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A SURVEY OF THE APPLICATIONS OF THE OPERATOR THEORY OF PARAMETRIC PROGRAMMING FOR THE TRANSPORTATION AND GENERALIZED TRANSPORTATION PROBLEMS

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ABSTRACT

Many models in operations research can be solved by the transportation (or generalized transportation) model of linear programming together with parametric programming of the rim conditions (warehouse availabilities and market requirements) and/or the unit costs. The authors have developed an operator theory for simultaneously performing such parametric programming calculations. The present paper surveys the application of this methodology to several classes of problems, e.g., traveling salesman problems, capacity expansion problems, stochastic transportation problems, transportation problems with nonlinear costs, multi-modal multi-objective transportation problems, time transportation problems, cash management problems, production smoothing problems, constrained transportation problems, etc..

1. INTRODUCTION

A recent survey of the industrial and government applications of mathematical programming revealed that perhaps as great as 70% of such applications fell in the category of network flow problems, in particular Transportation Problems (TP) and Generalized Transportation Problems (GTP) [26]. The intuitive appeal of these models, the impressive computational performance of their solution algorithms [25,37] and the ability to reformulate several more complex O.R. problems as single or sequence of TP (or GTP) are probable reasons for this popularity.

Our purpose in this paper is to suggest ways to further increase the applicability of TP and GTP. The special mathematical structures of TP and GTP have prevented their applications to a broader class of problems and have limited the considerations that can be taken into account in solving practical problems. Specifically, our focus is on a class of problems which cannot be solved directly as TP or GTP but which can be solved by the TP or GTP together with parametric programming procedures for examining the effects on the optimal solution of continuous changes in the data of the problem. The present paper surveys our previous work in this area. For greater details, proofs and more rigorous discussion of the ideas presented here, appropriate references will be provided. In the remainder of this section, we give a flavor of the algorithms employed to perform parametric programming calculations using what we call "operators" and motivate the value of operator theoretic algorithms in increasing the applicability of TP and GTP. The remaining sections discuss the different classes of applications where operator theoretic methods have proved useful in conjunction with a TP or a GTP formulation.

To fix ideas, let us consider a TP with a set $I = \{i\} = \{1, 2, \ldots, m\}$ of warehouses, a set $J = \{j\} = \{1, 2, \ldots, n\}$ of markets with unit costs $\{c_{ij}\}$, availabilities $\{a_i\}$ and requirements $\{b_j\}$ where $\sum_{i\varepsilon I} a_i = \sum_{j\varepsilon J} b_j$. Denoting by x_{ij} the amount shipped from warehouse i to market j, we define the transportation problem P to be:

$$(1) \qquad \text{Min} \sum_{(i,j)\varepsilon[I \times J]} c_{ij}\, x_{ij} = Z$$

subject to

$$(2) \qquad \sum_{j\varepsilon J} x_{ij} = a_i \quad \text{for} \quad i\varepsilon I,$$

$$(3) \qquad \sum_{i\varepsilon I} x_{ij} = b_j \quad \text{for} \quad j\varepsilon J, \text{ and}$$

$$(4) \qquad x_{ij} \geq 0 \quad \text{for } (i,j)\varepsilon[I \times J].$$

The GTP is the same as the TP except that (2) is replaced by

$$(5) \qquad \sum_{j\varepsilon J} e_{ij}\, x_{ij} \leq a_i \quad \text{for} \quad i\varepsilon I.$$

(In the Machine Loading context [16], e_{ij} (≥ 0) refers to the per unit production time of product j on machine i. Also the requirement, $\sum_{i\in I} a_i = \sum_{j\in J} b_j$ does not arise in GTP. Further a_i and b_j may be indifferent units). For expositional ease, our discussion in the rest of this section will be the context of the TP. The reader is referred to [33,34] for the more general capacitated (or upper bounded) transportation problem (where (4) is replaced by $0 \leq x_{ij} \leq U_{ij}$) and to [7,8,9,10] for the results in the case of the GTP.

We assume that the reader is familiar with the MODI [15, pp. 308-313] or stepping-stone method [13, Chapter II] for solving the TP (adaptation of the primal simplex method to the TP) with its terminology such as transportation tableau, basis B, cycle, basic solution (i.e., $\{x_{ij}\}$ satisfying (2)-(3) with $x_{ij} = 0$ for nonbasic cells), primal feasibility (i.e., (4)) and dual feasibility, i.e.,

(6) $$u_i + v_j \leq c_{ij} \quad \text{for} \quad (i,j)\varepsilon[I \times J] \text{ where}$$

(7) $$u_i + v_j = c_{ij} \quad \text{for} \quad (i,j)\varepsilon B.$$

By a <u>cell</u> we mean an index pair (p,q) with row $p\varepsilon I$ and column $q\varepsilon J$. The basis B can also be represented as a tree Q in the graph $[I \cup J, (I \times J)]$. When any basic cell $(p,q)\varepsilon B$ is dropped, the tree splits into two subtrees with row p in one subtree Q_R and column q in the second subtree Q_C. The set of rows and columns in Q_R (Q_C) are denoted by I_R and J_R (I_C and J_C) respectively. The subsets I_R and I_C (J_R and J_C) partition I(J) [32, pp. 217-218].

An <u>operator</u> $\delta \underset{\sim}{T}(P)$ determines the sequence of optimum solutions (i.e., $\{x_{ij}\}$, $\{u_i\}$, $\{v_j\}$ and Z) as the problem P with data $\{a_i\}$, $\{b_j\}$ and $\{c_{ij}\}$ is transformed into problems $P^T(\delta)$ with data $\{a_i^T\}$, $\{b_j^T\}$ and

$\{c^T_{ij}\}$ which are linear functions (8)-(10) below of a single parameter δ for all $0 \leq \delta \leq \infty$:

(8) $$a^T_i = a_i + \delta\alpha_i \quad \text{for} \quad i\varepsilon I,$$

(9) $$b^T_j = b_j + \delta\beta_j \quad \text{for} \quad j\varepsilon J, \text{ and}$$

(10) $$c^T_{ij} = c_{ij} + \delta\gamma_{ij} \text{ for } \quad (i,j)\varepsilon[I \times J]$$

where the prespecified values for $\{\alpha_i\}$, $\{\beta_j\}$ and $\{\gamma_{ij}\}$ are unconstrained in sign but such that

(11) $$\sum_{i\varepsilon I} \alpha_i = \sum_{j\varepsilon J} \beta_j.$$

Constraint (11) is required so as to satisfy $\sum_{i\varepsilon I} a^T_i = \sum_{j\varepsilon J} b^T_j$.

By imposing restrictions on the values for $\{\alpha_i\}$, $\{\beta_j\}$ and $\{\gamma_{ij}\}$, we get some important special cases of the operator $\delta\underset{\sim}{T}(P)$:

(A) Rim operators $\delta\underset{\sim}{R}(P)$ where $\gamma_{ij} = 0$ for all $(i,j)\varepsilon[I \times J]$, i.e., the data changes are only in the rim conditions $\{a_i\}$ and $\{b_j\}$.

(B) Cost operators $\delta\underset{\sim}{C}(P)$ where $\alpha_i = 0$ for $i\varepsilon I$ and $\beta_j = 0$ for $j\varepsilon J$, i.e., the data changes are only in the cost coefficients $\{c_{ij}\}$.

The Rim operators are further classified into

(A1) (Plus) Cell Rim Operator $\delta \underset{\sim}{R}^{+}_{pq}(P)$ where
$\alpha_i = 0$ for $i \varepsilon I - \{p\}$ with $\alpha_p = 1$ and
$\beta_j = 0$ for $j \varepsilon J - \{q\}$ with $\beta_q = 1$.
i.e., all data remain the same except $a^T_p = a_p + \delta$ and $b^T_q = b_q + \delta$.
The name cell operator arises from the fact that (p,q) is a cell in the transportation tableau.

(A2) (Minus) Cell Rim Operator $\delta \underset{\sim}{R}^{-}_{pq}(P)$.
This is the same as (A1) before except $\alpha_p = -1$ and $\beta_q = -1$,
i.e., all data remain the same except $a^T_p = a_p - \delta$ and $b^T_q = b_q - \delta$.

(A3) Area Rim Operator $\delta \underset{\sim}{R}_A(P)$ which is any rim operator, without any restrictions on $\{\alpha_i\}$ and $\{\beta_j\}$ aside from (11).

The Cost Operators are likewise classified into

(B1) (Plus) Cell Cost Operator $\delta \underset{\sim}{C}^{+}_{pq}(P)$ where
$\gamma_{ij} = 0$ for $(i,j) \varepsilon [I \times J]$ except that $\gamma_{pq} = 1$,
i.e., only data change is $c^T_{pq} = c_{pq} + \delta$.

(B2) (Minus) Cell Cost Operator $\delta \underset{\sim}{C}^{-}_{pq}(P)$
Same as (B1) above except that $\gamma_{pq} = -1$, i.e., all data remain the same except that $c^T_{pq} = c_{pq} - \delta$.

(B3) Area Cost Operator $\delta \underset{\sim}{C}_A(P)$ which is any cost operator without any restrictions on $\{\gamma_{ij}\}$.

The cell rim operators consider a simultaneous increase (or decrease) in warehouse p and market q. This concept is generalized to that of binary rim operators in [18] where the possibility of increasing a_p to $a_p + \delta$ and decreasing a_q to $a_q - \delta$ with all other data remaining the same are considered. (Likewise b_p can be changed to $b_p + \delta$ and b_q changed to $b_q - \delta$ with all other data remaining the same). In the case of capacitated transportation problems with upper bounds $\{U_{ij}\}$, we have bound operators as well which can be reduced to rim operators [32,, pp. 221-223]. For the GTP, we also have weight operators which examine the effects on the optimal solution of changes in the $\{e_{ij}\}$ [9].

Algorithms for implementing rim and cost cell and area operators are discussed in [33 ,34] for the TP and in [7,8,9,10] for the GTP. (The references [8,33] also discuss the application of the more general operator $\delta \underset{\sim}{T}(P)$.) These algorithms can roughly be described as follows:

<u>Algorithm 1 (for applying $\delta \underset{\sim}{T}(P)$)</u>

(i) Determine the basic optimum solution to the problem P. Let B_1 be the optimal basis. Let $k = 1$ and $\delta_1 = 0$.

(ii) Determine the maximum extent μ_k such that for $\delta_k \leq \delta \leq \delta_k + \mu_k$ the basis B_k continues to be optimal for the problem $P^T(\delta)$. For this range of δ, the optimal primal and dual solutions can be easily determined since the optimal basis is known. If $\mu_k = \infty$, stop. Otherwise go to (iii).

(iii) Determine an alternate optimal basis B_{k+1} for the problem $P^T(\delta)$ for $\delta = \delta_k + \mu_k$. If no such basis can be found, the problem $P^T(\delta)$ is infeasible for $\delta > \delta_k + \mu_k$; stop. If such a basis can be found, set $\delta_{k+1} = \delta_k + \mu_k$, $k = k + 1$ and go to (ii).

In step (ii) of the above algorithm, we determine the optimal solution as a function of δ with the optimal basis remaining the same. Such operators are referred to as <u>basis preserving operators</u> and are denoted by light face letters such as $\delta T(P)$, $\delta R(P)$, $\delta C^-_{pq}(P)$, etc. In the case of cell rim operators with $(p,q) \notin B_k$, $\delta R^+_{pq}(P)$ amounts to shifting the amounts $\{x_{ij}\}$ around the cycle created by adding (p,q) to B_k. (If $(p,q) \varepsilon B_k$ only x_{pq} gets altered.) The objective function Z increases by $(\delta - \delta_k)\ (u_p + v_q)$. In the case of area rim operators, the transformed solution is obtained as

$$(12) \qquad x_{ij} = x^0_{ij} + (\delta - \delta_k)\ y_{ij}$$

where $\{y_{ij}\}$ satisfy (2)-(3) with a_i and b_j replaced by α_i and β_j and such that y_{ij} is zero for nonbasic cells. The $\{x^0_{ij}\}$ correspond to values of $\{x_{ij}\}$ for $P^T(\delta_k)$. The objective function Z increases by $(\delta - \delta_k)[\sum_{i \varepsilon I} \alpha_i u_i + \sum_{j \varepsilon J} \beta_j v_j]$. Since the optimal basis remains the same and the costs $\{c_{ij}\}$ are not changing, the optimal dual variables do not change for basis preserving rim operators. The value for μ_k is determined as the maximum value for $(\delta - \delta_k)$ so that the transformed $\{x_{ij}\}$

satisfy the nonnegativity constraints (4). Thus for the problem $P^T(\delta_k + \mu_k)$ at least one basic cell, say, (r,s) reaches a value $x_{rs} = 0$ and any further application of the basis preserving operator with basis B_k would drive x_{rs} negative. The alternate basis B_{k+1} is determined from B_k by finding a cell (e,f) such that $B_{k+1} = B_k - \{(r,s)\} + \{(e,f)\}$ is also an optimal basis for $P^T(\delta_k + \mu_k)$. The cell (e,f) is determined so that

$$(13) \qquad c_{ef} - u_e - v_f = \min_{[I_C \times J_R]} (c_{ij} - u_i - v_j)$$

where the sets I_C and J_R are determined by dropping (r,s) from B_k (see earlier discussion). If the set $[I_C \times J_R]$ is empty, it can be shown that the problem $P^T(\delta)$ has no feasible solution for $\delta > \delta_k + \mu_k$.

For cell cost operators δC^+_{pq}, step (ii) of the above algorithm amounts determining the sets I_R, J_R, I_C, J_C obtained by dropping (p,q) from B_k. The optimal dual values for the basis preserving operator are obtained by merely increasing the $\{u_i\}$ to $\{u_i + (\delta - \delta_k)\}$ for $i \varepsilon I_R$ and decreasing $\{v_j\}$ to $\{v_j - (\delta - \delta_k)\}$ for $j \varepsilon J_R$ and leaving the remaining duals unchanged. It can also be shown that the objective function increases by $(\delta - \delta_k)x_{pq}$. (Similar remarks apply to δC^-_{pq}. A much simpler result holds for the case $(p,q) \notin B$.) For the area cost operator δC_A, we define

$$(14) \qquad u_i = u_i^0 + (\delta - \delta_k)u_i^* \quad \text{for} \quad i \varepsilon I \quad \text{and}$$

(15) $$v_j = v_j^0 + (\delta - \delta_k)v_j^* \quad \text{for} \quad j \in J$$

where u_i^* and v_j^* satisfy (7) with c_{ij} replaced by γ_{ij}. The $\{u_i^0\}$ and $\{v_j^0\}$ correspond to the duals for $P^T(\delta_k)$. The objective function Z increases by $(\delta - \delta_k) \sum_{(i,j)\in[I \times J]} \gamma_{ij} x_{ij}$. For the basis preserving cost operators, since the basis and the rim conditions do not change, the primal solution $\{x_{ij}\}$ does not change for $\delta_k \leq \delta \leq \delta_k + \mu_k$. The value μ_k is determined as the maximum value for $(\delta - \delta_k)$ so that the transformed $\{u_i\}$ and $\{v_j\}$ satisfy the dual feasibility conditions (6) for the problem $P^T(\delta)$. Thus for the problem $P^T(\delta_k + \mu_k)$, at least one of the nonbasic cells (e,f) satisfies $c_{ef} = u_e + v_f$ and any further application of the basis preserving cost operator with basis B_k would make $c_{ef} > u_e + v_f$ thus violating the dual feasibility condition (6). The alternate optimal basis B_{k+1} is determined by adding (e,f) to B_k and eliminating the minimum "giver" cell (r,s). (In the cycle created by adding (e,f) to B_k we mark alternate cells of the cycle as "getters" and "givers" starting with (e,f) as a "getter.")

The operators are computationally easy to apply. In particular the computational steps involved in applying cell operators are even easier and this is the reason we have provided specialized algorithms for the cell operators rather than treating them as special cases of area operators. As will be seen in the rest of this paper, cell operators tend to arise quite frequently in many applications.

Parametric programming is much more valuable for transportation problems compared to general linear programs. To see this, consider the linear program

(16) Min $C'X$ subject to $AX = b$ and $X \geq 0$

where C and X are $(N \times 1)$, b is $(M \times 1)$ and A is $(M \times N)$. Now assume that the requirements vector b can be changed to $b + \delta d$ where d is an $(M \times 1)$ vector and $\delta \geq 0$ is a scalar, but at a cost $g\delta$ (g is a scalar). The problem of determining the optimal δ can be formulated as

(17) Min $C'X + g\delta$ subject to $AX - \delta d = b$; $X, \delta \geq 0$.

We note that (17) is also a linear program in the variables X,δ so that it can be solved directly as such and no special parametric programming is necessary. However, such is not the case if (16) is a transportation problem. In that case, the constraint matrix A has a special "echelon-diagonal" pattern [31, pp. 227-228] with $N = mn$ and $M = m + n$ and the primal transportation algorithm effectively uses this structure. However, the presence of the vector d in (17) makes the coefficient matrix $[A \quad -d]$ not possess the special structure any longer. Thus (17) is not a transportation problem. Of course, (17) can be solved directly as a linear program but it will be computationally more efficient to solve (17) by applying an area rim operator (the α_i and β_j will be directly determined from d) to the optimum solution of the transportation problem (16). As remarked earlier, the objective function increases with a marginal cost of

$\sum_{i\varepsilon I} \alpha_i u_i + \sum_{j\varepsilon J} \beta_j v_j = f$ (say). Consequently, the overall marginal cost is $(f + g)$ and we apply the area rim operator until $(f + g)$ becomes nonnegative. The above example illustrates as to why parametric programming is likely to prove more valuable for transportation problems compared to general linear programs.

In addition to providing practical benefits in solving problems which do not directly fit as TP or GTP, the operator theoretic algorithms also provide theoretical insights. For instance:

(i) In certain linear cost capacity expansion problems [17,35] where market demands are monotone nondecreasing over time it can be shown that there exists an optimal solution in which the warehouse capacities are nondecreasing even though such constraints are not explicitly imposed. Although some of these insights may be obtained by alternate means such as lattice theory [43], our operator theoretic algorithms have the advantage of obtaining such a solution (in addition to proving existence).

(ii) By defining an additional warehouse $(m+1)$ and an additional market $(n+1)$ (with $c_{m+1,j} = 0$ for $j\varepsilon J \cup \{(n+1)\}$, $c_{i,n+1} = 0$ for $i\varepsilon I$, $a_{m+1} = 0$ and $b_{n+1} = 0$), the cell rim operator $\delta R^{+}_{\sim m+1,n+1}$ provides a means for determining the downward marginal cost $(= u_{m+1} + v_{n+1})$ of the transportation system as a whole (i.e., the marginal rate at which the total optimal cost will go down if the volume handled in the system is reduced) [34, p. 250].

(iii) The cell cost operators can be used in an algorithm for solving the transportation problem itself with costs translated if necessary so that $c_{ij} \geq 0$ for all (i,j). Consider any primal basic feasible solution to this problem. By temporarily defining $c_{ij} = 0$ for $(i,j)\varepsilon B$, the feasible solution is optimal to the transformed problem. We can now restore the costs for $(i,j)\varepsilon B$ one by one using the (plus) cell cost operator. It can be shown that this algorithm converges to an optimum within $2 \sum_{i\varepsilon I} a_i$ iterations (assuming a_i to be integer and the primal problem to be nondegenerate) [32]. This is the first primal basic algorithm that we know of with a polynomial bound in the number of iterations. Another interesting property of this cell cost operator algorithm is that, even if the problem is primal degenerate it will converge to an optimum without perturbation.

The operators also have the theoretical properties of commutativity; associativity, distributivity, etc., just as most other operators in mathematics.

Although parametric programming has been well investigated in the context of linear programming [20,21] much less results were available in the context of TP and GTP before our papers [7,8,9,10,33,34]. For instance [1,41] concern themselves with an analysis of the "stability" of an optimal basis with respect to data variations. To examine the maximum value δ for which the current basis is feasible when c_{pq} is changed to $c_{pq} + \delta$ these approaches would involve the determination of (m-1) x (n-1) cycles and solving a set of (m-1) x (n-1) inequalities in δ. The cell cost operator algorithm, on the contrary, does not involve any determination of cycles at all and evaluates only (m x n)/4 inequalities, on the average. Consequently, the procedures in [1,41] for continuous data variations are not computationally efficient for the parametric programming of transportation problems, although admittedly there is some overlap in the underlying theory. The determination of B_{k+1} from B_k in step (iii) of the Algorithm for rim operators is the same as finding an adjacent dual feasible basis in the dual simplex method for the transportation problem [11,14,27]. It should be emphasized that these last referenced papers are concerned with discrete changes in rim conditions whereas the operator theoretic algorithms deal with continuous variations in all the data (rims, costs, upper bounds, weights) of the TP and GTP.

2. Managerial and Economic Significance of Operators

One of the most surprising results of the operator theory approach to parametric programming for the TP and GTP was the discovery of the existence of shadow prices for each of the rim operators. These permit much richer interpretations and managerial uses than would be expected from standard linear programming theory. Among other things they permit the identification and complete explanation of the "transportation paradox" for the TP case and the related "production paradox" for the GTP case.

Before discussing these rim operator results we note that the dual variables for the cell costs c_{ij} are just the corresponding primal variables x_{ij}, as would be expected from ordinary parametric linear programming. Hence we do not discuss these further.

In [33,34] for the TP and [7, 8, 9,10] for the GTP, given an mxn problem the first step was to add an additional row, m+1, and column, n+1, before solving the problem. All costs in these new rows and columns are zero, except $c_{m+1,n+1} = M$ for the TP case. In the GTP, $c_{m+1,j} = M$ for all $j = 1,\ldots,n$ and the rim data a_{m+1} and b_{n+1} are chosen to make the sum of supplies equal the sum of demands in the TP case. In the GTP case $a_{m+1} = M$ and b_{n+1} is not needed and hence not defined. We define $I' = I \cup \{m+1\}$ and $J' = J \cup \{n+1\}$. These extra rows and columns serve somewhat different purposes in the two cases, and so will be discussed separately. However, in either case, once the enlarged problem is solved, we can define the optimal dual matrix D with entries

(18) $$d_{pq} = e_{pq}u_p + v_q, \quad \text{for } p \in I' \text{ and } q \in J'$$

where $e_{pq} = 1$ for all p and q in the TP case, and u_p and v_q are the optimal solutions to the dual problem.

We will now indicate how each of the $(m+1) \times (n+1)$ entries d_{pq} in (18) has at least one (and sometimes more than one) shadow price interpretation. This could not be predicted from ordinary parametric linear programming theory, because a TP or GTP problem has at most $m+n+2$ constraints, and that theory predicts the existence of only the same number of shadow prices.

We concentrate now on the TP case. The optimal dual matrix defined in equation (18) can be partitioned into the four areas as shown in Figure 1. In addition there are special row and column indices, k and ℓ, such that cells $(k,n+1)$ and $(m+1,\ell)$ are in the optimal basis. (Every basis must have such cells in the last row and column.)

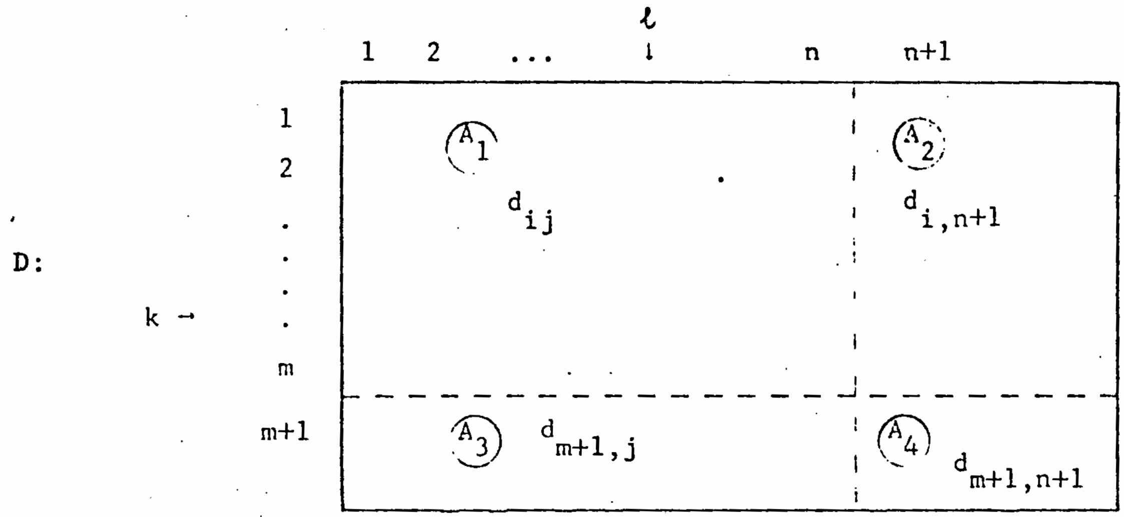


Figure 1

The following facts can be shown concerning matrix D, shown in Figure 1 .

(a) Except in the case of dual degeneracy, the entries d_{pq} (defined in (18) are unique (even though u_p and v_q are never unique in the TP case.)

(b) The d_{pq} in A_1, for $p \epsilon I$ and $q \epsilon J$, can be either positive or negative.

(c) In areas A_2, A_3, and A_4 we have $d_{p,n+1} \leq 0$, $d_{m+1,q} \leq 0$ for $p \epsilon I'$ and $q \epsilon J'$.

(d) In A_4 we have $d_{m+1,n+1} = -d_{k\ell}$.

The most important interpretations of these quantities is given in Figure 1 which applies only when the sum of the supplies initially equals the sum of the demands. (A more complete table is given on page 250 of [34].) The interpretations in Table 1 hold only over a finite range of δ, say $0 \leq \delta \leq \mu$. The <u>extent</u>, μ, can be calculated by methods given in [33].

Table 1

Area	Operator	Shadow Price	Operator	Shadow Price
A_1: $p\varepsilon I, q\varepsilon J$	δR^+_{pq}	d_{pq}	δR^-_{pq}	$-d_{pq}$
A_2: $p\varepsilon I, q=n+1$	$\delta R^+_{p,n+1}$	$d_{p,n+1}$	$\delta R^-_{p,n+1}$	$-d_{p\ell}$
A_3: $p = m+1,\ q\varepsilon J$	$\delta R^+_{m+1,q}$	$d_{m+1,q}$	$\delta R^-_{m+1,q}$	$-d_{kq}$
A_4: $p = m+1,\ q = n+1$	$\delta R^+_{m+1,n+1}$	$d_{m+1,n+1}$	---	---

Assumption: $\sum_{i\varepsilon I} a_i = \sum_{j\varepsilon J} b_j$ before operator is applied.

Indices k and ℓ are chosen so that $(k,n+1)$ and $(m+1,\ell)$ are in the current basis.

When $d_{pq} < 0$ in A_1, i.e., $p\varepsilon I$, $q\varepsilon J$, and the extent of the operator δR^+_{pq} is positive $(\mu > 0)$ then we observe that "transportation paradox" by applying δR^+_{pq} with $\delta > 0$. That is, we can "ship more (total tonnage) for less (total cost)."

When the extent of an operator $\mu = 0$, we have the degenerate case. By a finite number of basis changes it was shown in [34] how to obtain a shadow price d_{pq} and corresponding operator δR^+_{pq} having positive extent for a given p and q. In [18] Fong and Srinivasan show how to simultaneously calculate nondegenerate shadow price for all p and q.

The interpretations of the entries of the dual matrix for the GTP case are similar and simpler to state. The following facts can be proved, [7]; (a) u_p and v_q are unique (except in the dual degenerate case); (b) $u_p \leq 0$ for $p\varepsilon I$; and (c) $v_q \geq 0$ for $q\varepsilon J$. There is no downward marginal cost evaluator.

Again when $d_{pq} < 0$ and δR^{+}_{pq} has positive extent we have the "production paradox," that is, it is possible to "produce more (total products) for less (total cost)."

In the GTP case we have a new kind of operator, the weight operator, which acts (see [9]) to change a cell weight e_{pq}. This operator is, in general, non linear, which is not surprising since it acts to change the efficiency of a single machine.

A final remark is that each kind of operator, rim, cost, or weight gives rise to new algorithms for solving TP and GTP problems. One of them, the cost operator algorithm for the TP case has been extensively described and tested by Srinivasan and Thompson [38].

3. Workload Smoothing and Cash Management Models

(A) A New Approach to Workload Smoothing

The well-known HMMS [28] model of productivtion planning takes into account the following costs: hiring, firing, inventory, and stockout. It gives a way of smoothing production over time while minimizing the total of all these costs. Although those authors noted that these costs are actually piecewise linear, they approximated these cost functions by quadratic functions. In so doing they were able to get analytic expressions and to derive the "linear decision rule."

Here we shall derive a similar smoothing model that takes account of all of the same costs, but in their original piecewise linear form. We use the generalized transportation model in its "goal programming" version [13,28] and show how the smoothing model can be put directly into this form. We also sketch how the model can be extended to convex, piecewise linear, cost functions; and we show how operator theory can be used to impose other kinds of managerial constraints on total hiring, total firing, total inventories, and total stockouts.

We need the following quantities to state the model:

$I \quad = \{1,2,\ldots,m\}$ = the set of machines

$J \quad = \{1,2,\ldots,n\}$ = the set of goods to be produced

a_i = number of men already available for machine i

b_j = number of units of good j to be produced

p_{ij} = net profit on good j if produced by machine i

x_{ij} = number of units of good j produced by machine i

e_{ij} = number of machine i man-days needed to produce one unit of good j

H_i = number of workers hired to run machine i

h_i = cost per worker hired to run machine i

H = total number of workers hired for all machines

F_i = number of workers for machine i to be fired

f_i = cost per worker for machine i to be fired

F = total number of workers fired for all machines

G_j = inventory of good j to be carried over

g_j = inventory holding cost for good j

G = total inventory of all goods to be carried over

U_j = amount of stockout of good j

u_j = stockout cost for good j

U = total stockout of all goods

We use a maximizing version of the machine loading model.

(19) Maximize $Z = \sum_I \sum_J p_{ij}x_{ij} - \sum_I (f_iF_i + h_iH_i) - \sum_J (u_jU_j + g_jG_j)$

(20) Subject to: $\sum_J e_{ij}x_{ij} = a_i - F_i + H_i$

(21) $\sum_I x_{ij} = b_j - U_j + G_j$

(22) $\sum_J U_j = U$

(23) $\sum_J G_j = G$

(24) $\sum_I F_i = F$

(25) $\sum_I H_i = H$

(26) $x_{ij}, F_i, H_i, U_j, G_j, F, H, U, G \geq 0$

We shall convert this linear programming production smoothing model into a bounded variables goal programming machine loading model by adding suitable multiples of N, where N is a number definitely larger than any subscripted variable. Specifically, we add N to both sides of equations (20) and (21), we add nN to both sides of (23), and we add mN to both sides of (25). We also multiply by -1 the nonnegativity constraints for H_i and U_j by -1 and add N to both sides of these inequalities. After transposing suitable terms and rearranging we have the following model:

(27) Maximize: $Z = \sum_I \sum_J p_{ij}x_{ij} - \sum_I [f_iF_i - h_i(N-H_i)]$

$$- \sum_J [u_jU_j - g_j(N-G_j)] - N(\sum_I h_i + \sum_J g_j)$$

Subject to:

(28) $\sum_J e_{ij}x_{ij} + F_i + (N-H_i) = a_i + N$

(29) $\sum_I x_{ij} + U_j + (N-G_j) = b_j + N$

$$\sum_{J} U_j = U \tag{30}$$

$$\sum_{J} (N-G_j) = nN - G \tag{31}$$

$$\sum_{I} F_i = F \tag{32}$$

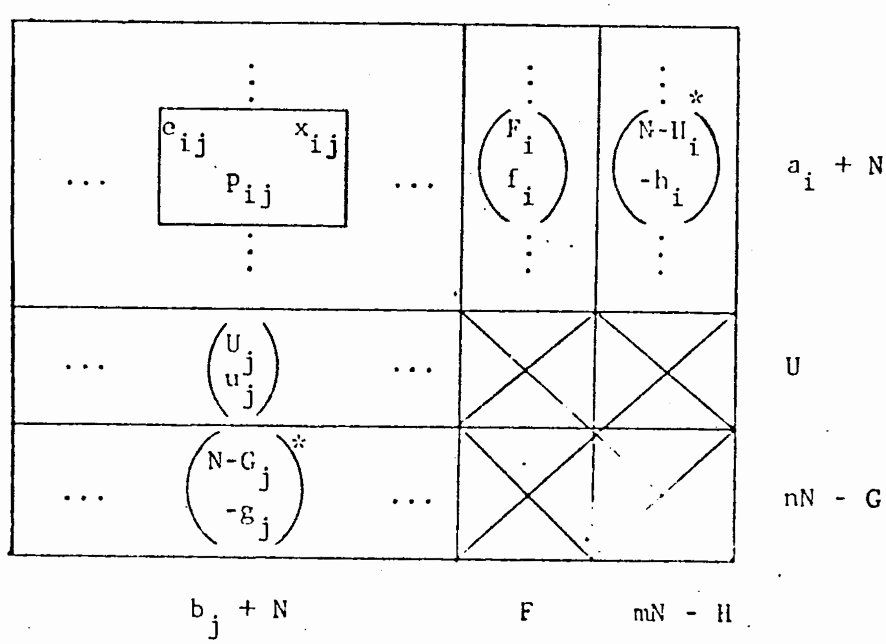
$$\sum_{I} (N-H_i) = mN - H \tag{33}$$

$$N - G_j \leq N \tag{34}$$

$$N - H_i \leq N \tag{35}$$

$$x_{ij},\ F_i,\ U_j,\ F,\ U \geq 0 \tag{36}$$

If we drop the constant term in (27) this is clearly the machine loading, upper bounded, goal programming model whose tableau is shown in Figure 2.



* Denotes variable upper bounded at N

Figure 2

By applying rim operators we can find the managerial consequences of changing the rim quantities U, G, F, and H.

Further variants of the model can be made as follows:

(1) By adding another row and column we can put the variables F, H, U, G into the tableau and determine them as in [2,35]

(2) By repeating the rows for U_j, $N-G_j$ and the columns for F_i and $N-H_i$ and imposing suitable upperbounds and changing cost coefficients, we can impose convex piecewise linear cost functions.

(3) By imposing structure in the upper lefthand corner we can take into account multistage production over time.

(B) Cash Management Models

Recently, a model for making the cash management decisions for a firm, was reformulated as an ordinary transshipment model by Srinivasan in [32]. In that paper he also noted that the generalized transportation model would be even better to take account of the varying yields on different kinds of accounts and securities.

There is not space here to go into all the ramifications of the model as described in [32], so we will content ourselves with illustrating the model with the example shown in Figure 3, which was taken from [13]. That figure gives a simple two-period model in which we have cash coming in each period, and securities we own coming due each period. Also we have cash requirements each period and accounts payable each period. It is assumed that there is a 2% discount for cash, and a 2% penalty for late payment. Cash invested for one period receives a 1% interest payment. Securities due at a given time period receive a 16% interest

Figure 3: Cash Management Example

		Cash due 1	Cash due 2	Accounts Payable 1	Accounts Payable 2	Slack	
Cash in	1	1 6.379 0	1.01 -.01	1.02 23 -.02	1.03 .156 -.03	1 0	30
	2	1 9999	1 11.66 0	.98 .02	1.02 20.84 -.02	1 2.08 0	35
Securities Owned	1	1.16 8.62 -.16	1.17 -.17	1.18 -.18	1.19 -.19	1 0	10
	2	1.12 -.12	1.16 10.31 -.16	1.10 -.10	1.18 -.18	1 0	12
		15	23	23	21		

Optimal Value = -3.916

0	-.0098	-.02	-.03	-.0097
.0097	0	-.01	-.02	0
-.16	-.171	-.182	-.194	-.146
-.145	-.16	-.168	-.183	-.138
-.0097	0	-.08	-.02	0

Figure 4. Optimal Dual Matrix

payment, but if cashed in one period early, get only 12% interest. Thus a bill (accounts receivable) is due in period 1 is paid with cash from period 1 it has an e factor of 1.02 because of the 2% discount for cash. But if it is paid (late) with cash received in period 2, its e factor is .98 because of a penalty for late payment. Note that c_{ij} is the negative of the interest rate received. The rest of the entries in Figure 2 can be deduced from the above rules, with e_{ij} in the northwest, c_{ij} in the middle and x^*_{ij} in the northeast for a cell (i,j).

Note that the optimal objective value is -3.916, so that the rate of return on all cash and securities to be received in the two periods is 3.918/87 = .045%.

The optimum dual matrix is shown in Figure 4. From the last column we see that more cash in period 2 is of no value in affecting the objective function, but more cash in period 1 is worth \$.146 per \$ and more cash in period 2 is worth \$.138 per \$. Note also that if we can simultaneously increase cash inflow in period 1 from 30 to 31.01 and cash payments in period 2 from 22 to 23 we will increase profits by .0098. However if we can increase cash income in period 2 from 35 to 36 and increase cash payments in period 1 from 15 to 16 we will decrease profits by .0097. Many other interpretations of these dual evaluators are evident from Figure 4.

4. Multiple Objective Models

If we look at the cost coefficients, the c_{ij}'s of equation (1), we can easily see that different interpretations for them will lead to data giving quite different solutions. Consider these two interpretations:

(i) c_{ij} is the cost of shipping a unit from warehouse i to market j.

(ii) $c_{ij} = t_{ij}$ is the time to ship a unit from warehouse i to market j.

With interpretation (i) the solution to the TP or GTP problem achieves a minimum total cost, while with interpretation (ii) the solution minimizes the total shipping time. (Later we will discuss still other objectives.)

Since we cannot simultaneously optimize both objectives (except for a rare numerical accident) we instead follow a suggestion of Geoffrion [24] and optimize the following composite objective:

$$\text{(37)} \qquad \text{Minimize} \quad (1-\delta)C(X) + \delta T(X)$$

where $C(X)$ is the total cost and $T(X)$ the total time of shipping plan X and the parameter δ is a given number in the unit interval $(0 \leq \delta \leq 1)$. As we let δ vary in this interval we can trace out an <u>efficient</u> or <u>pareto optimal</u> <u>surface</u> (to be defined in the next paragraph) in the cost-time space. A manager interested in applying the result can then choose one of the points on this surface and implement the corresponding solution. In so doing he must choose, directly or indirectly, a <u>trade-off</u> between the two competing objectives of cost and time.

In order to define efficiency, let $F_1, F_2, \ldots, F_k$ be k different minimizing objective functions, i.e., lower values of F_i represent more desirable outcomes. A solution X is said to be <u>efficient</u> (<u>pareto-optimal</u> or <u>non-dominated</u>) if and only if there does <u>not</u> exist any other solution X' which <u>dominates</u> X, i.e.,

$$\text{(38)} \qquad F_i(X') \leq F_i(X) \quad \text{for} \quad i = 1,2,\ldots,k$$

and with strict inequality holding in (38) for at least one i. We will refer to the above property by the condensed notation, "the solution k-tuple $(F_1(X), F_2(X), \ldots, F_k(X))$ is <u>efficient</u>."

We now discuss two applications involving multiple objectives for which operator parametric programming of the TP provides a solution technique.

(A) <u>Determining Cost vs. Time pareto-optimal frontiers in multi-modal transportation problems.</u>

In [40] Srinivasan and Thompson discussed the problem of choosing modes of transportation (railroad, highway, air) while taking into account the conflicting objectives of minimizing total transportation costs and average shipment times. To provide a mathematical formulation of the problem, we first define the index sets

(39) $I = \{1,2,\ldots,m\}$ the set of warehouses (rows),

(40) $J = \{1,2,\ldots,n\}$ the set of markets (columns),

(41) $K = \{1,2,\ldots,p\}$ the set of transportation modes.

For $i\epsilon I$, $j\epsilon J$, $k\epsilon K$ we define

x_{ijk} = amount shipped from warehouse i to market j via mode k

c_{ijk} = shipping cost per unit from warehouse i to market j via mode k

t_{ijk} = time of shipping from warehouse i to market j via mode k.

a_i = supply at warehouse i $(a_i \geq 0)$

b_j = demand at market j $(b_j \geq 0)$

We assume supplies equal demands, i.e. $\sum_{i\epsilon I} a_i = \sum_{j\epsilon J} b_j$, and now state the constraints of the problem.

(42) $$\sum_{j\epsilon J} \sum_{k\epsilon K} x_{ijk} = a_i \quad \text{for } i\epsilon I$$

(43) $$\sum_{i\epsilon I} \sum_{k\epsilon K} x_{ijk} = b_j \quad \text{for } j\epsilon J$$

(44) $$x_{ijk} \geq 0 \quad \text{for } i\epsilon I,\ j\epsilon J, \text{ and } k\epsilon K.$$

We say a solution $X = \{x_{ijk}\}$ is <u>feasible</u> if it satisfies (42)-(44). Corresponding to any feasible solution the total cost C and the (weighted) total shipment time T are:

(45) $$C(X) = \sum_{i\epsilon I} \sum_{j\epsilon J} \sum_{k\epsilon K} c_{ijk} x_{ijk}, \quad \text{and}$$

(46) $$T(X) = \sum_{i\epsilon I} \sum_{j\epsilon J} \sum_{k\epsilon K} t_{ijk} x_{ijk}$$

In computing the average shipment times each of the t_{ijk} is weighted by the corresponding shipments x_{ijk}. It is easy to show, see [40], that if instead of using total shipment times we used average shipment times as an objective, the same optimal solutions, would be obtained.

If we substitute (45) and (46) into (47) it is clear that in each cell we will choose a mode k so that the quantity

(47) $$(1-\delta)c_{ijk} + \delta t_{ijk}$$

is minimized. Define new cell costs

(48) $$d_{ij}(\delta) = \operatorname*{Min}_{k\epsilon K}[(1-\delta)c_{ijk} + \delta t_{ijk}]$$

and let $K_{ij}(\delta)$ be the set of indices in K where this minimum occurs. Given reasonable assumptions on the original costs the set $K_{ij}(\delta)$ changes only a finite number of times in the interval $0 \leq \delta \leq 1$. Thus the objective function in (48) is

(49) $$d_{ij}(\delta) = (1-\delta)c_{ijk_1} + \delta t_{ijk_1}$$

for a specific index $k_1 \in K_{ij}(\delta)$ as δ varies in a certain interval (δ_1, δ_2) of positive length $0 \leq \delta_1 \leq \delta \leq \delta_2 \leq 1$.

In [40] it was shown that the objective functions $d_{ij}(\delta)$ in (49) can be used as the objective for an ordinary transportation problem, and its solution for every δ can be used to trace out the effiency frontier. Moreover an area cost operator can be used to explore the optimal solutions as δ

changes. In this way a finite algorithm is developed that changes the objective function, with (49) changing as δ changes, and also as the index set $K_{ij}(\delta)$ changes to maintain a solution to the problem (48).

A simple example is worked in detail in [40].

(B) Algorithms for Minimizing Total Cost, Bottleneck Time and Bottleneck Shipment in Transportation Problems.

Consider a TP that has unit cost c_{ij} and time t_{ij} when a good is transported from warehouse i to market j. We can define the following objectives for every feasible solution satisfying (2)-(4):

(50) Total Cost $$TC = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

(51) Bottleneck Time $$BT = \underset{\{(i,j) \mid x_{ij} > 0\}}{\text{Maximum}} t_{ij}$$

(52) Shipment on Bottleneck Routes $$SB = \sum_{\{(i,j) \mid t_{ij} = BT\}} x_{ij}.$$

Thus BT measures the maximum time taken by any shipping route actually used ($x_{ij} > 0$) in the solution, while SB gives the total shipping amount over all routes having this maximum time.

As before we denote by TP the problem with the TC objective function. We also call the problem with the BT objective the Bottleneck Transportation Problem (BTP).

In [39] Srinivasan and Thompson gives algorithms to solve each of the following problems:

(i) Trace the pareto-optimal or efficient solution pairs (TC,BT). The solution-procedure involves the application of a series of cell cost operators.

(ii) Solve the BTP. Here we have a single objective problem. The solution procedure involves the application of a series of cell cost operators. This algorithm has been programmed and thoroughly tested. It is very efficient, being able to solve BTP's in about half the time as ordinary TP's of the same size and choices of data.

(iii) Solve the SB problem. Once the bottleneck time, BT, has been determined it was shown in [39] that the SB problem can subsequently be solved by a further application of an area cost operator.

Although the objective functions considered here have not been very widely applied so far, it is to be hoped that the existence of the algorithms discussed here for solving these problems will make applications to areas such as assembly line balancing and personnel selection [39] more widespread. In fact, it seems to us that for many applications the BT or SB objectives are much more appropriate than the more usual TC objective.

5. CAPACITY EXPANSION PROBLEMS

In this section we outline the application of area rim operators in solving a multi-period capacity expansion and shipment planning problem for a single product under a linear cost structure [17]. The product can be manufactured in the set $I = \{1, 2, \ldots, m\}$ of producing regions and is required by the set $J = \{1, 2, \ldots n\}$ of markets for each of the time periods (e.g., years) $K = \{1, 2, \ldots, T\}$. Let $K' = K - \{1\}$. Let $\{r_j^0\}$ be the initial demand in market j and let $r_j^t \geq 0$ be the known increment in market j's demand in time period t. Thus $\sum_{\tau=0}^{t} r_j^\tau$ represents the demand in market j at time t. Let $\{q_i^0\}$ be the initial production capacity in region i and let $\{q_i^t\}$ be the <u>cumulative</u> capacity added in region i from periods 1 to t. Thus the total production capacity in region i at time t is $q_i^0 + q_i^t$. The demand in a market can be satisfied by production and shipment from any of the regions, but must be met exactly during each time period (i.e., no back-logging or inventorying). Let c_{ij} be the unit cost of "shipping" from region i to market j. (This includes transportation costs, variable costs of production including costs of maintaining a unit of capacity.) Let k_i be the unit cost of capacity expansion in region i. Proportional capacity expansion costs may be realistic when the production capacity is rented or subcontracted or when the fixed costs are relatively small. Moreover, the optimum solution to the problem with linear costs can be used to provide a lower bound to the objective function of the problem with concave expansion costs. Let h_i be the unit cost of maintaining s_i units of idle capacity in region i. We have assumed that all the costs are stationary but the model can be easily extended to take into account inflationary effects. Let g_i be the terminal (or resale) value of a unit of capacity in region i at time T. Let α be the discount factor per period. Then the problem of determining a schedule of capacity expansions for the regions and a schedule of shipments from the regions to the markets so as to minimize the discounted capacity expansion and shipment costs can be formulated as the problem P below:

$$(53)\quad \text{Min} \sum_{t\varepsilon K} \sum_{(i,j)\varepsilon[I \times J]} \alpha^{t-1} c_{ij} x_{ij}^t + \sum_{t\varepsilon K} \sum_{i\varepsilon I} \alpha^{t-1} h_i s_i^t$$

$$+ \sum_{i\varepsilon I} k_i q_i^1 + \sum_{t\varepsilon K'} \sum_{i\varepsilon I} \alpha^{t-1} k_i (q_i^t - q_i^{t-1}) - \sum_{i\varepsilon I} \alpha^T g_i q_i^T$$

subject to

(54) $$\sum_{j\epsilon J} x_{ij}^{t} + s_i^t - q_i^t = q_i^0 \quad \text{for} \quad i\epsilon I \text{ and } t\epsilon K,$$

(55) $$\sum_{i\epsilon I} x_{ij}^{t} = \sum_{\tau=0}^{t} r_j^{\tau} \quad \text{for} \quad j\epsilon J \text{ and } t\epsilon K,$$

(56) $$q_i^t - q_i^{t-1} \geq 0 \quad \text{for} \quad i\epsilon I \text{ and } t\epsilon K' \quad \text{and}$$

(57) $$x_{ij}^t,\ s_i^t,\ q_i^t \geq 0 \quad \text{for} \quad i\epsilon I,\ j\epsilon J \text{ and } t\epsilon K.$$

The objective function (57) gives the minimum total time discounted shipment, idle capacity maintenance and capacity expansion costs less the salvage value for the capacity. After making suitable assumptions on the salvage values g_i and rearranging terms, (53) reduces to

(58) $$\text{Min} \sum_{t\epsilon K} \sum_{(i,j)\epsilon[I \times J]} \alpha^{t-1} c_{ij} x_{ij}^t + \sum_{i\epsilon I} \alpha^{t-1} h_i s_i^t$$
$$+ \sum_{t\epsilon K} \sum_{i\epsilon I} \alpha^{t-1} k_i' q_i^t$$

where $k_i' = (1-\alpha)k_i$.

The constraints (54) state that the amount shipped out of region i at time t ($= \sum_{j\epsilon J} x_{ij}^t$) plus whatever is left as idle capacity ($= s_i^t$) should be equal to the net capacity $q_i^0 + q_i^t$. Equation (54) can be rewritten as

(59) $$\sum_{j\epsilon J} x_{ij}^t + s_i^t + (N - q_i^t) = q_i^0 + N \quad \text{for } i\epsilon I$$

where N is a large positive number. Thus if we define $x_{i,n+1}^t = s_i^t$ and $x_{i,n+2}^t = (N - q_i^t)$, Equation (59) becomes

(60) $$\sum_{j\varepsilon J''} x_{ij}^t = q_i^0 + N \quad \text{for} \quad i\varepsilon I$$

where $J'' = J \cup \{(n+1)\} \cup \{(n+2)\}$. We note that

(61) $$0 \leq x_{i,n+2}^t \leq N \quad \text{for} \quad i\varepsilon I$$

since $N \geq q_i^t \geq 0$. The constraints (56) become

(62) $$-x_{i,n+2}^t + x_{i,n+2}^{t-1} \geq 0 \quad \text{for} \quad i\varepsilon I,\ t\varepsilon K'.$$

Constraints (56) (and equivalently (62)) are the coupling constraints in P linking the time periods in K. Consequently, if we drop the constraints (62) , the problem P splits into a sequence of problems P_t ($t\varepsilon K$) below (the set I is augmented by a "dummy" region (m+1) to pick up any extra amounts in the (n+1) and (n+2) columns; let $I' = I \cup \{m+1\}$):
This sequence of problems can be solved by effective and repeated use of area rim operators sequentially from the solution of problem p_1 to get $p_2 \cdots p_T$.

(63) $$\text{Min} \sum_{(i,j)\varepsilon[I' \times J'']} c_{ij}\, x_{ij}^t$$

subject to

(64) $$\sum_{j\varepsilon J''} x_{ij}^t = q_i^0 + N \quad \text{for} \quad i\varepsilon I',$$

(65) $$\sum_{i\varepsilon I'} x_{ij}^t = \sum_{\tau=0}^{t} r_j^\tau \quad \text{for} \quad j\varepsilon J'',$$

(66) $$x_{i,n+2}^t \leq N \quad \text{for} \quad i\varepsilon I' \quad \text{and}$$

(67) $$x_{ij}^t \geq 0 \quad \text{for} \quad (i,j)\varepsilon[I' \times J'']$$

where the values for c_{ij}, q_i^0 and r_j^T for $i = m+1$; $j = (n+1), (n+2)$ are to be appropriately defined.

The interesting thing to note is that P_t is a transportation problem and, what is more, problem P_t differs from P_{t-1} only in the right hand side of constraints (65). The requirement of market j increases by r_j^t is going from P_{t-1} to P_t. Thus the optimal solution to P_t can be obtained from that of P_{t-1} by the application of the area rim operator $\delta \underset{\sim}{R}_A(P_{t-1})$ with $\delta = 1$, $\alpha_i = 0$ for $i \varepsilon I'$ and $\beta_j = r_j^t$ for $j \varepsilon J$. (The definition of r_{n+2}^t is such that $\sum_{j \varepsilon J''} \beta_j = \sum_{i \varepsilon I'} \alpha_i = 0$.) It can be proved that the solutions so obtained satisfy the constraints (62) (equivalently (56)) thus providing the optimal solution to the original problem P. Consequently, a 10 period, 10 region, 200 market problem is reduced from a linear program P with 2190 constraints and 20,200 variables to a transportation problem with 11 regions and 202 markets together with the application of the area rim operator to obtain the transportation solutions for the next 10 periods. Illustrations of this approach are given in [17].

Rim operators prove useful in other capacity expansion problems also. In [19], Fong and Srinivasan extend the formulation discussed above to the case where the costs may be nonstationary, demands not necessarily increasing and capacity expansion costs having fixed components as well. The heuristic algorithms used in [19] start with a feasible solution and improve it by swapping capacities between two regions over the planning horizon if that would reduce the total cost. The binary rim operators discussed earlier [18] prove useful in this context. The heuristic solutions so obtained are only about .8% away from the optimum and considerably faster (often by factor of more than 10) compared to mixed integer exact procedures.

Other applications of rim operators in the context of capacity expansion include determination of growth paths in logistics operations [35], growth models in machine loading problems [2] and the continuous time plant location procedures of Rao and Rutenberg [30].

6. Post Optimization in TP and GTP

In this section we will consider the application of operator theory on certain well-known problems which do not have a direct solution procedure. It will be shown that each problem considered can be solved by solving an initial transportation-type problem possessing a direct algorithm, on which solution operator theory will be sequentially applied to generate a sequence of solutions converging to the required optimal solution. In each problem the crucial step where the operator theory is applied will be emphasized. Appropriate references where a detailed discussion is made will be given.

a. Transportation type problems with quantity discounts

The transportation type problem with quantity discounts may be formulated as follows:

(68) Minimize $$Z = \sum_{i \in I} \sum_{j \in J} c^*_{ij} x_{ij}$$

(69) Subject to: $$\sum_{j \in J} x_{ij} = a_i \qquad \text{for } i \in I$$

(70) $$\sum_{i \in I} x_{ij} = b_j \qquad \text{for } j \in J$$

(71) $$0 \leq x_{ij} \leq \lambda^r_{ij} \qquad \text{for } i \in I \text{ and } j \in J$$

where: $I = \{1,2,\ldots,i,\ldots,m\}$ set of sources

$J = \{1,2,\ldots,j,\ldots,n\}$ set of sinks.

For each (i,j):

$$c^*_{ij} = \begin{cases} c^1_{ij} & \text{if} \quad 0 = \lambda^0_{ij} \le x_{ij} < \lambda^1_{ij} \\ c^2_{ij} & \text{if} \quad \lambda^1_{ij} \le x_{ij} < \lambda^2_{ij} \\ \cdots & \qquad \cdots \cdot \cdots \\ c^k_{ij} & \text{if} \quad \lambda^{k-1}_{ij} \le x_{ij} < \lambda^k_{ij} \\ \cdots & \qquad \cdots\cdot \quad \cdots \\ c^r_{ij} & \text{if} \quad \lambda^{r-1}_{ij} \le x_{ij} < \lambda^r_{ij} \le \infty \end{cases} \tag{72}$$

and $c^k_{ij} \ge c^{k+1}_{ij}$ for $k = 1,2,\ldots,r-1$.

Let us refer the problem (68)-(71) as P^1.

The problem is solved by an algorithm [6], where initially one solves a problem with $c^*_{ij} = c^r_{ij}$ for $\forall$ (i,j). Obviously if the optimal solution $X = \{x^*_{ij}\}$ satisfies (72), then the solution is optimal. Else, a heuristic is provided to find a cell (s,t) from among the "interval infeasible" cells from which the branch and bound procedure can be applied. Let us say that the current optimal x^*_{st} should be such that $\lambda^{k-1}_{st} \le x^*_{st} < \lambda^k_{st}$, due to the cost $c^*_{st} = c^k_{st}$ used, but in fact, it is interval infeasible in that $x^*_{st} < \lambda^{k-1}_{st}$. This condition leads to two branches (or subproblems)

(i) In branch 1, a lower bound restriction of the form $x_{st} \ge \lambda^{k-1}_{st}$ is imposed while the current c^*_{st} remains unchanged.

(ii) In branch 2, the current c^*_{st} is replaced by the "interval feasible" c^*_{st}, i.e., the c^{ℓ}_{st} corresponding to the current x^*_{st} and an upper bound restriction in the form of $x_{st} < \lambda^{k-1}_{st}$ is imposed.

It is shown in [6] that the branch 1 problems are solved via cell rim operator [33], δR^-_{st} where $\delta = \lambda^{k-1}_{st}$ while the branch 2 problems are solved by cell cost operator [33], δC^+_{st} where $\delta = c^{\ell}_{st} - c^*_{st}$. Computational efficiencies in the branch selection are given in the paper with suitable illustrations. It is also shown that the "fixed change Transportation Problem" is a special case of this model, so that operator theory is applicable for that problem as well.

b. Optimal facility location under random demand with general cost structure

Operator theory has also been the primary method in [5] to solve the Facility Location problem to decide the location of plants, their respective capacities or production levels and also the distribution plan to different demand centers. In their paper [5], Balachandran and Jain consider the following problem.

A firm manufactures a product which is required at n different demand centers. The demand b_j $(j = 1,2,\ldots,n)$, at each center, is assumed to be a random variable with known marginal density $f(b_j)$. The firm has the option of setting up facilities at m different sites, i, $(i = 1,2,\ldots,m)$. The possible capacities of the facility at site i could be any one element from the ordered set A_i, where $A_i = \{a_{ir} | r = 1,2,\ldots,m_i\}$. The first element a_{i1} of each of the sets A_i is assumed to be 0 and corresponds to the decision of not locating a facility at site i and the last element a_{im_i} corresponds to the maximum possible production level at site i. The cost of building c capacity of y_i units at site i is $f_i(y_i)$ where

$$f_i(y_i) = \begin{cases} 0 & \text{if} \quad y_i = a_{i1} = 0 \\ K_{ir} + v_{ir} y_i & \text{if} \quad a_{ir} < y_i \le a_{i,r+1} \text{ for } r = 1,2,\ldots,m_{i-1} \\ \infty & \text{if} \quad y_i > a_{im_i} \end{cases} \tag{73}$$

K_{ir} may be considered as the fixed component of the cost associated with setting up a plant of maximum capacity $a_{i,r+1}$, and v_{ir} is the per unit variable cost. Thus the cost structure at any particular site is a piece-wise linear function of the capacity of the plant. The cost of distributing x_{ij} units from a facility at site i to demand center j is $t_{ij}x_{ij}$ where $\{t_{ij}\}$ are given. These costs may be considered as the discounted costs if a multi-period planning horizon is considered. With the above notation the problem can be formulated as follows:

(74) Minimize $Z = \sum_{i=1}^{m} f_i(y_i) + \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij}x_{ij}$

(75) subject to $\sum_{i=1}^{m} x_{ij} = B_j$ for $j \in J$,

(76) $\sum_{j=1}^{n} x_{ij} = y_i \leq a_{im_i}$ for $i \in I$,

(77) $x_{ij} \geq 0,\ y_i \geq 0,$

where $J = \{1,2,\ldots,n\}$, $I = \{1,2,\ldots,m\}$ and B_j represents the realization of the random variable b_j. In order to ensure a feasible solution we assume

$$\sum_{i=1}^{m} a_{im_i} \geq \sum_{j=1}^{n} B_j \ .$$

We will present here the application of operator theory [33,34] for the deterministic demand problem [B_j's are known], solved by a branch and bound procedure where in each branch one requires a solution of the transportation problem. In each node of the branch and bound tree the solution is generated from that of the parent node by cost operators [33].

First, for each production function $f_i(y_i)$ for site 'i', substitute $\ell_{i0}^1 + c_i^1 y_i$ the best linear underestimate (see Definitions 1, 2 of [5]) of $f_i(y_i)$ for $a_{i1} \leq y_i < a_{im_i}$. This substitution in equation (74) leads to an initial transportation problem that is solved to yield an optimal solution $X^1 = \{x_{ij}^1\}$ and a current lower bound $Z_1(X^1)$ the objective function value of the transportation problem and $Z(X^1)$ as the current upper bound where X^1 is substituted in equation (74).

At a typical step of the algorithm, choose the open node with the smallest lower bound. If lower and upper bounds are equal, then the solution at this node is optimal. If not, partition this node on the basis of an index k identifying the location with the greatest difference between the actual and the best linear underestimate, so that two constraints of the form $y_k \leq a_{k,t}$ and $y_k > a_{k,t}$ are imposed yielding two branches. Substitution of best linear underestimates revised to results in changes correspond to the new ranges of y_k in the objective function coefficients. Further, the constraint $y_k > a_{k,t}$ implies that the slack variable for the k^{th} plant $x_{k,n+1} < a_{km_i} - a_{k,t}$ thus making that slack variable upper-bounded. The changes in the costs as well as the changes in upperbounds can be effectively implemented by the "operator theory" for the transportation problem [33,34]. The new solutions for branch nodes can be generated by the operators and thus two new open nodes are created with the parent open node now being closed. If any of the branch problems has no feasible solution also one closes that node. Determine the current lower and upper bounds for the two newly opened nodes and continue the algorithm.

c. The stochastic transportation and generalized transportation problem

Garstka [22] has presented an algorithm for solving the Stochastic Transportation Problem utilizing the concepts of "Stochastic Programming with Simple Recourse". He allows the demands b_j's to have a known marginal distribution and utilizing the permit penalties of under and over production and introducing an equivalent convex function as the objective he solves the "simple recourse" problem. Later Balachandran [4] has provided similar results for the generalized transportation problem and applied operator theory to provide the solution procedure. We will consider the stochastic generalized transportation and the use of operator theory here. A similar discussion is applicable in the case of transportation problem.

Consider the following problem:

$$\text{(78)} \qquad \text{Min} \quad \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

s.t.

$$\text{(79)} \qquad \sum_{j \in J} e_{ij} x_{ij} \leq a_i \qquad \forall\, i \in I$$

$$\text{(80)} \qquad \sum_{i \in I} x_{ij} = b_j \qquad j = 1, \ldots, n$$

$$\text{(81)} \qquad x_{ij} \geq 0 \qquad \forall\, i \in I \text{ and } j \in J$$

Following the usual assumptions of "Stochastic Programming" with recourse [22], with b_j having a marginal density $f(b_j)$, let us introduce a per unit penalty for $p_j \geq 0$ and $d_j \geq 0$, for under and over production. Then the equivalent stochastic generalized transportation problem will be (see [4] for, details).

$$\text{(82)} \qquad \text{Minimize} \quad Z_1 + Z_2$$

subject to

$$\text{(83)} \qquad \sum_{j \in J} e_{ij} x_{ij} \leq a_i \quad \forall\, i \in I$$

$$x_{ij} \geq 0$$

$$\text{(84)} \qquad \text{where} \quad Z_1 = \sum_{i \in I} \sum_{j \in J} \tfrac{1}{2}(2c_{ij} - b_j + d_j) x_{ij}$$

$$\text{(85)} \qquad \text{and} \quad Z_2 = \sum_{j \in J} (p_j + d_j) \int_{\sum_{i \in I} x_{ij}}^{b_{jm}} (b_j - \sum_{i \in I} x_{ij}) f(b_j)\, db_j$$

where b_{jm} is the median of b_j.

The following properties can be shown to be true [5]:

1. The objective function is convex.
2. For any $i \in I$ and $j \in J$, if $c_{ij} > p_j$, then $x_{ij} = 0$.
3. For any $j \in J$ and all $i \in I$, if $c_{ij} \geq (p_j - d_j)/2$, then $\sum_{i \in I} x_{ij} \leq b_{jm}$ in the optimal solution. Note that if $p_j < d_j$ the above inequality trivially holds since $c_{ij} \geq 0$.
4. For any $i \in I$ and all $j \in J$ if $c_{ij} + d_j < 0$, then $\sum_{j \in J} c_{ij} x_{ij} = a_i$.

With these properties, it is shown, [5], that the optimal solution to the stochastic generalized transportation problem can be obtained by solving an "initial" generalized transportation problem where the rim conditions for the demands (columns) can be first set equal to b_{jm} (the median of the random variable b_j) for every j and the objective function coefficient for x_{ij} as $\frac{1}{2}(2c_{ij} - p_j + d_j)$. The future rim conditions for b_j's are successively obtained by solving from the following relationship

(86) $$v_j^k = (p_j + d_j) \int_{b_j^{k+1}}^{b_{jm}} f(b_j) db_j$$

where v_j^k is the "known" optimal duals corresponding to iteration 'k' and b_j^{k+1} is the unknown rim condition for the j^{th} column to be evaluated from the above relationship. The newly evaluated b_j^{k+1} are used to obtain the next set of optimal solutions $X^{k+1} = \{x_{ij}^{k+1}\}$ and the duals u_i^{k+1} and v_j^{k+1}. However, it is easily seen that the next set of optimal primal and dual solutions for iteration (k+1) can be obtained utilizing "area rim operators" [8,10] instead of re-solving. The algorithm terminates when $v_j^{t+1} = v_j^t$ (or $b_j^{t+1} = b_j^t$) for every j. A convergence proof is also indicated [5]. Similar results are equally applicable for the stochastic transportation problem by utilizing the "area rim operators" [33,34] of the ordinary transportation problem.

Applications of operator theory are also seen in "One Machine Job Shop Scheduling Decision" [23] where branch solutions of transportation problems can be obtained by operator theory, in "Lock Box" decision models [29] and also in solving transportation problems with convex costs [12]. Zero-one decision problems are used in optimal assignment of sources to uses [36] and in allocation of jobs to be processed by computers in a computer network [3]. These problems use extensively rim and cost operators in ensuring either the zero-one requirement [36] or the unique source requirement [3]. Interested readers can get more details from these references.

7. Algorithms for Transportation and Generalized Transportation Problems

(i) Cost Operator Algorithm for the Transportation Problem

Operator theory can also be used to devise algorithms for the transportation problem and the Generalized Transportation Problem. Srinivasan and Thompson have developed such an algorithm [38] using cost operators. Their procedure starts with the determination of a primal basic feasible solution. The unit costs corresponding to the basic cells in the initial solution are then altered so that the solution is dual feasible as well and hence optimal to the problem with modified costs. The altered costs are then successively restored to their true values with appropriate changes in the primal and dual solutions using cell cost operators. When all the altered costs (at most, $m+n-1$ of them corresponding to the basis cells) are restored to their true values, one obtains an optimum solution for the original problem.

The cell cost operator algorithm has many interesting theoretical features. First, it converges in a finite number of steps to an optimum even without perturbation of the rim conditions as against other primal basic methods. Second, it converges to an optimum in $(2T-1)$ iterations for primal nondegenerate transportation problems where T denotes the sum of the (integer) warehouse availabilities (also the sum of the (integer) market requirements). This bound on the number of steps is much smaller than the bound for the Ford-Fulkerson algorithm for the transportation problem [38] by a factor of approximately min $(m,n)/2$ where m and n are the number of warehouses and markets respectively. For primal degenerate transportation problems, however, the cell cost operator bound is slightly weaker than the Ford-Fulkerson bound by a factor of approximately two. As against most primal basic algorithms which have exponential bounds, the cell cost operator algorithm has the more desirable polynomial bound. However, in terms of <u>average</u> computation times, the primal MODI algorithm is faster then the cell cost operator algorithm. Details of the cell cost operator algorithm as well as the related area cost operator algorithm are given in [38].

In a similar manner, one can devise an algorithm for the transportation problem using rim operators. For instance, any dual basic feasible solution can be used to start the algorithm. The rim conditions can be altered so that the solution is primal feasible and hence optimal to the altered problem. The rim conditions can then be restored to their true values using rim operators with appropriate changes in the primal and dual solution. When all the rim conditions are restored to their true values, an optimum solution to the original problem would have been determined.

(ii) Weight Operator Algorithm for the Generalized Transportation Problem

Similar to the cost operator algorithm for the Transportation Problem [38], one could start with a primal basic feasible solution to a Generalized Transportation problem by inspection. From this <u>initial</u> solution to GTP, the per unit cost coefficients corresponding to basic solution are then altered so that dual feasibility and hence optimality is attained. From this stage the altered costs are successively restored to the original problem values with appropriate changes in the primal and dual solutions using cell cost operators which lead to the optimal solution to the original problem. The same procedural steps could be done by using rim operators as well as starting with any dual basic feasible solution.

An interesting approach is in the cell weight operator application. Since a transportation problem is easier in basis structure, we can solve the GTP with the given unit costs and rim values but keeping all the weight coefficients as identically equal to unity. This converts the GTP as an ordinary transportation problem so that the solution is obtained. Since the cell weight operator application affects both primal and dual feasibility concurrently, we need not worry about the starting solution to be either primal or dual feasible. Thus from the optimal basis of the reduced GTP we had, one restores successively the original weights of the GTP from the current unit weights. At this time, the authors have not any computational results and feel that this approach is promising.

Recently Cacetta [12] has provided an algorithm to solve the Transportation problem having costs which are convex. He is able to show that, by appropriate values of cost changes and the application of cost operators sequentially with these cost changes, the optimal solution for the convex transportation problem is attained. A similar algorithm for the GTP is straightforward.

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