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THE ORIGIN OF CYCLING IN DYNAMIC ECONOMIC MODELS ARISING FROM MAXIMISING BEHAVIOUR*

BY

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I. INTRODUCTION

Recent research[1] on the behaviour of the dynamical processes that arise from the maximising behaviour of economic agents, has centered around the problem of characterising the stability or instability of equilibria. No attempt has been made to characterise more precisely the qualitative behaviour of the trajectories in a neighborhood of a stable equilibrium point.[2] This paper takes a step in this direction by presenting conditions under which the trajectories in a neighborhood of a stable equilibrium point exhibit cyclical behaviour.

These results may be viewed as part of a broader attempt to understand the forces that lead to cyclical behaviour in dynamic economic models. In a recent paper [14] I showed how the introduction of uncertainty leads to cycling, when cycling is absent in a purely deterministic model. But we can also seek to understand the sources of cyclical behaviour in a purely deterministic environment. Instead of the system as a whole interacting with an uncertain environment, the separate sectors or commodities that form the components of the system, interact with one another to cause cycling. In short, interaction between sectors or commodities can cause cycling.

In the classical investigations of Lotka [8] and Volterra [24] on the evolution of interacting, predator-prey biological species, it is the presence of a fundamental skew-symmetric matrix arising from the predator-prey interactions, that gives rise to the cyclical behaviour in the number of individuals of each species.[3] I will show that a similar skew-symmetric matrix is present in a large class of dynamic economic models arising from maximising behaviour and that it is to the presence of this skew-symmetric matrix that the origin of cycling may be traced. This skew-symmetric matrix arises from certain asymmetric interactions induced by the process of investment.

The basic class of maximum problems considered is outlined in section II. The problem is then transformed into a normal form in which the forces that give rise to cycling stand out with especial clarity (section III). This normal form is used to present a precise characterisation of cycling in the two-dimensional case, as well as a sufficient condition for cycling in the n-dimensional case (section IV). These results are used to throw light on the forces that lead to cycling in a rational expectations equilibrium for a competitive industry (section V). These latter results may be viewed as part of a preliminary attempt to develop a theory of the business cycle based on the theory of resource allocation, that meets the two important tenets proposed by Lucas [9, 11]: first, that the sequence of prices and quantities be determined through a process of competitive equilibrium and second, that the expectations of agents be rational, in the sense that the anticipated sequence of prices formed on the basis of their expectations, coincides with the actual sequence of prices generated on the markets by their maximising behaviour.

II. THE BASIC MAXIMUM PROBLEM

In order to present the problem in its simplest form I shall consider the following class of undiscounted extremum problems. The objective is to find a continuous path $k(t) \in R^n$ which maximises the undiscounted stream of future profits (welfare)

$$\int_0^\infty L\big(k(t), \dot{k}(t)\big)\,dt \qquad (\mathcal{P})$$

where $L(k,\dot{k}) \in C^r$, $r \geq 2$ is a real-valued strictly concave function in $(k,\dot{k})$. The Euler-Lagrange equations for $(\mathcal{P})$ are given by

$$L_{\dot{k}\dot{k}}\ddot{k} + L_{\dot{k}k}\dot{k} - L_k = 0 \tag{1}$$

ASSUMPTION I (Existence of Equilibrium Point). There exists $k^* \in \overset{\circ}{R}{}^{n+}$, where $\overset{\circ}{R}{}^{n+}$ denotes the interior of the non-negative orthant satisfying

$$L_k(k^*,0) = 0 \tag{2}$$

k^* is an equilibrium point $(\ddot{k}^* = \dot{k}^* = 0)$ for the Euler-Lagrange equations (1). In view of the strict concavity of $L(k,\dot{k})$, k^* is unique. We introduce local coordinates around the equilibrium point k^*

$$x = k - k^*$$

and consider the following quadratic form

$$L^\circ(x,\dot{x}) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}' \begin{bmatrix} L^*_{kk} & L^*_{k\dot{k}} \\ L^*_{\dot{k}k} & L^*_{\dot{k}\dot{k}} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

where the asterisk denotes evaluation at k^* so that $L^*_{kk} = L_{kk}(k^*,0)$, for which we make the following assumption

ASSUMPTION II (Strong Concavity at Equilibrium Point). The quadratic form $L^\circ(x,\dot{x})$ is negative definite.

In a neighborhood of k^* paths which are solutions of $(\mathcal{P})$ minimise

$$-\int_0^\infty L^\circ\big(x(t), \dot{x}(t)\big)dt \tag{$\mathcal{L}$}$$

The Euler-Lagrange equations for $(\mathcal{L})$ are just the linearised Euler-Lagrange equations for (1) around the equilibrium point k^* and are given by

$$L^*_{\dot{k}\dot{k}}\ddot{x} + (L^*_{\dot{k}k} - L^*_{k\dot{k}})\dot{x} - L^*_{kk}x = 0 \tag{3}$$

It is well-known that under Assumptions I and II for each initial condition in a neighborhood of k^* a unique solution of $(\mathcal{P})$ exists which is locally asymptotically stable. This paper extends our understanding of the qualitative behaviour of the solutions of $(\mathcal{P})$ in a neighborhood of k^* beyond the property of stability, by a more precise analysis of the qualitative behaviour of the solutions of $(\mathcal{L})$. This in turn is undertaken by an analysis of the linearised equations (3).

DEFINITION. $(\mathcal{L})$ is called a symmetric (nonsymmetric) variational problem if

$$L^*_{\dot{k}k} - L^*_{k\dot{k}} = 0 \ (\neq 0)$$

Remark. For a nonsymmetric variational problem since $L^*_{\dot{k}k} = L^{*\prime}_{k\dot{k}}$ we have

$$(L^*_{\dot{k}k} - L^*_{k\dot{k}})' = -(L^*_{\dot{k}k} - L^*_{k\dot{k}}) \tag{4}$$

so that the matrix $(L^*_{\dot{k}k} - L^*_{k\dot{k}})$ in (3) is a skew-symmetric matrix.

DEFINITION. The variational problem $(\mathcal{L})$ is said to exhibit cycling if the characteristic polynomial

$$D(\lambda) = |L^*_{\dot{k}\dot{k}}\lambda^2 + (L^*_{\dot{k}k} - L^*_{k\dot{k}})\lambda - L^*_{kk}| = 0 \tag{5}$$

has at least one pair of complex conjugate roots.

Remark. If $(\mathcal{L})$ is a symmetric variational problem then there is no cycling.[4]

The main object of this paper is to find conditions under which cycling arises in the variational problem $(\mathcal{L})$.

III. TRANSFORMATION TO NORMAL FORM

To simplify the notation in the analysis that follows we let

$$A = -L^*_{kk}, \qquad B = -L^*_{\dot{k}\dot{k}}$$

$$N = -L^*_{k\dot{k}}, \qquad C = N - N'$$

so that (3) reduces to

$$B\ddot{x} - C\dot{x} - Ax = 0 \tag{6}$$

where A and B are positive definite in view of Assumption II and C is skew-symmetric.

DEFINITION. We say that α_i is an eigenvalue of A in the metric of B and $w^i \varepsilon R^n$, $w^i \neq 0$ is an associated eigenvector if $Aw^i = \alpha_i Bw^i$.

Let $\alpha_1,\ldots,\alpha_n$ and $w^1,\ldots,w^n$ denote the eigenvalues and associated eigenvectors of A in the metric of B. $\alpha_1,\ldots,\alpha_n$ may also be called the curvature coefficients of A since they provide measures of the curvature of A in the directions $w^1,\ldots,w^n$, in the metric induced by B.

Remark. If ($\mathcal{L}$) is a symmetric variational problem then (5) implies that the eigenvalues of A in the metric of B are related to the eigenvalues of the linearised equations (6) in the following way

$$\sqrt{\alpha_i} = \lambda_i, \qquad i = 1,\ldots,n$$

Remark. $\alpha_1,\ldots,\alpha_n$, $w^1,\ldots,w^n$ may be characterised as the solutions of the following sequence of extremum problems [7, pp. 317-320]

$$\alpha_i = \max_{x \varepsilon L_i} \frac{x'Ax}{x'Bx} = \frac{w^{i'}Aw^i}{w^{i'}Bw^i} \qquad i = 1,\ldots,n$$

$$L_i = \{x \varepsilon R^n,\ x \neq 0 \mid x'Bw^j = 0,\quad j = 1,\ldots,i-1\}$$

We recall from the theory of pencils of quadratic forms [7, pp. 310-312] that the $n \times n$ matrix of eigenvectors $W = [w^1 \ldots w^n]$, where $w^1, \ldots, w^n$ denote n column vectors, may be chosen in such a way that

$$W'BW = I, \quad W'AW = \mathcal{A} \tag{7}$$

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix}$$

The nonsingular transformation to principal coordinates $y = (y_1, \ldots, y_n)$

$$x = Wy$$

reduces the linearised equations (6) to the normal form

$$W'BW\ddot{y} - W'CW\dot{y} - W'AWy = \ddot{y} - \Gamma\dot{y} - \mathcal{A}y = 0 \tag{8}$$

where

$$W'CW = \Gamma = \begin{bmatrix} 0 & \gamma_{12} & \cdots & \gamma_{1n} \\ -\gamma_{12} & 0 & \cdots & \gamma_{2n} \\ \vdots & \vdots & & \vdots \\ -\gamma_{1n} & -\gamma_{2n} & \cdots & 0 \end{bmatrix} \tag{9}$$

Remark. If $(\mathcal{L})$ is a symmetric variational problem then the linearised equations in principal coordinates (8) separate into n independent one-sector systems.

Remark. If $(\mathcal{L})$ is a nonsymmetric variational problem then the linearised equations in principal coordinates no longer separate into independent one-sector systems. There is in fact a skew-symmetric interaction between the n sectors. The term $\Gamma\dot{y}$ imposes velocity dependent rotational forces on the system which, under conditions to be examined in the next section, lead to cycling. In the symmetric case no rotational forces act on the system so that no cycling can arise.

Remark. No artificial introduction of rotational forces is involved here: the skew-symmetric forces $\Gamma\dot{y}$ arise naturally from the structure of the extremum problem and are present in extremum problems of a quite general form. These skew-symmetric forces may be viewed as the cause of cycling for the class of maximising problems considered in this paper.

IV. CHARACTERISATION OF CYCLING

The principal result of this section is a complete characterisation of cycling for the two-dimensional case. There is however one n-dimensional case where sufficient conditions for cycling are readily established.

PROPOSITION 1. If the variational problem ($\mathcal{L}$) is nonsymmetric and if $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha^*$ then there is cycling.

Proof. Consider the eigenvalue problem induced by the dynamical equations (8)

$$\left(\lambda^2 I - \Gamma\lambda - \mathcal{A}\right)v = 0 \tag{10}$$

The assumption $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha^*$ implies that $\mathcal{A} = \alpha^* I$ so that (10) reduces to an eigenvalue problem for Γ

$$\left[\Gamma - \left(\frac{\lambda^2 - \alpha^*}{\lambda}\right)I\right]v = 0 \tag{11}$$

Since Γ is a real skew-symmetric matrix its eigenvalues are pure imaginary

$$\pm i\gamma_1, \ldots, \pm i\gamma_k, 0, \ldots, 0 \tag{12}$$

where 2k are pure imaginary with non-zero frequencies $\gamma_1, \ldots, \gamma_k$ and $n - 2k$ are zero [7, p. 285]. Let $\lambda = \mu + i\nu$ then (11) implies that for each eigenvalue in

(12) induced by γ_j

$$\frac{\lambda^2 - \alpha^*}{\lambda} = i\gamma_j$$

so that $$\mu = \pm\sqrt{\alpha^* - \left(\frac{\gamma_j}{2}\right)^2}, \qquad \nu = \left(\frac{\gamma_j}{2}\right)$$

The eigenvalues of (10) are thus given by

$$\pm\sqrt{\alpha^* - \left(\frac{\gamma_1}{2}\right)^2} \pm i\left(\frac{\gamma_1}{2}\right), \ldots, \pm\sqrt{\alpha^* - \left(\frac{\gamma_k}{2}\right)^2} \pm i\left(\frac{\gamma_k}{2}\right), \pm\sqrt{\alpha^*}, \ldots, \pm\sqrt{\alpha^*} \qquad \Delta$$

Consider the two-dimensional case. In this case the linearised Euler-Lagrange equations (8) reduce to

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} - \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} - \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \tag{13}$$

for which the characteristic polynomial is

$$D(\lambda) = \lambda^4 + (\gamma^2 - \alpha_1 - \alpha_2)\lambda^2 + \alpha_1\alpha_2 = 0 \tag{14}$$

The four eigenvalues of the dynamical system (13) are thus[5]

$$\lambda = \frac{\pm\sqrt{-H} \pm \sqrt{-J}}{2} \tag{15}$$

where $$H = \gamma^2 - \left(\sqrt{\alpha_1} + \sqrt{\alpha_2}\right)^2$$

$$J = \gamma^2 - \left(\sqrt{\alpha_1} - \sqrt{\alpha_2}\right)^2$$

(15) leads at once to the following

PROPOSITION 2 (Characterisation of cycling for $n = 2$). <u>Under Assumption II, if</u>

n = 2 the variational problem ($\mathcal{L}$) exhibits cycling if and only if

$$|\gamma| > |\sqrt{\alpha_1} - \sqrt{\alpha_2}| \tag{16}$$

Proof. Assumption II implies $\sqrt{\alpha_1} > 0$, $\sqrt{\alpha_2} > 0$ so that (15) leads to three distinct regions in the parameter space $(\sqrt{\alpha_1}, \sqrt{\alpha_2})$ as shown in Figure 1

$$\mathcal{A}_1 = \{(\sqrt{\alpha_1}, \sqrt{\alpha_2}) \in \mathring{R}^{2+} \mid H < 0, \quad J < 0\}$$

$$\mathcal{A}_2 = \{(\sqrt{\alpha_1}, \sqrt{\alpha_2}) \in \mathring{R}^{2+} \mid H < 0, \quad J > 0\}$$

$$\mathcal{A}_3 = \{(\sqrt{\alpha_1}, \sqrt{\alpha_2}) \in \mathring{R}^{2+} \mid H > 0, \quad J > 0\}$$

The eigenvalues are real in $\mathcal{A}_1$ and complex in $\mathcal{A}_2$ and $\mathcal{A}_3$. Δ

Remark. Assumption II precludes purely imaginary eigenvalues. Thus, under Assumption II, only parameter values in $\mathcal{A}_1$ and $\mathcal{A}_2$ are feasible.

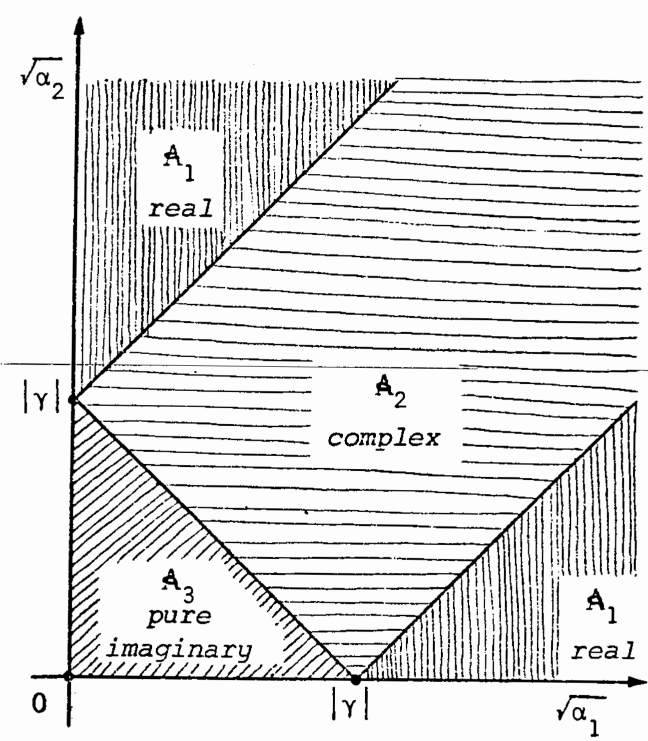

Figure 1. The regions of real and complex eigenvalues in the parameter space $(\sqrt{\alpha_1}, \sqrt{\alpha_2})$.

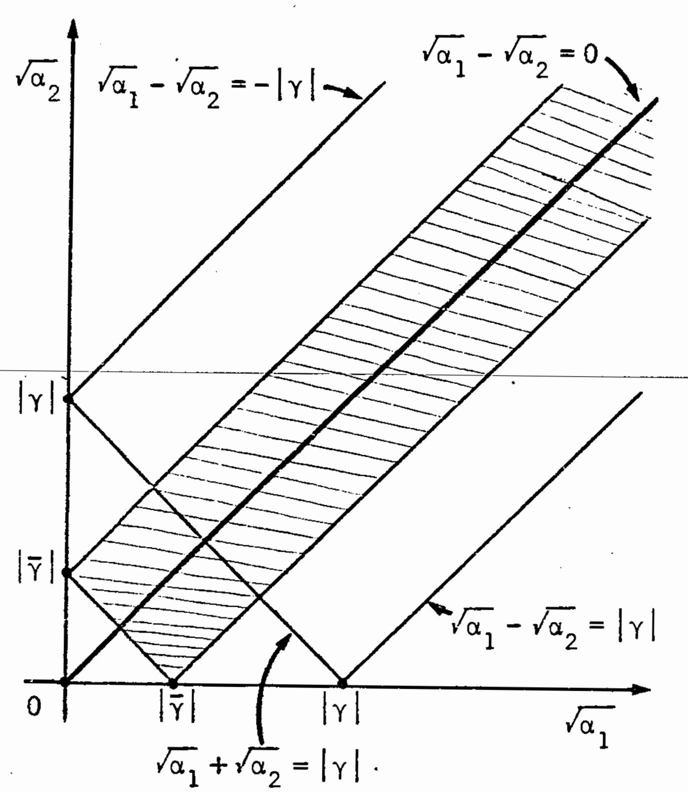

Figure 2. Effect of reduction of skew-symmetry from $|\gamma|$ to $|\bar{\gamma}|$ on the region of complex eigenvalues.

Remark. Since the trajectory which minimises ($\mathcal{L}$) is asymptotically stable the cyclical motion is damped and the two of the four eigenvalues (15) which characterise the trajectory have negative real parts. It is of some interest to know how a change in the magnitude of the skew-symmetry $|\gamma|$ affects the real parts and hence the magnitude of the damping. In a similar way we may ask how a change in skew-symmetry affects the imaginary parts and hence the period of the cycles.

PROPOSITION 3. Along the trajectory which minimises ($\mathcal{L}$), an increase in the magnitude of the skew-symmetry $|\gamma|$ leads to (i) a reduction in the exponential damping and (ii) a reduction in the period of the cycles.

Proof. If the eigenvalues are complex then the two eigenvalues which characterise the trajectory which minimises ($\mathcal{L}$) are given by $\lambda = \mu \pm i\nu$ where

$$\mu = -\frac{1}{2}\sqrt{\left(\sqrt{\alpha_1}+\sqrt{\alpha_2}\right)^2 - \gamma^2}, \qquad \nu = \frac{1}{2}\sqrt{\gamma^2 - \left(\sqrt{\alpha_1}-\sqrt{\alpha_2}\right)^2}$$

from which (i) is immediate and (ii) follows by recalling that if ν is the frequency of cycling then the period of each cycle is $\left(\frac{2\pi}{\nu}\right)$. Δ

Remark. To complete the solution of the problem for the two-dimensional case we need to relate the derived parameters $(\alpha_1, \alpha_2; \gamma)$ to the original matrices $(A, B; C)$. To this end we write

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = -\begin{bmatrix} L^*_{k_1k_1} & L^*_{k_1k_2} \\ L^*_{k_2k_1} & L^*_{k_2k_2} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} = -\begin{bmatrix} L^*_{\dot{k}_1\dot{k}_1} & L^*_{\dot{k}_1\dot{k}_2} \\ L^*_{\dot{k}_2\dot{k}_1} & L^*_{\dot{k}_2\dot{k}_2} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix}, \quad g = \left(L^*_{k_2\dot{k}_1} - L^*_{k_1\dot{k}_2}\right)$$

By (7), $$|W'BW| = |W'||B||W| = |W|^2|B| = 1$$

so that $$|W| = \frac{1}{\sqrt{|B|}}$$

Thus by (9), $$\gamma = |W|g = \frac{g}{\sqrt{|B|}}$$

The eigenvalues of A in the metric of B are given by

$$\alpha = \frac{\frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 - |A||B|}}{|B|}, \qquad \Delta = a_{11}b_{22} + a_{22}b_{11} - 2a_{12}b_{12}$$

so that the basic cycling condition (16) becomes

$$|g| > \left|\Delta - 2\sqrt{|A||B|}\right| \tag{17}$$

which reduces to the following especially simple condition when $a_{12} = b_{12} = 0$

$$|g| > \left|\sqrt{a_{11}b_{22}} - \sqrt{a_{22}b_{11}}\right|$$

or in terms of the quadratic form $L^{\circ}(x,\dot{x})$

$$\left|L^*_{k_2\dot{k}_1} - L^*_{k_1\dot{k}_2}\right| > \left|\sqrt{L^*_{k_1k_1}L^*_{\dot{k}_2\dot{k}_2}} - \sqrt{L^*_{k_2k_2}L^*_{\dot{k}_1\dot{k}_1}}\right| \tag{18}$$

V. CYCLING IN RATIONAL EXPECTATIONS EQUILIBRIUM

In several recent contributions [9, 11] Lucas has emphasised the importance of developing a theory of the business cycle in which prices and quantities are determined at each instant of time through <u>competitive equilibrium</u> and in which the expectations of agents are <u>rational</u> in the sense of Muth [18]. Lucas has also emphasised the role of <u>uncertainty</u> in generating the observed pattern of business cycles.

In this section I will use the results of the previous sections to examine a rational expectations equilibrium for a competitive industry with a fixed finite number of firms in which each firm behaves according to the standard Lucas-Mortensen adjustment cost theory of the firm [10, 19]. The analysis of rational expectations for the industry is made possible by the introduction of an extended integrand similar to that employed by Brock [1], Brock-Magill [2] and Scheinkman [23] and originally introduced by Lucas and Prescott [12].

While the presence of uncertainty is of undisputed importance in generating the observed pattern of business cycles, it may well be of interest to seek causes of cycling which are independent of the presence of uncertainty but which are consistent with the postulates of competitive equilibrium and rational expectations. Thus while the analysis of this section in no way pretends to form a theory of the business cycle, it seeks to explore the ways in which technological forces arising from the recursive nature of the production process may act on representative firms within an industry so as to cause cycling in the process of competitive equilibrium over time.

Consider therefore an industry composed of N representative firms, each producing the same industry good with the aid of n capital goods. I assume that each firm forms identical expectations about the industry product's price path which is a measurable function

$$r(t) : [0,\infty) \longrightarrow R^+ \tag{19}$$

The instantaneous flow of profit of the representative firm is given by $r(t)f\left(k(t), \dot{k}(t)\right)$, where $f(k,\dot{k}) \in C^2$, incorporates both adjustment costs and the cost of purchasing new capital equipment and is a strictly concave function in $(k,\dot{k})$, where $k = (k_1,\ldots,k_n)$ denotes the vector of capital goods. To simplify the analysis and to make possible the application of the results of the previous sections I assume that the representative firm faces a zero interest rate and that it seeks to find a capital expansion path which is a continuous function

$$k(t) : [0,\infty) \longrightarrow R^{n+} \tag{20}$$

which maximises its future stream of profit

$$\int_0^\infty r(t) f\big(k(t), \dot{k}(t)\big) dt \tag{$\mathcal{R}$}$$

given the fixed initial capital endowment $k(0) = k_0 \varepsilon R^{n+}$.

The total market supply forthcoming at each instant on the product market

$$Q_S(t) = Nf\big(k(t), \dot{k}(t)\big), \quad t \varepsilon [0,\infty)$$

has a complex functional dependence on the price path (19), since it arises as a by-product of the solution of the basic maximum problem ($\mathcal{R}$) by each firm. On the demand side of the market I make the simplifying assumption that the total market demand depends only on the current market price

$$Q_D(t) = \psi^{-1}\big(r(t)\big), \quad t \varepsilon [0,\infty)$$

where $\psi \varepsilon C^1$ and $\psi(Q) > 0$, $\psi'(Q) < 0$, $Q \geqq 0$.

DEFINITION. A rational expectations equilibrium for the product market of the industry is a measurable price path (19) such that

$$Q_D(t) = Q_S(t) \quad \text{for almost all } t \varepsilon [0,\infty) \tag{E}$$

DEFINITION. The function $\Psi\big(Nf(k,\dot{k})\big)$ where

$$\Psi(Q) = \int_0^Q \psi(y) dy, \quad Q \geqq 0$$

denotes the integral of the demand function, so that

$$\Psi \in C^2, \quad \Psi'(Q) = \psi(Q), \quad \Psi''(Q) = \psi'(Q) < 0, \quad Q \geqq 0$$

is called the extended integrand. The problem of finding a continuous function (20) which maximises

$$\int_0^\infty \Psi\left(Nf(k(t), \dot{k}(t))\right)dt \tag{$\mathcal{E}$}$$

is called the extended integrand problem.

Our analysis of the cyclical properties of rational expectations equilibrium will be based on the following proposition which is a straightforward adaptation of the result of Brock-Magill [2, Th. 5] and Scheinkman [23, Sect. 4] to the undiscounted case. This proposition transforms the analysis of rational expectations equilibrium from a direct analysis of the representative firms problem ($\mathcal{R}$) and the market equilibrium condition (E) to an indirect analysis of the extended integrand problem ($\mathcal{E}$).

PROPOSITION 4. If $k(t)$ is the solution of the Euler-Lagrange equation for ($\mathcal{E}$)

$$\Psi'\left(Nf(k,\dot{k})\right)f_k(k,\dot{k}) - \frac{d}{dt}\left\{\Psi'\left(Nf(k,\dot{k})\right)f_{\dot{k}}(k,\dot{k})\right\} = 0 \tag{21}$$

which satisfies the initial condition $k(0) = k_0$ and the transversality condition

$$\overline{\lim_{t\to\infty}} \left[-\Psi'\left\{Nf\left(k(t), \dot{k}(t)\right)\right\}f_{\dot{k}}\left(k(t), \dot{k}(t)\right)\right]k(t) < \bar{\alpha} \tag{22}$$

and if for any alternative continuous path $\tilde{k}(t)$ with $\tilde{k}(0) = k_0$

$$\underline{\lim_{t\to\infty}} \left[-\Psi'\left\{Nf\left(k(t), \dot{k}(t)\right)\right\}f_{\dot{k}}\left(k(t), \dot{k}(t)\right)\right]\tilde{k}(t) > \underline{\alpha} \tag{23}$$

for constants $\bar{\alpha}$, $\underline{\alpha}$ then the price path

$$r(t) = \Psi'\left\{Nf\left(k(t), \dot{k}(t)\right)\right\} \tag{24}$$

is a rational expectations equilibrium for the product market of the industry.

Proof. (21) and (24) imply that the Euler-Lagrange equation for ($\mathcal{R}$) is satisfied

$$r(t)f_k(k,\dot{k}) - \frac{d}{dt}\left(r(t)f_{\dot{k}}(k,\dot{k})\right) = 0$$

(22)-(24) imply that the standard transversality conditions for ($\mathcal{R}$) are satisfied [15, Lemma 2]. Since $\Psi' = \psi$, (24) implies that (E) holds. Thus each firm maximises ($\mathcal{R}$) and the market equilibrium condition (E) is satisfied. Δ

Remark. In order that Assumption I be satisfied, we assume the existence of an equilibrium point $k^* \in \mathring{R}^{n+}$

$$f_k(k^*,0) = 0$$

With this assumption the conditions of Proposition 4 will be satisfied by a path k(t) for which

$$\|k(t) - k^*\| \longrightarrow 0 \quad \text{as } t \to \infty$$

The analysis of rational expectations equilibrium may thus be viewed as a problem ($\mathcal{P}$) with basic integrand

$$L(k,\dot{k}) = \Psi\left(Nf(k,\dot{k})\right)$$

while the associated problem ($\mathcal{L}$) has the integrand

$$L^\circ(x,\dot{x}) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}' \begin{bmatrix} \Psi'^*f^*_{kk} & \Psi'^*f^*_{k\dot{k}} \\ \Psi'^*f^*_{\dot{k}k} & \left(\Psi'^*f^*_{\dot{k}\dot{k}} + N\Psi''^*f^*_{\dot{k}}f^{*\prime}_{\dot{k}}\right) \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Remark. If $f^*_{k\dot{k}}$ is symmetric, in particular if $f(k,\dot{k}) = u(k) + v(\dot{k})$ as in Scheinkman [23] or if $n = 1$, then there is no cycling in the rational expectations equilibrium process in the neighborhood of k^*.

Remark. Proposition 1 allows us to deduce the following: if the eigenvalues of $\Psi'^*f^*_{kk}$ in the metric of $-(\Psi'^*f^*_{\dot{k}\dot{k}} + N\Psi''^*f^*_{\dot{k}}f^{*\prime}_{\dot{k}})$ differ but little, then asymmetry of $f^*_{k\dot{k}}$ causes cycling.

When $n = 2$ the reader may readily apply (17) to obtain a necessary and sufficient condition for cycling. If in addition the production function $f(k,\dot{k})$ satisfies $f^*_{k_1k_2} = f^*_{\dot{k}_1\dot{k}_2} = 0$ and $f^*_{\dot{k}} = 0$ then the condition reduces to

$$\left| f^*_{k_2\dot{k}_1} - f^*_{k_1\dot{k}_2} \right| > \left| \sqrt{f^*_{k_1k_1} f^*_{\dot{k}_2\dot{k}_2}} - \sqrt{f^*_{k_2k_2} f^*_{\dot{k}_1\dot{k}_1}} \right|$$

The basic cause of cycling is thus technological and arises from the presence of asymmetry in the effect of investment in one good on the marginal product of another capital good.

Remark. Lucas [9] has emphasised the difficulty of generating an equilibrium process in which there is cycling and in which "persistent, recurrent, unexploited profit opportunities" are absent. In the present context, if the industry good is storable there is an incentive for speculators to carry the commodity over from periods of relative abundance when the price is low to periods of relative scarcity when the price is high, the extent of such arbitrage activity depending on the cost of storage. This arbitrage activity reduces the extent of cycling in both the price and the quantity traded. However if the commodity is perishable or if the cost of storage is sufficiently high the cycles will tend to persist.[6]

VI. CONCLUDING REMARKS

In this paper I have attempted to throw some light on the origin of cycling in a class of dynamic maximisation problems that arises in economics. It would be of interest to have a precise characterisation of cycling for the n-dimensional case, both for the class of models considered here and for the more general class of discounted models. In these discounted models the stability problem is much more delicate, but it is precisely here, in conjunction with the emergence of instability, that the presence of cycling is likely to have its most interesting consequences.[7]

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FOOTNOTES

1. See [2, 3, 4, 13, 15, 17, 22, 23].

2. An exception is the interesting paper of Ryder and Heal [20]. But their results are restricted to a particular discounted model.

3. For an excellent exposition of the Lotka-Volterra theory in English the reader is referred to the book by D'Ancona [5]. See also Samuelson's discussion [21].

4. Magill-Scheinkman [17] show that symmetric discounted variational problems have real roots.

5. (15) is derived as follows. Let $\xi = \alpha + i\beta = \lambda^2$ then

$$\alpha = \frac{1}{2}(\alpha_1 + \alpha_2 - \gamma^2), \qquad \beta = \frac{1}{2}\sqrt{(-H)J}$$

The relation $(\alpha + i\beta) = (\mu + i\nu)^2$ implies

$$\mu = \frac{1}{\sqrt{2}}\left(\frac{\beta}{\theta}\right), \qquad \nu = \frac{\theta}{\sqrt{2}}, \qquad \theta = \sqrt{-\alpha + \sqrt{\alpha^2 + \beta^2}}$$

which in turn implies

$$\mu = \frac{1}{2}\sqrt{(-H)}, \qquad \nu = \frac{1}{2}\sqrt{J}$$

6. For an analysis of the way transactions costs influence inventory behaviour when the inventory consists of a portfolio of assets held by an investor, see Magill-Constantinides [16]. For a further discussion of the relation between inventories and investment the reader is referred to the original discussion of Eisner and Strotz [6].

7. I refer to the Hopf bifurcation and the emergence of a limit cycle.