DISCUSSION PAPER NO. 299

# Overhead Allocation Via Mathematical Programming*

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September 1977

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A B S T R A C T

OVERHEAD ALLOCATION VIA MATHEMATICAL PROGRAMMING

This manuscript builds upon the work of Kaplan and Thompson [8] by pointing out its restrictions, building upon its strengths and offering an improved method of overhead allocation. We reformulate their example as a Generalized Transportation Problem. Unique duals which reflect the product-resource form interaction are determined and utilized to allocate common and traceable overhead to individual products. Lastly, sensitivity analysis is introduced to address the problem of avoidable overhead and full absorption costing.

Overhead Allocation Via Mathematical Programming.

## 1. INTRODUCTIONS

In recent years articles have appeared in the accounting literature which address the issue of allocating (assigning) overhead cost (i.e., cost not directly traceable to individual production centers or products) to individual production centers or productions of a multi product firm. This literature can be divided into two groups. The first group which concentrated on allocation via matrix algebra to address the reciprocity of service centers in the allocation of overhead cost to production centers [8,10,11,12,13]. The second group was concerned with finding a method of allocation which would assign overhead cost based upon their economic contribution to the individual products produced. Kaplan and Welam [9] and Jensen [6] utilize the duals of non-linear optimization models to develop allocation alogrithms.

Kaplan and Thompson [8], here after K & T, developed rules to allocate a firm's overhead within the context of a linear programming model. Concentrating on the optimum output mix decision they developed an allocation alogrithm which only under specific conditions does not perturb the optimal product mix decision while assigning overhead to each product line.

The purpose of this paper is to build upon the work of K & T [7] by pointing out its restrictions, building upon its strengths and offering an improved method of overhead allocation. We reformulate their example as a Generalized Transportation Problem. Unique duals which reflect the product-resource from interaction are determined and utilized to allocate common and traceable overhead to individual products. Lastly sensitivity analysis is introduced to address the problem of avoidable overhead and full absorption costing, the case which eliminates the restrictions of the K & T model.

Like K & T [7] we will take as given the present historical accounting system and the product and period cost it generates. We will also adopt the Hadley [4] notation for the linear programming model.

Let

$m$ = the total number of resources: $i = 1,2, \ldots,m$

$n$ = the total number of products: $j = 1,2, \ldots,n$

During a given period we can therefore let

$x_j$: be the number of units of product j to be produced.

$b_i$: be the amount of resource i available

$a_{ij}$: is the per unit utilization of the $i^{th}$ resource by the $j^{th}$ product

$p_j = (r_j - c_j)$ is the per unit contribution to profit of the production and sale of one unit of product j, i.e., per unit selling price $r_j$ minus per unit variable production cost $c_j$.

The one period decision problem for maximizing profits will be:

(1) $\text{MAX } px$ subject to $\{Ax \leq b;\ x \geq 0\}$

The corresponding dual to the problem is

(2) $\text{MIN } wb$ subject to $\{wA \geq p;\ w \geq 0\}$

COMMON MANUFACTURING OVERHEAD

Here we are interested in analyzing the assignment of manufacturing cost which cannot be directly traced to an individual product or resource but instead is common to the production of two or more products. Well known examples are depreciating of factory and building and multipurpose equipment, janitors, utilities, factory supplies, etc., i.e., cost of items we usually consider to be indirect manufacturing (factory) overhead.

Let the total manufacturing overhead amount be H dollars. Two situations exist for a given accounting period of concern:

(i) the firm may be operating at some $x^*$ so that its total variable contribution margin exceeds its common overhead H i.e., $H < px^*$

(ii) the firm may be operating at some $x^*$ so that $H \geq px^*$. Let us define $k_1 = \frac{H}{px^*}$ for $\frac{H}{px^*} < 1$, case (i), and $k_2 = \frac{H}{px^*}$ for $\frac{H}{px^*} \geq 1$, case (ii).

It can be seen that $w^*A$ is an $(1 \times n)$ vector whose $j^{th}$ component assigns the imputed value of the resources used to produce product j. Utilizing such an imputed value of resources, it is conceivable to make the following allocation of overhead. Here we briefly review and set out by example the Kaplan & Thompson [7] allocation system, which addresses only case (i). Let $k=k_1$ in this case.

The two product, two resource linear program they used [7] is reproduced below.

$$(3) \qquad \begin{aligned} \max \quad & 1x_1 + 1/2\, x_2 \\ \text{st.} \quad & 3x_1 + 2x_2 \leq 12 \\ & 5x_1 \qquad\ \ \leq 10 \\ & x_1, x_2 \quad \geq 0 \end{aligned}$$

The rule for allocating <u>common</u> <u>overhead</u> to each product unit is given by $kw^*A$, where $k = \frac{H}{px^*} = \frac{H}{w^*b} < 1$. Here H = \$2.5 Common overhead to be allocated and $px^*$ = \$3.5 total contribution at optimal feasible solution $x^* = [2,3]$ and $w^* = (.25,.05)$ is the optimal feasible dual solution. Using the example in [7], the LP formulation (4) and the numerical example (5) are:

$$(4) \qquad \begin{aligned} \text{MAX} \quad & (p - kw^*A)x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$$(5) \qquad \begin{aligned} \text{MAX} \quad & (1 - 5/7)x_1 + (1/2 - 5/14)x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \leq 12 \\ & 5x_1 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

For allocation of <u>traceable</u> <u>overhead</u> they define $B_i = C(b)/b_i$ where $B_i$ is the average cost per unit of capacity. $C(b_i)$ is total dollars traceable to resource i and $b_i$ a specific amount of resource capacity. The assumed $B_1$ = \$.10/unit and $B_2$ = .04/unit.

$$(6)\quad \text{MAX } (p - B'A)x \quad \text{s.t.} \quad Ax \leq b, \quad x \geq 0$$

Where $B' = \text{MIN } \{B_i, w_i^*\}$

$$(7)\quad \text{MAX } (1 - 1/2)x_1 + (1/2 - 2/10)x_2$$
$$\text{s.t.} \quad 3x_1 + 2x_2 \leq 12$$
$$5x_1 \leq 10$$
$$x_1, x_2 \geq 0$$

Taking their exact examples and combining the two types of overhead to be allocated in the K & T numerical examples, leads to the following accumulation problems (8) and (9) in which, $k^* = \dfrac{H}{(p - B'A)x^*}$ ;

$$(8)\quad \text{MAX } [p-(1-k^*)B'A - k^*w^*A]x$$
$$\text{s.t.} \quad Ax \leq b$$
$$x \geq 0$$

$$(9)\quad \text{MAX } (-.30/1.90)x_1 + (-.18/1.90)x_2$$
$$\text{s.t.} \quad 3x_1 + 2x_1 \leq 12$$
$$5x_1 \leq 10$$
$$x_1, x_2 \geq 0$$

One can easily see that the objective function of (9) is less than zero and the optimal production mix will not be maintained. This is so even though the exact rules and examples of K & T [7] were followed. This example violates their thesis that they have reconciled the direct and absorption problem by presenting an allocation procedure which does not distort the optimal product mix.

Therefore, it is important to have an allocation system which is operational under circumstances in which a firm will show an accounting loss for the period. Consequently we will also address this issue where k > 1, a condition assumed not to exist under the Kaplan and Thompson [7] allocation rules (we called it case (ii) with $k = k_2$).

## 2. Allocation of Common Overhead using the Generalized Transportation Model

Consider that the solution $\underline{x}^* = [2,3]$ to the linear program (3) was used for planning purposes. The duals $\underline{w}^* = (.25, .05)$ represent the increase in net contribution that an additional unit of resource could add. Let us further assume that H = \$2.5 is a common overhead pool and not directly traceable to a corresponding resource or product.

> Rule 1: Utilize the optimal duals corresponding to each resource i, the value $w_i^*$ the shadow price and the total quantity $b_i$ and obtain an economically meaningful position of the total overhead pool associated to each resource. Due to the dimensionality of $w_i^*$ as dollars per resource and $b_i$ as the number of resource units (available), $w_i^* b_i$ provides a common unit of measure, i.e., dollars which is comparable for all resources. If $w_i^* = 0$ for some i which implies the corresponding resource is not fully utilized then $w_i^* b_i = 0$. This provides an economic interpretation of zero value to the allocation system. For each resource i whose $w_i^* \neq 0$ implies the products produced in the system collectively utilizes the entire resource $b_i$.

This leads to the case of the resource i having an ability to bear a share of overhead out of H.

We propose this share to be $\dfrac{w_i^* b_i}{\sum_{i=1}^{m} w_i^* b_i} H = H_i$.

Thus $H_i$ is that part of the total common overhead H that is assigned to resource i. Consequently, each resource will be assigned the H = \$2.5 as follows: Resource 1: Labor is $\frac{15}{7} = H_1$ and

Resource 2: Machines is $\frac{2.5}{7} = H_2$

Further $H_i = 0$ if $w_i^* = 0$; $H_1 > 0$ if $w_i^* \neq 0$; so that $\sum_{i=1}^{m} H_i = H$.

Next we are concerned with the distribution of $H_i$ the overhead share corresponding to resource i to every product j. This is repeated for every resource i that has a positive $w_i^*$. Then all the quantities are added which provides the overhead allocation.

Let us now discuss the distribution allocation associated with a specific resource i such that $w_i^* > 0$. We will show that the generalized transportation problems discussed in Balachandran and Thompson [7] provides a means for such an allocation.

Realistically each resource is available in different forms, or different plants, buildups, types of labor resources, of machines, so that each resource i, is available in $m_i$ forms. Also the per unit utilization $a_{ij}$, (that is given the LP formulation) which provides the amount of resource i, that each unit of product j consumes, differs from form to form slightly around $a_{ij}$ such that $a_{ij}$ can be viewed as the average per unit utilization of resource i for product j. Let us call the standard per unit utilization of resource i, in form f, for a unit of product j as $a_{fij}$. To start with let us formulate the objective function as a maximization of the contribution margin so that we can solve the following or GTP for Resource i

## 3. The linear programming formulation for the Generalized Transportation Problem

Consider for example, that a specific resource say i, is available in m>1 forms. (We have examined the case where m = 1 elsewhere [1]). We will now present the LP formulation to the GTP. The related example is for resource i=1 (labor). We assume that two forms of labor exist; skilled f=1 semi-skilled, f=2. Consequently the following maximization problem for the optimal production schedule at the maximum contribution margin can be formulated as follows:

$$(10) \quad \text{MAX} \sum_{f=1}^{m_i} \sum_{j=1}^{n} p_{fj}\, x_{fj}$$

$$(11) \quad \text{s.t.} \quad \sum_{j=1}^{n} a_{fj}\, x_{fj} = b_{fj} \quad ; \ f=1, \ldots , m_i$$

$$(12) \quad \sum_{f=1}^{m_i} x_{fj} = x_j^* \quad ; \quad j = 1, \ldots , n$$

$$x_{fj} \geq 0$$

Note: The equality is needed if, in the LP, the resource $b_i$ is fully utilized. Otherwise, we will have less than or equal to inequalities in expression (11).

Where:

$p_{fj}$= $(r_{fj} - c_{fj})$ is the per unit contribution margin of the $j^{th}$ product which is produced using the $f^{th}$ form of resource i (i is dropped since it is constant throughout the formulation

$r_{fj}$= is the per unit selling price of product j

$c_{fj}$= is the per unit variable cost of production of product j by form f.

$x_{fj}$= is the amount of product j produced by form f.

$a_{fj}$= is the perunit standard utilization of form f in the production of product j.

The optimal feasible solution $x_{fj}^*$ indicates the optimal use of the resource forms forcasted for use.

For example let us assume that their are two forms of labor available. Skilled labor is forcasted at $3.41 per hour in the amount of 60,000 (.6 in hundred thousands) hours. And semi skilled labor is forcasted at $3.29 per hour also in the amount of 60,000 (also .6 in hundred thousands). Let us assume that the selling price for product 1 is $2.00/unit and for product 2 is $1.17/unit. Then a numerical example of (10) - (12) is

$$
\begin{array}{llrcrcrcrl}
(10') & \text{MAX} & 1.15\,x_{11} & + & .575\,x_{12} & + & .85\,x_{21} & + & .35\,x_{22} & \\
(11') & \text{s.t.} & 2.5\,x_{11} & + & 1.75\,x_{12} & & & & & = .6 \\
 & & & & & & 3.5\,x_{21} & + & .25\,x_{22} & = .6 \\
(12') & & x_{11} & & & + & x_{21} & & & = 2 \\
 & & & & x_{12} & & & + & x_{22} & = 3 \\
 & & & & & & \text{all } x_{fj} & & & \geq 0
\end{array}
$$

When the problem is solved we obtain the optimum primal $x^*_{fj}$ and duals of $\alpha^*_f$ and $\beta^*_j$ for f=1, ..., $m_i$ resource forms and j=1, ..., n products. We can now construct the dual vector $d^*_{fj} = \{a_{fij}\ \alpha^*_f + \beta^*_j\}$. The $d_{fj}$ provides us with the interaction of the $j^{th}$ product and the $f^{th}$ resource form. For (10') - (12') $x^* = [1, 2, 1, 1]$ and $\alpha^*_1 = -3$ $\alpha^*_2 = -3$, $\beta^*_1 = 1.9$ and $\beta^*_2 = 1.1$ and $px^* = \$3.5$.

PRODUCTS

| Duals Labor Forms $\alpha_f$ \ Duals for Products $\beta_j$ | | $\beta^*_1 = 1.9$ | $\beta^*_1 = 1.1$ | Slack | Capacity |
|---|---|---|---|---|---|
| $\alpha^*_1 = -3$ | Form 1 | $\boxed{2.5}$ 1.15 | $\boxed{1.75}$ .575 | $\boxed{1}$ 0 | 0.6 |
| $\alpha^*_2 = -3$ | Form 2 | $\boxed{3.5}$ .85 | $\boxed{.25}$ .35 | $\boxed{1}$ 0 | 0.6 |
| $\alpha^*_3 = 0$ | Ficti- tious | $\boxed{1}$ -100 | $\boxed{1}$ -100 | $\boxed{1}$ 0 | 100 |
| Number of Products | | 2 | 3 | — | |

Now this set of duals can be used to allocate $H_i$ (the overhead associated with resource i to each product j and form f.

Rule a: Each product j shall bear its share of common overhead associated with resource i. The per unit overhead vector $\overline{h}_i = k\omega^* G$, where G is the matrix of technological coefficients for the GTP (similar to 11' and 12') and

$\omega = (\alpha_1^*\ldots, \alpha_f^*,\ldots, \beta_1^*,\ldots, \beta_j^*)$. This charge is uniquely determined in advance by an alogrithm which uses the unique set of product-resource-from duals determined by optimizing the GTP. It is easy to see that additivity holds since $k\omega^* Gx^* = \frac{H_i}{px^*}\omega^* Gx^* = H_i$. The allocation is unambigious since a predetermined alogrithm can be applied "expost" which takes into account the product-resource form interaction and economic weights. Lastly it assigns common overhead using the duals of the GTP which represent the net marginal revenue of product j and the resource form f of i. It indicates whether each unit of product generates positive economic rent on the common overhead. Also when $k_1 < 1$ the alogrithm has the added property of maintaining the optimal mix (5')- (8'). Therefore within our example

$k_1 = \frac{H_i}{px^*} = \frac{15}{24.5}$ then the per unit charge vector is $\bar{h}_1 =$ $(\frac{17.25}{24.5}, \frac{8.625}{24.5}, \frac{12.75}{24.5}, \frac{5.25}{24.5})$ with $\bar{h}_1 x^* = \frac{15}{7} = H_1$.

When $k \geq 1$ the alogrithm will not guarantee non-perturbation of it optimal production mix decision of the GTP. However if the manager is interested in an allocation alogrithm which does not peturb the GTP optimal production mix decision then he may choose to compute an average per unit overhead charge for each product j based upon the following rule

Rule b: If $k \geq 1$ determin $\bar{h}_i$ as above; allocate to each product j a per unit overhead charge $\delta_j$ determined by

$$\frac{\sum_i h_{ij}\, x_{ij}^*}{\sum_i x_{ij}} = \delta_j$$

It is easily demonstrated that $\delta_j$ $(j = 1,\ldots,n)$ will not perturb the optimal product mix decision of the GTP if it is subtracted from each contribution margin associated with a specified column, say $j = k$. Let the new contribution margins be $p_{fk}' = p_{fk} - \delta_k$ (for kth col.) so that $p'_{fj} = p'_{fj}$ $j \neq k$; and for all f

Then $\sum_f \sum_j p_{fj} x_{fj} = \sum_f \sum_j p_{fj} x_{fj} - \sum \delta_k x_{fk}$

$$= \sum_f \sum_j p_{fj} x_{fj} - \delta_k \sum_f x_{fk}$$

$$= \sum_f \sum_j p_{fj} x_{fj} - b_k \delta_k$$

Since $b_k \delta_k$ is constant, the optimal solution $x^*_{fj}$ of problem (10) - (12) will be optimal if $p_{fj}$ is replaced by $p'_{fj}$ in equation (10); also the optimal solution found will be same except for the optimal objective function value decreased by $b_k \delta_k$

4. Traceable Overhead: A Generalized Approach

In this section we will present a generalized approach which can incorporate the identifiable cause and effect relationships between resources and products inherent in traceable overhead and which is also based on the principles of economics and duality. First we develop an allocation alogrithm within the context of the Generalized Transportation model, and then demonstrate how the Kaplan and Thompson [7] model is a specialized case of this approach.

In practical situations, there may be different production processes and different types of machines, shifts, factories, etc. which can produce the same set of products though the machines, laborers, and factories may differ in their relative efficiences. Thus, they share the load of production collectively. Thus we consider a specific resource i which is available in a quantity $b_i$ (though different for different processes), with the per unit cost of $B_i$, so that the total cost is $C(b_i) = B_i b_i$ as given earlier. This total resource $b_i$ is generally available in more than one form, say m in number. Let us consider $b_{fi}$ as the total amount available in form f(or from the $f^{th}$ labor level etc.) such that $b_i = \sum_{f=1}^{m} b_{fi}$

Since there are different forms available, the per unit repuirement of the $j^{th}$ product for this resource may be form dependent. This means that $a_{fij} \neq a_{gij}$ where f and g are any two forms of resource i. There are several explanations for this phenomenon. One can view the $a_{ij}$ defined in K.T. model as an average value of all $a_{fij}$ due to different forms of f. Similarly due to different efficiencies, production priorities, labor differences, shift changes, location of facilities, there may be variable cost of production $c_{fij}$ that can be defined as a function of form f and product j. Thus, for this resource i, one can ascertain variable costs $c_{fij}$ specific to the form of resource i. Thus the constribution margin $p_{fij}$ changes due to changes in $c_{fij}$. A list of cases for the types of different firms in different contexts and their impact on $a_{ji}$, $p_{fi}$, are given in the longer version of this paper [1]. Due to existence of different forms of resource raw materials, labor types, the following four categories arise:

(i) the resource is available only in one form.

(ii) there are many forms with differences due to variable production costs only ($c_{fij}$).

(iii) There are many forms with differences due to the per unit resource requirements ($a_{fij}$).

(iv) there are many forms with differences in both variable production costs and per unit resource requirements.

If the allocation decision warrants case (i), our earlier discussion on traceable overhead is applicable. We will show that for the remaining three cases a Generalized Transportation Allication model is applicable thus generalizing the K.T. Model. We will also show that this model reduces to an ordinary transportation model through an appropriate transformation when case (ii) is applicable.

The original production-mix decision provided us with the optimal planned product mix of $x_1^*, x_2^*, \ldots, x_n^* \geq 0$ based on the LP optimization. It is to be noted that certain of these products may be at zero level. Let us ignore these products since for directly traceable overhead one cannot assign an overhead to a product that is not produced. Then the traceable overhead allocation problem can be solved by a set of generalized transportation problems, GTP, associated with each resource i that is available in different forms. For convenience we will eliminate the subscript i, since the GTP is specific to resource i (for every i available in $m_i \geq 2$ forms). For our general discussion let us consider case (iv) and later derive the procedure for cases (ii) and (iii). Thus, the problem of allocation depends upon the solution to the following GTP (subscript i is eliminated) for the resource i, i = 1,2,...

(13) Maximize $\sum_{f=1}^{m} \sum_{j=1}^{n} p_{fj} x_{fj}$ where $p_{fj} = r_j - c_{fj}$

(14) Such that $\sum_{j=1}^{n} a_{fj} x_{fj} \leq b_f; \; f = 1, \ldots, m$

(15) $\sum_{f=1}^{m} x_{fj} = x_j^*; \; j = 1, \ldots, n$

(16) $x_{fj} \geq 0; \quad f = 1, \ldots, m; \; j = 1, \ldots, n$

It is to be noted that the original optimal production mix is not altered due to the constraint set (15). Further, if in the original L. P. the $b_i$ was fully utilized, then in (14) the constraints will be equalities for each f. On the contrary, if there was slack in $b_i$, then the constraint set (14) is not changed.

Solution procedures for solving such GTP's are available in the literature (see Balachandran and Thompson [2])

There must be a feasible solution since each constraint for resource i was satisfied for the optimal solution $x_1^* \ldots x_j^* \ldots x_n^*$. The optimal solution to (13) - (16) provide us the partitioning of each $x_j^*$ allocated to different forms $x_{fj}^*$ and also the duals $\alpha_f^*$ associated with each form f. It should be noticed that the Kaplan-Thompson model considers the <u>average</u> price $B_i = C(b_i)/b_i$ whereas our allocation decision gives <u>an economic weight</u> to the overhead allocation due to the optimal duals $\alpha_f^*$. This method also produces an optimal dual corresponding to the $j^{th}$ product which we will call $\beta_j^*$. Since $\alpha_f^*$ are related due to the fact that $a_{fj}\alpha_f^* + \beta_j^* \leq d_{fj}$ the form-resource interactions are <u>implicitly</u> considered.

The overhead for each form $H_{fi}$ will be allocated by the following criterion: (note we are putting back the subscript i for resource i)

$$H_{fi} = \left\{ \left[ \frac{\alpha_{fi}^*}{\sum_{f=1}^{m_i} \alpha_{fi}^*} \right] C(b_i) \right\}$$

where $\alpha_{fi}^*$ is the optimal dual for form f of the $i^{th}$ resource.

Thus, the following overhead allocation is made:

Rule 2: (i) For the $i^{th}$ resource, all the units of product j that were produced ($x_j^*$) will collectively bear an overhead of:

$$\sum_{f=1}^{m_i} \left[ \frac{a_{fij} x_{fij}^*}{b_{fi}} \right] H_{fi}; \quad \text{where}$$

$b_i$: amount of the $i^{th}$ resource available/used

$b_{fi}$: amount of the $i^{th}$ resource in the form f available/used

$m_i$: the number of forms f for $i^{th}$ resource so that

$$\sum_{f=1}^{m_i} b_{fi} = b_i$$

$C(b_i)$: the traceable overhead for resource i

$a_{fij}^{*}$: the per unit $i^{th}$ resource requirement in form f for the $j^{th}$ product

$x_{fij}^{*}$: the optimal solution to the GTP

$\alpha_{fi}^{*}$ : the optimal dual of form f for resource i

The same rule is also applicable for case (iii) since the rule is independent of the variable production costs. However, the optimal solution $x_{fij}^{*}$ or $\alpha_{fi}^{*}$ may differ if all the variable production costs are the same and equal to a constant value, e.g. unity. Thus cases (iii) and (iv) are similar and solved utilizing the GTP allocation.

Let us now consider case (ii). In this case we believe that there are no differences in the per unit resource requirement $a_{fij}$. Thus the original $a_{ij}$ are <u>form independent</u>. Thus the earlier problem (13) - (16) for the $i^{th}$ resource reduces to the following (subscript i is again dropped).

(17) Maximize $\sum_{f=1}^{m} \sum_{j=1}^{n} p_{fj}x_{fj}$ where $p_{fj} = r_j - c_{fj}$

(18) Subject to $\sum_{j=1}^{n} a_j x_{fj} \leq b_f$ ; $f = 1, \ldots, m$

(19) $\sum_{f=1}^{m} x_{fj} = x_j^{*}$; $j = 1, \ldots, n$

(20) $x_{fj} \geq 0$ ; $f = 1, \ldots, m$; $j = 1, \ldots, n$

This GTP can be shown to be equivalent to the following <u>ordinary transportation problem</u> due to the transformation that:

(21) $y_{fj} = a_j x_{fj}$ or $x_{fj} = \dfrac{y_{fj}}{a_j}$

Thus the equivalent transportation problem is

(22) Maximize $\sum_{f=1}^{m} \sum_{j=1}^{n} p'_{fj} y_{fj}$

(23) s.t. $\sum_{j=1}^{n} y_{fj} \leq b_f$ ; $f = 1, \ldots, m$

(24) $$\sum_{f=1}^{m} y_{fj} = y_j^* \; ; \; j = 1, \ldots, n$$

(25) $$y_{fj} \geq 0 \; ; \; f = 1, \ldots, m \; ; \quad j = 1, \ldots, n$$

(26) where $$y_j^* = a_j x_j^* \quad \text{and} \quad p'_{fj} = - \frac{p_{fj}}{a_j}$$

Rule (2 ii): All units of product j produced ($x_j^*$) using the $i^{th}$ resource will collectively bear an overhead of:

$$\sum_{f=1}^{m_i} \frac{y_{fij}^{**}}{b_{fi}} \; \frac{\alpha_{fi}^*}{\sum_{f=1}^{m_i} \alpha_{fi}^*} \; . \; C(b_i)$$

where $y_{fij}^*$ is the optimal solution to the TP (22-26) and $\alpha_{fi}$ is the optimal dual for form f of the $i^{th}$ resource and everything else the same as defined earlier.[2]

Example 1:

We consider the general problem where differences in both per unit resource requirements for the various forms (e.g. $a_{fij} \neq a_{gij}$) and variable production costs $c_{fj}$ for forms of a resource (e.g. $c_{fij} \neq c_{kij}$). This is really a combination of case (ii) and case (iii) or case (iv). (Again in this example $r_j$'s are the same so that the problem on Max of total profit is equivalent to minimization of total variable costs.

Consider the original two resource, two product overhead allocation example as given by Kaplan and Thompson, given on page 4 of this paper

Let us again assume that resource 1 which is available in 12 units is available in the form of two machines (two forms), f = 1 or 2, with $b_{11} = 7$ and $b_{21} = 5$. The main difference in this example as compared to the earlier example (2) is that the technological coefficients are not the same for both forms, and variable production costs also differ. Thus the problem is: (after dropping the resource # subscript)

$$\begin{array}{llllllll}
\text{Min} & 1x_{11} & + & 2x_{12} & + & 1x_{21} & + & 1x_{22} \\
\text{s.t.} & 2.5x_{11} & + & 2.25x_{12} & & & & = 7 \text{ (form 1)} \\
 & & & & & 3.25x_{21} & + 1.75x_{22} & = 5 \text{ (form 2)} \\
 & x_{11} & & & + & x_{21} & & = 2 \ (x_1^{*} = 2) \\
 & & & x_{12} & & & + \ x_{22} & = 3 \ (x_2^{*} = 3) \\
 & & & & & & & \text{all } x_{ij} \geq 0
\end{array}$$

Using the algorithm provided by Balachandran and Thompson [2] the optimal solution is $x_{11}^{*} = 1$, $x_{12}^{*} = 2$, $x_{21}^{*} = 1$, $x_{22}^{*} = 1$. The optimal duals (unique) for the forms (rows) are: $\alpha_1^{*} = -140/423$, $\alpha_2^{*} = \dfrac{-1400}{5499} \sim \dfrac{-14}{55}$ and the optimal duals for the products (columns) are $\beta_1^{*} = 773/423$ and $\beta_2^{*} = \dfrac{7949}{5499} \sim \dfrac{159}{110}$ are calculated in the following tableau for resource 1.

PRODUCTS

| Duals for Forms $\alpha_f$ \ Duals for Products $\beta_j$ | | $\beta_1^{*} = \frac{773}{423}$ <br> 1 | $\beta_2^{*} = \frac{7949}{5499}$ <br> 2 | slack | capacity |
|---|---|---|---|---|---|
| $\alpha_{11}^{*} = \frac{-140}{433}$ | 1 | 2.5 <br> 1 <br> 1 | 2.25 <br> 2 <br> .5 | 1 <br> 0 | 7 |
| $\alpha_{21}^{*} = \frac{-1400}{5499}$ | 2 | 3.25 <br> 1 <br> 1 | 1.75 <br> 1 <br> 1 | 1 <br> 0 | 5 |
| $\alpha_{31}^{*} = 0$ | | 1 <br> M | 1 <br> M | 1 <br> 100 <br> 0 | 100 |
| # of products | | 2 | 3 | | |

FORMS

with each cell containing the following information

| $e_{ij}$ | $x_{ij}$ |
|---|---|
| | $d_{ij}$ |

The traceable overhead is then allocated to individual forms according to Rule 2: (iii):

Rule 2: (iii) Form $f_i$ used to produce product $x_j$ will be assigned its portion $(H)_{fi}$ of the $i^{th}$ resource's traceable overhead $C(b_i)$ as follows:

$$(H)_{fi} = \frac{\alpha_{fi}^*}{\sum_{f=1}^{m} \alpha_{fi}} \quad C(b_i)$$

Thus using the optimal duals $\omega_{fi}^*$ as weights which take the product form interaction into consideration the total traceable overhead of \$1.20 is distributed to the forms as follows:

Form 1 is assigned for its 7 units of utilized capacity

$$\frac{\alpha_{11}^*}{\alpha_{11}^* + \alpha_{21}^*} \quad C(b_i) = \frac{-140/433}{-6199} \quad (\$1.20) \qquad = \$0.626$$

Form 2 is assigned for its 5 units of utilized capacity

$$\frac{\alpha_{11}^*}{\alpha_{11}^* + \alpha_{21}^*} \quad C(b_i) = \frac{-1400/5499}{6199} \quad (\$1.20 \qquad = \$0.574$$

Total traceable cost of Resource #1 to Account For = $\underline{\underline{\$1.200}}$

Now we can allocate the \$1.20 of overhead traceable to production according to the following restated rule:

Rule 2(iv): Each unit of product j produced using form fi (of resource i), $x_{fj}^*$, is assigned a per unit overhead of

$$\frac{a_{fij}}{b_{fij}} \quad (H)_{fi}$$

where $a_{fij}$ is the per unit utilization of form f of resource i used to produce product j, $(H)_{fi}$ is the overhead assigned using rule 2 (iii) and $b_{fi}$ is the capacity of form fi

Product 1 (2 units produced)

| | |
|---|---|
| 1 unit with machine 1 [(1)(2.5)/7] [\$0.626] = | \$0.2236 |
| 1 unit with machine 2 [(1)(3.25)/5] [\$0.574] = | 0.3731 |
| Total Traceable Cost allocated to Product 1's Production | 0.5967 |

Product 2 (3 units produced)

| | |
|---|---|
| 2 units with machine 1 [2(2.25)/7] [\$0.626] = | \$0.4024 |
| 1 unit with machine 2 [1(1.75)/5] [\$0.574] = | 0.2009 |
| Total Traceable Cost allocated to Product 2's Production | 0.6033 |
| Total traceable overhead to Account For | \$1.2000 |

5. AVOIDABLE OR ESCAPABLE OVERHEAD:

Situations arise where overhead charges can be escaped entirely if production of certain products, or utilization of particular facilities or resources are avoided. For example, setup costs are avoided if particular equipment is not used or the product requiring the setup is not produced.

Kaplan and Thompson have shown that this case can be analyzed through a two stage process. In the first stage they formulate the given linear programming problem as a "fixed charge problem" and use zero one decision variables for every product or resource that can lead to a possible fixed charge ($F_i$ if the $j^{th}$ product is produced and $G_i$ if the $i^{th}$ resource is used). This fixed charge problem is shown to be equivalent to a mixed integer problem which is solved to yield the optimal $x^*$. For every j, where $x_j^* = 0$, the original problem is reduced by dropping the $p_j$ and the associated vector of technological coefficients, $\bar{a}_j$. Also the corresponding resource constraint is dropped if the resource is not utilized. In the second stage they solve this reduced linear programming problem to get the duals $w^*$ for those resources that are utilized together with the same $x^*$ of the earlier fixed charge problem. They then provide the following rule which is quoted assuming the fixed charge as $F_j$ (for the product j if product $x_j^* > 0$).

> "... In order to assign avoidable cost $F_j$ associated with the positive production of product j, treat as much as possible of F as a variable cost,..." ([7] p. 363)

The above condition of "as much as possible of $F_j$" was operationalized as being that much of $F_j$ so that the original optimal solution $x^*$ is not distorted when the original objective function "$\Sigma p_j x_j$" is replaced by "$\Sigma(p_j - \frac{F_j}{x_j^*})x_j$".

Before we proceed to give our allocation rule for avoidable overhead certain comments are in order.

First, Kaplan and Thompson didn't provide a constructive procedure for finding the maximum extent of $F_j$ that can be allocated to product j and still maintain the original product in mix, $x^*$. (This maximum extent is defined as $F_j'$). We will provide a procedure to determine this value which is based upon the sensitivity analysis of the contribution margin vector.

Second, if the entire fixed charge, $F_j$, cannot be absorbed by the $x_j^*$ produced, Kaplan and Thompson state that all the extra charges that are unallocated by their Rule 3 should go to the "common overhead pool" which uses Rule 1. Our main concern is when $F_j$, an avoidable overhead, is directly traceable to produce $x_j^*$ it is incorrect from a profit maximization standpoint to allocate the remainder $(F_j - F_j')$ to the other products $x_k$ where $k \neq j$. Further if the maximum extent that can be allocated for product j is only $F_j'$ (when $F_j' < F_j$) then any positive allocation for the common overhead pool H to product j after $F_j'$ is allocated will certainly distort the optimal product mix $x^*$. In fact, the economic decision (assuming away all interdependencies with other products $x_k$

for $\forall\ k \neq j$) is <u>not</u> to produce $x_j$ and avoid the overhead $F_j$. However, Kaplan and Thompson ([7], p. 364) state that "if escapable overhead is too high, it may lower this contribution to the point that it no longer seems profitable to generate additional sales by producing an unprofitable product. To avoid this error, we impose the constraint that escapable overhead should be directly allocated to a product only up to the point where it does not distort the optimal solution."

Third, consider the situation of a linear program in the second stage (after removing all $x_j^* = 0$) where we have $n'$ products all produced at positive levels. For this to happen the number of resources (after elimination of all totally unused resources) say $m' > n'$, this implies $(m' - n')$ number of resources will have unused capacity thus providing corresponding duals to be zero. It <u>is not justifiable to pool all the unassigned avoidable overhead and co-mingle it with the unavoidable</u> overhead which is assigned under earlier rules (i.e., rules 1 or 2) utilizing the duals, since overhead due to resources <u>whose duals are zero are not allocated as per Kaplan and Thompson [7] duals</u>.

As stated earlier, our purpose is to present an allocation system which provides management information concerning the cost and profitability of products which at the same time would lead management to choose a production mix which optimizes the profitability of the firm. Therefore an accounting information system which highlights products which are not covering their avoidable (escapable) cost and also indicates a per unit deficit helps managers evaluate pricing decisions as well as directs their attention to products where cost "shaving" may be necessary.

Therefore we can make the following observation. If $F_j$ is the fixed charge that is due to avoidable overhead for product j with $x_j^*$, the optimal solution from stage one (the planning stage), then based upon the sensitivity of the $\bar{p}$ vector charges, the optimal basis will change or $x^*$ will be perturbed whenever the amount subtracted from $p_j$, for every j, reduces $p_j$ beyond its lower limit that perserves the same basis obtained from sensitivity analysis.

Since we are subtracting the avoidable overhead of $F_j/x_j^* = \rho_j$ from each $p_j$, let us first find the maximum proportion of $\rho_j$ that can be absorbed. Let us define $\theta_c$ as this proportion. If a $\theta_c \geq 1$ is obtained from parametric programming then we can conclude that the entire avoidable overhead can be absorbed. Conversely, if $\theta_c < 1$, the total avoidable overhead cannot be absorbed without perturbing the product mix $x^*$. This indicates <u>an out of pocket loss</u> will occur unless the contribution margin is increased either by increasing selling price or decreasing variable or avoidable production cost. As alluded to earlier the only possible reason for producing $x_j$ when $\theta_c < 1$ is due to an interdependency with another product which when sold produces a contribution to profit which offset the out of the pocket loss of producing $x_j$.

The information needed for the accounting information system can be obtained by addressing the objective function alone. We know $x^* = \{x_j^*\}$ is primal feasible. Let p be the original contribution vector and $\rho$ be the vector $\{\rho_j = F_j/x_j^*\}$ of the avoidable overhead to be absorbed (where every $\rho_j \geq 0$). Let the new $p' = p - \theta\rho$. When $\theta$ varies from 0, the original optimal solution $x^* = \{x^*_j\}$ associated with that basis matrix B, remains primal feasible but may cease to be optimal (i.e., may cease to be dual feasible). The <u>relative</u> contribution vector [see Hadley [4]].

$$(z_j - p_j) \text{ New} = p_B' B^{-1} a_j - p_j'$$
$$= (p_B - \theta_{\rho_B})\, B^{-1} a_j - (p_j - \theta\rho_j)$$

where $p_B$ or $\rho_B$ is the subvector corresponding to those variable contribution margins or avoidable overhead for $j^{th}$ product where $x_j^* > 0$ (i.e., the basic variables)

Therefore $(z_j - p_j)_{New} = (z_j - p_j)_{old} - \theta(\xi_j - \rho_j)$

where $\xi_j = \rho_B B^{-1} a_j$.

Since $\theta \geq 0$, the quantity $(z_j - p_j)_{New} \geq 0$ for all $\theta \geq 0$ for all $\theta \geq 0$, if and only if $(\xi_j - p_j) \leq 0$ for every j. This is obvious since $(z_j - p_j) = 0$ for product $x_j^* > 0$.

When $\theta = 1$, then $(z_j - p_j) \geq 0$ implying that the entire avoidable overhead can be absorbed by product $x_j$. Moreover, if there exists at least one $\xi_j = \rho_B B^{-1} a_j$ where $(\xi_j - \rho_j) > 0$ then there exists a critical value of $\theta \geq 0$ beyond which B ceases to be an optimal basis (i.e., $x^*$ is distorted). We will call this critical value $\theta_c$ which is defined below.

For every j such that $(\xi_j - \rho) > 0$, define $\theta_j = \dfrac{(z_j - p_j)}{(\xi_j - \rho_j)}$ old

Then $\theta_c = \min\limits_{j|(\xi_j - \rho_j) > 0} \theta_j$

Now when $\theta_c \geq 1$, we can absorb the entire avoidable overhead into each contribution margin without distorting $x^*$. However, if $0 \leq \theta_c < 1$ the following cases arise:

(i) For every $\theta_j \geq 1$, we can absorb the entire avoidable overhead $F_j/x^*$ can be completely absorbed.

(ii) For every product j when $\theta_c \leq \theta_j < 1$ then $(\theta_j F_j)/x_j^*$ amount of the per unit avoidable overhead can be absorbed.

The unabsorbed quantity of the avoidable overhead may be recouped either by increasing selling price, decreasing variable cost of those product(s) j where $\theta_c \leq \theta_j \leq 1$. The per unit contribution margin for product j must be raised to $p_{jNew}$ from its original value $p_j$. The new value of $p_{jNew} = p_j + (1 - \theta_j) F_j/x_j^*$ will maintain the original product mix of $x^*$.

We can now formulate allocation Rule 4 under conditions of breakeven and loss.

Rule 5: If $F_j$ is the fixed charge that is due to avoidable overhead for product j with $x_j^*$ the optimal solution from stage one then the overhead is allocated as follows:

(a) If $\theta_c \geq 1$ allocate $F_j/x_j^*$ to product j

(b) If $0 \leq \theta_c < 1$ allocate the entire per unit avoidable overhead as below.

(i) For every $\theta_j \geq 1$ allocate the entire per unit avoidable overhead $F_j/x_j^*$

(ii) For every product j where $\theta_c \leq \theta_j < 1$ then assign $(\theta_j F_j)/x_j^*$ part of the per unit avoidable overhead to each unit of j; and if this product j where $\theta_j < 1$ is produced due to complementary product k whose $\theta_k > 1$ then assign the remaining amount $(1 - \theta_j)F_j$ to the product k if $(\theta_k - 1)F_k \geq (1 - \theta_j)F_j$. Else, either the contribution margin $p_j$ of product j is increased to absorb the excess avoidable overhead, or production for product j is not initiated. If $\theta_k < 1$, then the profit margin for both j and k has to be increased or their production is not initiated.

Footnotes

1. Many defense contracts permit a firm to allocate manufacturing overhead as well as general administrative expenses to the contracting agency for reimbursement. The revenues of health care institutions to a large extent are dependent upon allocations under Medicare and Medicaid.

2. It is to be noted that in any transportation problem, the duals for the forms (row) $\alpha_{fi}$ and those for the products (columns) $\beta_j$ form a one-parameter family [4]. In other words for any arbitrary constant $\delta$, $\alpha_{fi} + \delta$ and $\beta_j - \delta$ are all duals. It is customary [4] to arbitrarily specify $\alpha_{1i} = 0$ initially and solve for the rest of duals from the known basis utilizing the relationship $\alpha_{fi} + \beta_j = p_{fij}$ where (fi,j) is a cell in the given basis. In our overhead allocation problem, it is economically meaningful to find the economic weights $H_{fi}$ as a ratio of the dual of form $f_j$ to the entire duals sum $\Sigma\, a_{fi}$. If we set any dual, say $\alpha_{fi} = 0$, that doesn't make sense. So, in a relative sense we will choose that form $f_j$ which is used to the maximum and set its dual $\alpha_{fi} = 1$. With this convention we obtain the relative importance of all other forms by evaluating their respective duals sequentially. It is to be reiterated that $H_{fi}$ can have different values for different $\delta$. We set them unique by making that $\alpha_{fi} = 1$ where f is that form of the $i^{th}$ resource that is used to a maximum. This problem does not arise in the Generalized Transportation Problem since all the duals are uniquely determined.

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