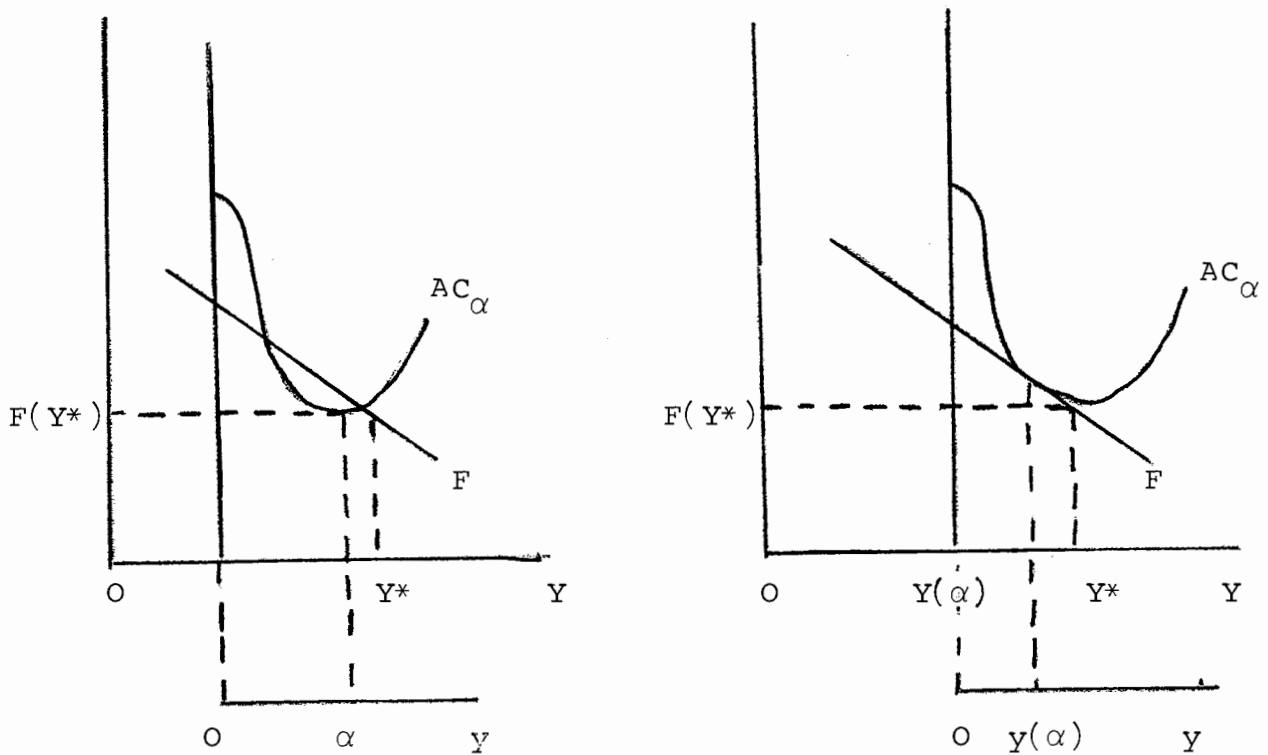


FIGURE 2

Step 2: If marginal revenue, $F'(Y+y)y + F(Y+y)$, is decreasing in both Y and y , for all Y and y , then regardless of the cost function, the largest optimal response to Y_2 is less than or equal to the smallest optimal response to Y_1 whenever $Y_1 < Y_2$ (see figure 3); if it does not increase profit to expand output from y_1 to y_2 in response to Y_1 , then it decreases profit to make that same expansion in response to Y_2 (the change in costs is the same, but revenue does not increase as much).

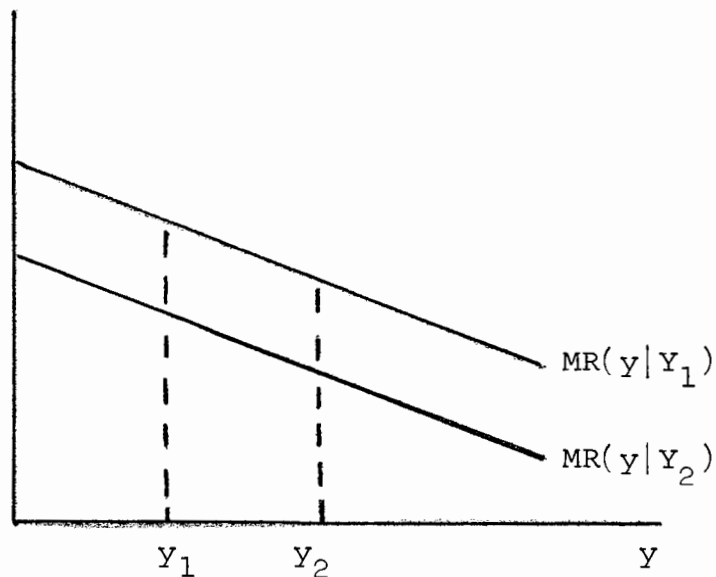


FIGURE 3

We show that there exists a $K > 0$ such that $F'(Y+y)y + F(Y+y)$ is decreasing in both Y and y for $Y \in [Y^*-K, Y^*]$, $y \in [0, Y^*-Y]$, and show that for α sufficiently small these intervals contain all pertinent values of Y and y . Thus property (ii) holds.

Step 3: We show that for all α sufficiently small, the relevant values of Y and y are such that marginal revenue is decreasing and marginal cost is nondecreasing in y (i.e., the optimal values of y lie in $[\alpha, t\alpha]$). In general, at the optimal response marginal revenue equals marginal cost, so there is a unique optimal response, and the reaction correspondence is a continuous function for Y in the desired interval. Hence property (iii) holds. Once properties (i) - (iii) hold, the proof follows as in example B: given a continuous function defined on an interval whose graph lies above a certain line at one endpoint, and below it at the other, the graph must intersect the line.

Proof of Theorem 2: Define the following notation.

- a) The profit function $P(y|Y, \alpha) := F(Y+y)y - C_\alpha(y)$. If $\frac{Y}{\alpha} \in \mathbb{I} \setminus \{0\}$, $\alpha > 0$ and $Y > 0$ then the profit function is continuous at (y, Y, α) (and $P(0|Y, \alpha) \geq \lim_{y \rightarrow 0^+} P(y|Y, \alpha)$ with equality if and only if C_α is continuous at 0).
- b) The reaction correspondence
 $y^*(Y|\alpha) := \{y^* \in [0, \infty) | P(y^*|Y, \alpha) \geq P(y|Y, \alpha) \forall y \in [0, \infty)\}$.
 If $Y \in (0, Y^*]$ then $y^*(Y|\alpha)$ is a nonempty compact subset of $[0, Y^* - Y]$.
- c) The smallest reaction $y(Y|\alpha) := \min y^*(Y|\alpha)$.
- d) The critical aggregate output
 $Y(\alpha) := \sup\{Y \in [0, \infty) | y(Y|\alpha) > 0\} = \sup\{Y \in [0, \infty) | P(y(Y|\alpha)|Y, \alpha) > 0\}$.
 For $\alpha < Y^*$, $Y(\alpha)$ is well defined and $Y(\alpha) \in [Y^* - \alpha, Y^*]$
 (similar to the proof of Theorem 1). If $Y_T = \sum_{i=1}^n y_i > Y(\alpha)$
 then $n, \{y_1, \dots, y_n\}$ satisfies the free entry condition.

e) The critical single firm output $y(\alpha) := \max y^*(Y(\alpha)|\alpha)$.

For $\alpha < Y^*$, $y(\alpha)$ is well defined and

$y^*(Y(\alpha)|\alpha) \subset [0, Y^* - Y(\alpha)] \subset [0, \alpha]$, so $y(\alpha) \leq \alpha$.

f) The number of firms $n(\alpha) := \lfloor \frac{Y(\alpha) + y(\alpha)}{y(\alpha)} \rfloor$ (the greatest integer less than or equal to $\frac{Y(\alpha) + y(\alpha)}{y(\alpha)}$). For $\alpha < Y^*$ this is a well defined, finite, positive integer if $y(\alpha) > 0$. For convenience define $N(\alpha) := \frac{n(\alpha) - 1}{n(\alpha)}$. $Y(\alpha) + y(\alpha) \in ((\frac{n(\alpha) + 1}{n(\alpha)}Y(\alpha), (\frac{n(\alpha)}{n(\alpha) - 1})Y(\alpha))$ and $N(\alpha)Y(\alpha) > Y(\alpha) - y(\alpha) \geq Y(\alpha) - \alpha \geq Y^* - 2\alpha$.

For any fixed $\alpha < Y^*$, assume for the moment that $y(\alpha) > 0$, and

consider the cumulative reaction graph shown in figure 4. By

f), $Y(\alpha) + y(\alpha) \in ((\frac{n(\alpha) + 1}{n(\alpha)}Y(\alpha), (\frac{n(\alpha)}{n(\alpha) - 1})Y(\alpha))$ so $(Y(\alpha), Y(\alpha) + y(\alpha))$

lies in the cone between $L_1(\alpha)$ and $L_2(\alpha)$ (possibly on $L_1(\alpha)$). If

$(Y_0, Y_0 + y_0) \in L_1(\alpha) \cap \{(Y, Y + y^*(Y|\alpha)) | Y \in (N(\alpha)Y(\alpha), Y(\alpha))\}$

consider $n(\alpha), \{y_1, \dots, y_{n(\alpha)}\}$ where $y_i = y_0$ for all i .

Then for all i , $\sum_{\substack{j=1 \\ j \neq i}}^{n(\alpha)} y_j = (n(\alpha) - 1)y_0 = Y_0$. Hence

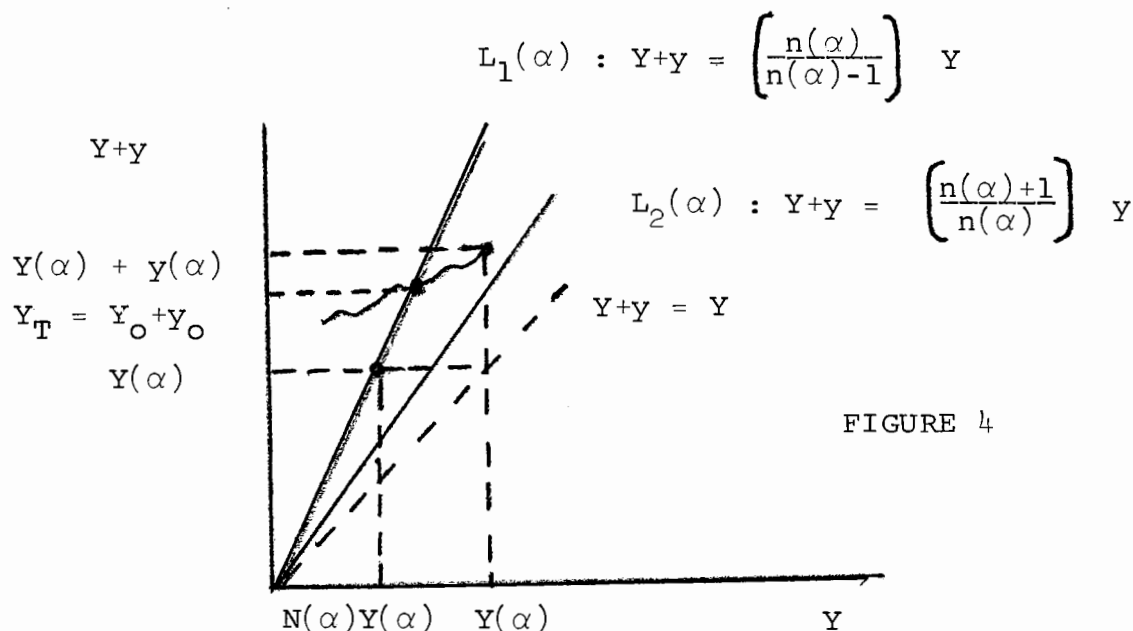
(g) each y_i is an optimal response to the other firms' outputs

since $y_i = y_0 \in y^*(Y_0|\alpha)$, and

(h) there is no incentive for additional firms to enter since

$$Y_T = \sum_{i=1}^{n(\alpha)} y_i = n(\alpha)y_0 = Y_0/N(\alpha) > N(\alpha)(Y(\alpha)/N(\alpha)) = Y(\alpha).$$

Thus $n(\alpha), \{y_1, \dots, y_{n(\alpha)}\}$ is an element of $E(\alpha, C, F)$.



Therefore in order to complete the proof it is sufficient to prove that there exists $\alpha^* > 0$ such that for all $\alpha \in (0, \alpha^*]$, $y(\alpha) > 0$ and $L_1(\alpha) \cap \{(Y, Y+y^*(Y|\alpha)) | Y \in (N(\alpha)Y(\alpha), Y(\alpha))\} \neq \emptyset$. This is done in three steps.

Step 1 shows $y(\alpha) > 0$ for all α sufficiently small.

Step 2 shows $y(N(\alpha)Y(\alpha)|\alpha) \geq y(\alpha)$, and $y(Y|\alpha) \rightarrow y(\alpha)$ as $Y \rightarrow Y(\alpha)^-$ for all α sufficiently small.

Step 3 shows $y(Y|\alpha)$ is continuous for all $Y \in [N(\alpha)Y(\alpha), Y(\alpha))$, for all α sufficiently small.

Hence for α small enough, $Y+y(Y|\alpha)$ is a continuous function which lies above $L_1(\alpha)$ at $Y = N(\alpha)Y(\alpha)$, and converges to a point below $L_1(\alpha)$ as $Y \rightarrow Y(\alpha)^-$, so the required intersection must be nonempty (if $(Y(\alpha), Y(\alpha)+y(\alpha)) \in L_1(\alpha)$ then there is no need to go

through steps 2 and 3, so in step 3 we may assume $(Y(\alpha), Y(\alpha) + y(\alpha))$ lies below $L_1(\alpha)$ and not on it).

Step 1: By (C1) there exist $\epsilon, \delta > 0$ such that $AC(y) > AC(1) + \epsilon$ for all $y \in (0, \delta]$. Pick $Y_1 \in (0, Y^*)$ such that $F(Y_1) \in (AC(1), AC(1) + \epsilon)$ and set $\alpha_1 := (Y^* - Y_1)/2$.

- a) For all $\alpha \in (0, \alpha_1]$, $Y(\alpha) > 0$ and $P(y(\alpha) | Y(\alpha), \alpha) = 0$ by the continuity properties of P .
- b) For all $\alpha \in (0, \alpha_1]$, for all $Y \in [Y^* - 2\alpha, Y(\alpha))$, $P(y(Y|\alpha) | Y, \alpha) \geq P(y(\bar{Y}|\alpha) | Y, \alpha) > P(y(\bar{Y}|\alpha) | \bar{Y}, \alpha) \geq 0$, where $\bar{Y} \in (Y, Y(\alpha)]$ is such that $y(\bar{Y}|\alpha) > 0$ (\bar{Y} exists by the definition of $Y(\alpha)$, and the strict inequality follows from $F' < 0$). Thus $y(Y|\alpha) > 0$ and, by the choice of α_1 , $y(Y|\alpha) > \delta\alpha$, for all $Y \in [Y^* - 2\alpha, Y(\alpha))$, for all $\alpha \in (0, \alpha_1]$.
- c) As Y converges to $Y(\alpha)-$, there is a limit point of $y^*(Y|\alpha)$ which lies in $[\delta\alpha, Y^*]$. This limit point is an optimal response to $Y(\alpha)$ by the continuity properties of the profit function. Thus for all $\alpha \in (0, \alpha_1]$, $y(\alpha) \geq \delta\alpha > 0$. This completes Step 1.

Step 2.

Let $K := \min_{Y \in [Y^* - 2\alpha_1, Y^*]} \left\{ \frac{-F'(Y)}{\max\{0, F''(Y)\}} \right\}$. Then $K > 0$

($K = +\infty$ is possible) since F'' is bounded and F' is bounded away from zero and negative on the compact set $[Y^* - 2\alpha_1, Y^*]$.

Let $\alpha_2 := \min\{\frac{K}{3}, \alpha_1\} > 0$. Then for $\alpha \in (0, \alpha_2]$, $F''(Y+y)y + F'(Y+y) < C$ for all $Y \in [Y^* - 2\alpha, Y(\alpha)]$, $y \in [0, Y^* - Y]$ by the choice of K .

Thus marginal revenue, $F'(Y+y)y + F(Y+y)$, is decreasing as a function of Y and as a function of y for all relevant values of Y and y (recall $[N(\alpha)Y(\alpha), Y(\alpha)] \subset [Y^*-2\alpha, Y(\alpha)]$, and $y^*(Y|\alpha) \subset [0, Y^*-Y]$) for all $\alpha \in (0, \alpha_2]$.

For any $\alpha \in (0, \alpha_2]$, and any $Y_1, Y_2 \in [Y^*-2\alpha, Y(\alpha)]$ with $Y_1 < Y_2$, the marginal revenue curve $F'(Y_1+y)y + F(Y_1+y)$ lies strictly above the marginal revenue curve $F'(Y_2+y)y + F(Y_2+y)$. If $y_2 \in y^*(Y_2|\alpha)$, every optimal response to Y_1 must be at least as large as y_2 otherwise, if $y_1 < y_2$ is an optimal response, $y_1 \in y^*(Y_1|\alpha)$, then

$$\int_{Y_1}^{Y_2} MR(y|Y_2, \alpha) dy < \int_{Y_1}^{Y_2} MR(y|Y_1, \alpha) dy \leq C_\alpha(Y_2) - C_\alpha(Y_1),$$

and $P(y_1|Y_2, \alpha) > P(y_2|Y_2, \alpha)$ contrary to $y_2 \in y^*(Y_2|\alpha)$. Thus for all $\alpha \in (0, \alpha_2]$, for all $Y_1, Y_2 \in [Y^*-2\alpha, Y(\alpha)]$ with $Y_1 < Y_2$, $y^*(Y_2|\alpha) \subset [0, y(Y_1|\alpha)]$, and in particular $y(N(\alpha)Y(\alpha)|\alpha) \geq y(\alpha)$.

It is now easy to see that $y(Y|\alpha)$ converges to $y(\alpha)$ as Y converges to $Y(\alpha)-$. As in c) of step 1, as Y converges to $Y(\alpha)-$, $y(Y|\alpha)$ has a limit point which is an optimal response to $Y(\alpha)$, α . By the previous paragraph, this limit point must be greater than or equal to $y(\alpha)$. Thus, by the definition of $y(\alpha)$ as the largest optimal response to $Y(\alpha)$, α , there can be only one limit point, and it must be $y(\alpha)$. This completes Step 2.

Step 3. For $\alpha \in (0, \alpha_2]$, the reaction correspondence $y^*(Y|\alpha)$ is upper hemi continuous in Y on $[N(\alpha)Y(\alpha), Y(\alpha)]$ (using the previously derived properties of P , $Y(\alpha)$ and $y^*(Y|\alpha)$, the problem can be restricted so that the Maximum Theorem-Berge [1] p. 116 - can be applied). Therefore if $y^*(Y|\alpha)$ is single valued on $[N(\alpha)Y(\alpha), Y(\alpha)]$ then it is continuous there, and step 3 will be complete.

By assumption (C3), there exist $s \in (0, 1), t \in (1, \infty)$ such that $C''(y) \geq 0$ for all $y \in [s, t] \cap I$. To eliminate the intersection with I , if $t \notin I$ redefine t as $(1 + \sup I)/2$, so $t \in I$, and $t \geq 1$, with equality if and only if $\sup I = 1$. We now show that for α sufficiently small,

$y^*(Y|\alpha) \subset (s\alpha, t\alpha]$ for all $Y \in [N(\alpha)Y(\alpha), Y(\alpha)]$.

a) Since $s < 1$, $AC(s) > AC(1)$, and $\inf\{AC(y) | y \in (0, s]\} > AC(1)$.

As in b) of step 1, there exists $\alpha_3 \in (0, \alpha_2]$ such that

$F(Y) < \inf\{AC(y) | y \in (0, s]\}$ for all $Y \in [Y^* - 2\alpha_3, \infty)$. Then for

all $\alpha \in (0, \alpha_3]$, $y(Y|\alpha) > s\alpha$ for all $Y \in [N(\alpha)Y(\alpha), Y(\alpha)]$.

b) If $t = 1$ then $\sup I = 1$ so for all $\alpha \in (0, \alpha_3]$,

$y^*(Y|\alpha) \subset [0, t\alpha]$ for all Y . If $t \geq 2$ then for all

$\alpha \in (0, \alpha_3]$, $y^*(Y|\alpha) \subset [0, Y^* - Y] \subset [0, 2\alpha] \subset [0, t\alpha]$ for all

$Y \in [N(\alpha)Y(\alpha), Y(\alpha)]$. For $t \in (1, 2)$, by the first half of step 2,

it is sufficient to show that $y^*(Y^* - 2\alpha|\alpha) \subset [0, t\alpha]$. Let $z(\alpha)$

be a maximizer of $(F(Y^* - 2\alpha + z\alpha) - AC(1))z\alpha$ where z is restricted to $[t, \infty)$ (clearly $z(\alpha) \in [t, 2]$). Then $P(z\alpha | Y^* - 2\alpha, \alpha) \leq$

$(F(Y^* - 2\alpha + z(\alpha)\alpha) - AC(1))z(\alpha)\alpha$ for all $z \in [t, \infty)$ by choice of $z(\alpha)$

and the fact that $AC(z) \geq AC(1)$. Also for all $z \in [t, \infty)$,

$$R(\alpha) := \frac{(F(Y^* - 2\alpha + z(\alpha)\alpha) - AC(1))z(\alpha)\alpha}{(F(Y^* - 2\alpha + \alpha) - AC(1))\alpha} \geq \frac{P(z\alpha | Y^* - 2\alpha, \alpha)}{P(\alpha | Y^* - 2\alpha, \alpha)}$$

(both $F(Y^* - 2\alpha + t\alpha)$ and $F(Y^* - 2\alpha + \alpha)$ are greater than $AC(1)$ so $R(\alpha)$ is strictly positive). For $\alpha \in (0, \alpha_3]$, F is twice continuously differentiable on $[Y^* - 2\alpha, Y^*]$ so by Taylor expansion of F about Y^* , and noting $F(Y^*) = AC(1)$, we obtain

$$R(\alpha) = z(\alpha)(2 - z(\alpha)) \left[\frac{F'(Y^*) + \frac{\alpha(2 - z(\alpha))}{\alpha(2 - z(\alpha))}}{F'(Y^*) + \frac{\alpha}{\alpha}} \right].$$

Since $z(\alpha) \in [t, 2] \subset (1, 2]$, $z(\alpha)(2 - z(\alpha)) \leq t(2 - t) < 1$, so there exists $\alpha^* \in (0, \alpha_3]$ such that $R(\alpha) < 1$ for all $\alpha \in (0, \alpha^*]$, and response α is strictly better than any response $y > t\alpha$.

For $t = 1$ or $t \geq 2$ set $\alpha^* := \alpha_3$. In all cases, for all $\alpha \in (0, \alpha^*]$, $y^*(Y|\alpha) \subset [0, t\alpha]$ for all $Y \in [N(\alpha)Y(\alpha), Y(\alpha))$, and together with a), $y^*(Y|\alpha) \subset (s\alpha, t\alpha]$, with $y^*(Y|\alpha) \subset (s\alpha, t\alpha)$ if $t > 1$.

For $\alpha \in (0, \alpha^*]$, marginal revenue is decreasing, and marginal cost is nondecreasing for all $Y \in [N(\alpha)Y(\alpha), Y(\alpha))$, $y \in (s\alpha, t\alpha]$, so there is at most one element of $y^*(Y|\alpha)$ in $(s\alpha, t\alpha]$. On the other hand, $y^*(Y|\alpha)$ is a nonempty subset of $(s\alpha, t\alpha]$ so there is at least one element of $y^*(Y|\alpha)$ in $(s\alpha, t\alpha]$. Thus for all $\alpha \in (0, \alpha^*]$, $y^*(Y|\alpha)$ is single valued, and therefore continuous for $Y \in [N(\alpha)Y(\alpha), Y(\alpha))$, and step 3 is complete.

Hence for $\alpha \in (0, \alpha^*]$,

$L_1(\alpha) \cap \{(Y, Y + y(Y|\alpha)) \mid Y \in [N(\alpha)Y(\alpha), Y(\alpha))\} \neq \emptyset$, and $E(\alpha, C, F) \neq \emptyset$.

Q.E.D.

Section 5

(1) It has been assumed throughout the paper that firms act noncooperatively. Because of the possibility of entry, any cartel must practice limit pricing, so the total industry gains from collusion are small, and converge to zero as α converges to zero. Because of the problems and costs involved with collusion among a large number of firms, when α is small, the gains from collusion will not justify the formation of large cartels. On the other hand, if a small coalition of firms acts collusively, other producing firms will generally be even better off than the coalition members, so with a large pool of producing firms, a "free rider problem" works against the formation of small coalitions. Finally, for α sufficiently small, at an equilibrium, it is generally not profitable for a producing firm to act collusively with an entering firm. Thus the assumption of noncooperative behavior seems justified when α is small relative to Y^* .

(2) The argument in Theorem 2 is based on shrinking firms relative to a fixed inverse demand, but a trivial corollary proves the analogous result for fixed firm size and replicated consumer sector. Let F_r be the inverse demand when the consumer sector is replicated r times, $F_r(Y) := F(Y/r)$ (this assumes that the partial equilibrium analysis is not affected by the replication).

Corollary: Given a cost function satisfying (C1), (C2) and (C3), and an inverse demand function F satisfying (F1) and (F2), there exists $r^* < \infty$ such that for all $r \geq r^*$, $E(1, C, F_r) \neq \emptyset$.

Proof: Identify C, F_r with C_α, F in Theorem 2, where $\alpha = \frac{1}{r}$.
 If $n, \{y_1, \dots, y_n\}$ is an element of $E(\alpha, C, F)$ then
 $n, \{ry_1, \dots, ry_n\}$ is an element of $E(1, C, F_r)$. Q.E.D.

This Corollary serves to emphasize the fact that the existence of equilibrium depends on the firm (α) being small relative to the market (Y^*) rather than small in absolute terms.

(3) If $(\alpha_j)_{j=1}^\infty$ is a sequence converging to 0 and $n_j, \{y_1^j, \dots, y_{n_j}^j\} \in E(\alpha_j, C, F)$ for all j , then n_j converges to ∞ and $\max_{1 \leq i \leq n_j} \{y_i^j\}$ converges to 0. This follows from the optimality

of each firm's response and the fact that aggregate output $Y_T^j \in [Y^* - \alpha_j, Y^*]$ for all j . Thus as the firms become technologically small with respect to the market, the endogenously determined number of operating firms becomes large. Compared to the n -firm Cournot technique, where the number of firms is exogenously increased and the output of each firm becomes small, the method used in this paper offers a much more natural interpretation of the "Folk Theorem," and can be used to prove the "Folk Theorem" when average cost curves are U-shaped and n -firm Cournot equilibrium invariably fails to converge to the perfectly competitive equilibrium as n is increased.

(4) As α converges to 0, each firm's actions converge to price taking actions if the firms have strictly U-shaped average cost curves. As α converges to 0, Y_T converges to Y^* and price converges to $F(Y^*)$ so both the actual response y_i and the price taking response y_p lie in an interval $[r(\alpha)\alpha, r'(\alpha)\alpha]$ where

$r(\alpha) < 1 < r'(\alpha)$ and both $r(\alpha)$ and $r'(\alpha)$ converge to 1 as α converges to 0. Thus the relative difference between the two responses, converges to 0.

Notice that in general firm profit is strictly positive at equilibrium even with free entry, because only integral numbers of firms can operate. All the equilibria constructed in the proof have strictly positive profit except in the case where $n(\alpha) = \frac{Y(\alpha)+y(\alpha)}{y(\alpha)}$ (i.e., when it is not necessary to "round off" to obtain an integer), but even in that case there is another equilibrium with $n(\alpha)-1$ firms and positive profit for each firm.

(5) It is clear from the proof of Theorem 2 that the differentiability properties of F and C are only required in neighborhoods of Y^* and 1 respectively. A countable number of nondifferentiable "well behaved" kinks (i.e., where left and right hand first derivatives exist but are not equal, or left and right hand second derivatives exist but are not equal) may be allowed in the inverse demand and cost functions. This allows all the types of nondifferentiabilities commonly allowed in cost and inverse demand functions. With an extended definition of first and second derivatives (e.g., when left and right hand derivatives are not equal, the first derivative takes on both values, and the second derivative is $-\infty$ if the right hand derivative is less than the left hand derivative), sufficiently strong assumptions for the proof of Theorem 2 are: (F3) there must be a positive ϵ such

that for all Y in $[Y^* - \epsilon, Y^*]$, the inverse demand is continuous, with first derivative negative, bounded away from zero and upper semicontinuous from the left at Y^* , and second derivative bounded above; and (C4) there must be an s in $(0, 1)$ and a t in $(1, \infty)$ such that the second derivative of the cost function takes on only one sign (i.e., is either nonpositive or nonnegative) on each of $(s, 1)$ and $(1, t) \cap I$ (the sign need not be the same on both intervals). These assumptions are quite general, and the only cost functions ruled out by (C4) are clearly pathological functions.⁹

It is also clear from the proof that the family of average cost functions need not satisfy $AC_\alpha(y) = AC(y/\alpha)$. The only properties needed for the proof are:

- (i) $MES(AC_\alpha) = \alpha$ and $AC_\alpha(\alpha) = AC(1)$ for all $\alpha \in (0, \infty)$; and
- (ii) there exists $s \in (0, 1)$ (s independent of α) and $\delta > 0$ such that $AC_\alpha(y) \geq AC(1) + \delta$ for all $y \in (0, s\alpha]$, for all $\alpha \in (0, \infty)$, where s corresponds to the s used in (C3)
- b) ($C''_\alpha(y) \geq 0$ for all $y \in [s\alpha, t\alpha]$, for all $\alpha \in (0, \infty)$).

Let $AC^\alpha(y) := AC_\alpha(\alpha y)$, so $MES(AC^\alpha) = 1$ for all $\alpha \in (0, \infty)$. Let $AC^0(y) := \inf_{\alpha \in (0, \infty)} AC^\alpha(y)$. Then $MES(AC^0) \leq 1$, and property (ii) guarantees that $MES(AC^0) > s$, and in a certain sense places a bound on the discontinuity of minimum efficient scale in the limit.

Assumption (C4) can be stated so that it applies to a family of average cost functions satisfying (i) and (ii).

(6) Hart [3] proves a result similar in spirit to Theorem 1 for a general equilibrium model with endogenous product choice. Each firm from an infinite set of potential firms produces at most one commodity chosen from a set of differentiated commodities, using the homogeneous numeraire good as sole input. Average cost curves

are U-shaped, but vary over commodities and over firms, and there is zero cost for zero output. At an equilibrium, consumers maximize utility as price takers, while firms maximize profit in a Nash equilibrium with complete knowledge of demand, using commodity type and quantity as strategic variables. Firms are made small relative to the market by replicating consumers (rE is the economy with consumers replicated r times). Under certain assumptions it is shown that if an equilibrium exists for each rE , $r=1,2,\dots$ then for large r , the equilibrium allocation is approximately Pareto optimal (a specific welfare measure of the difference between the equilibrium and a Pareto optimal allocation converges to zero as r converges to infinity).

Conclusion

In the partial equilibrium analysis of a market for a single homogeneous good, with constant factor prices, the "Folk Theorem" is valid under quite general assumptions. With firm size measured by technology, and the number of firms endogenous, if firms are small relative to the market then Nash-Cournot equilibrium with free entry does exist, and the market outcome is approximately competitive. The treatment of free entry recognizes that free entry is not equivalent to a zero profit condition when firms are significant and properly handles the integer problem that does arise with free entry.

Theorems 1 and 2 show that it is not necessary to mix the significant and infinitesimal cases in the discussion of a single perfectly competitive market in the long run. When firms are

significant but small, they can be assumed to recognize their effect on price, and a Nash-Cournot equilibrium with entry still exists. The market outcome in this equilibrium is approximately perfectly competitive, with aggregate output, price, and individual firm profit near Y^* , $F(Y^*)$, and 0 respectively. If average cost is strictly U-shaped then individual firm output is also approximately equal to the output of a firm which views price as fixed. This provides a justification for use of the long run perfectly competitive model, with infinitesimal firms, as an idealization of markets with free entry where firms are technologically small relative to the market.

FOOTNOTES

*Conversations with Wayne Shafer led to the use of the long run perfectly competitive model in the presentation of the results, and also raised the question of possible incentives for collusion. Questions concerning the properties of equilibrium with entry evolved out of conversations with Hugo Sonnenschein, whose comments on earlier drafts of this paper greatly improved the presentation. Of course, all errors remain my own.

¹Most expositions of the model assume that the number of firms is large but finite, and firms are not infinitesimal. This paper shows that it is not necessary to mix the significant and infinitesimal cases. Firms can be treated as infinitesimal, yielding the perfectly competitive results, or firms can be treated as significant but small, yielding market results which approximate the perfectly competitive results.

²Unless stated otherwise, all assumptions and discussions henceforth deal exclusively with noninfinitesimal firms. When firms are noninfinitesimal, the terms "perfectly competitive price" and "perfectly competitive output" refer to minimum average cost, and the demand at price equal to minimum average cost respectively. To remain consistent with the perfectly competitive model, the cost functions used for imperfectly competitive firms are long run cost functions. This is done for consistency only, since the results of Theorems 1 and 2 hold for short run as well as long run cost functions.

³These assumptions are considerably stronger than necessary (see Remark 5), but are presented here in their standard form to maintain the correspondence with the common presentation of the long run perfectly competitive model.

⁴Several extremely strong assumptions on the demand and cost functions are commonly made in order to insure the existence of equilibrium for each n . The assumption that profit functions are concave for all outputs of the individual firms, for all aggregate outputs of the other firms, is almost universal. Roberts and Sonnenschein [6] have shown how unreasonably restrictive this assumption is. A second assumption that is frequently made requires that marginal revenue be everywhere decreasing as a function of both individual firm output and aggregate output of other firms. An example in which n -firm

Cournot equilibrium fails to exist for all n greater than 1 has cost and inverse demand functions

$$c(y) = \begin{cases} 0 & y = 0 \\ 10+y & y \in (0, 1] \\ \infty & y \in (1, \infty) \end{cases} \text{ and } F(Y) = \begin{cases} 60 & Y \in [0, \frac{1}{4}] \\ 14-2Y & Y \in (\frac{1}{4}, 7] \\ 0 & Y \in (7, \infty) \end{cases} .$$

The reaction correspondence is

$$y(Y) = \begin{cases} \frac{1}{4}-Y & Y \in [0, 15/228) \\ \{\frac{7}{38}, 1\} & Y = 15/228 \\ 1 & Y \in (15/228, \frac{1}{2}) \\ \{0, 1\} & Y = \frac{1}{2} \\ 0 & Y \in (\frac{1}{2}, \infty) \end{cases} .$$

The only Cournot equilibrium occurs when n equals 1. However, if entry is allowed, a second firm, assuming the output of the first firm is fixed, has profit incentive for entry. The discontinuity of F , the fact that F' is not negative initially, the discontinuity of C at 0 and the fact that output is bounded are not essential to the results of this example.

⁵Treatment of the convergence of n -firm Cournot output to competitive output can be found in [2], [5], and [7]. Ruffin [7] and Okuguchi [5] recognize the importance of minimum average cost = $C'(0+)$ (=AC(0+)) for C continuous at 0, with $C(0) = 0$ for the validity of the "Folk Theorem" in this context.

⁶In fact, $E(\alpha, C, F) \neq \emptyset$ for all $\alpha \in (0, \frac{1}{2}]$, but for $\alpha \in [\frac{1}{4}, \frac{1}{2}]$ the properties of the reaction correspondence are different. Since this example is to serve as a preview to the proof of Theorem 2, the proof for $\alpha \in [\frac{1}{4}, \frac{1}{2}]$ is extraneous (however the properties of $E(\alpha, C, F)$ for $\alpha \in [\frac{1}{4}, \frac{1}{2}]$ may be of interest in their own right). For $\alpha \in (\frac{1}{2}, \infty)$, $E(\alpha, C, F) = \emptyset$ since, as in Example A, the free entry condition is not satisfied for any finite number of firms.

⁷The bounds on Y_T cannot be improved without additional assumptions, as seen in Example B with $\alpha = 1/5$, where 4, $\{1/5, 1/5, 1/5, 1/5\}$ and 5, $\{1/5, 1/5, 1/5, 1/5, 1/5\}$ are both elements of $E(1/5, C, F)$ with $Y_T = 4/5 = Y^* - \alpha$ and $Y_T = 1 = Y^*$ respectively.

⁸In the example of footnote 4, the zero profit condition is satisfied with $n = 3/2$, but when the number of firms is rounded to either 1 or 2, equilibrium with free entry fails to exist.

⁹The following example of a cost function that violates (C4) also shows that the existence of the interval $[s, t]$ on which $C''(y)$ is nonnegative is an independent assumption in (C3)(b), and does not follow from the other parts of assumption (C3). Minimum average cost is attained at a limit point of the set of points where $C''(y)$ changes sign.

$$AC(y) = \left[\frac{(y-1)^6 - (y-1)^8}{2} \right] \left(1 + \sin\left(\frac{1}{y-1}\right) \right) + (y-1)^8 + 1$$

for $y \in (0, 2]$, $y \neq 1$, and $AC(1) = 1$.

It is easy to generate examples similar to this one for which (C4) is violated and step 3 of the proof of Theorem 2 fails.

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