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RESEARCH, DEVELOPMENT AND TECHNOLOGICAL CHANGE IN A GROWING ECONOMY

by

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INTRODUCTION

Technological change has been generally recognized as a major component of growth in the modern economy. The effects of improvements in productive efficiency have been explored under a wide range of assumptions (see the surveys by Hahn and Matthews [12]; and Kennedy and Thirlwall [17]). The occurrence of technical progress is needed to explain in a consistent manner the "stylized facts" of neoclassical growth theory (see Burmeister and Dobell [4] pp. 65-66). However, technological change is usually presented as "mana from heaven"; whether embodied or disembodied, technological change is the result of a continuous flow of improvements, costless, exogenous, whose entire future is known with certainty. Can a more accurate representation of the sources of technical progress be given within the context of a model of economic growth?

This paper will take the position that the properties of technological change must be derived from more basic assumptions concerning research and development processes which are observed and controlled by a decision maker, whether he is a firm manager or a government planner. A decision maker may choose what research and development projects will be undertaken, when they will begin and the time at which the results obtained will be introduced into the production process. The rate and direction of technical change will be related to decisions concerning when projects are begun and ended and what projects are chosen.

In a growth model a planner may allocate resources between consumption, investment in the capital stock and R and D expenditures. The costs of technical change will then depend on the costs incurred in R and D. Uncertainties associated with the costs, arrival times and quality of innovations will depend on uncertainties associated with the outcomes of basic research and the "operational" production techniques which may be created by the development process given a particular state of basic research.

A growth model will be presented here in which technological change depends upon the outcomes of a research and development project subject to periodic evaluation by a decision maker. The first part of this paper will present a new approach to the problem of modelling the innovative process. This approach will allow a separation of the process of developing a new technique from the process of basic research. The sequential nature of R and D decision-making will be emphasized.

The second part of the discussion will focus on dynamic resource allocation and control of an R and D project when the outcome is a discrete shift in productive efficiency. The purpose of the analysis will be to show how the costs of R and D and the arrival dates of innovations are the result of a sequential decision process which involves choosing at each stage whether or not to continue the R and D process. If the decision is made to discontinue the R and D process, the innovation can be introduced; otherwise, if the R and D process is continued, a decision must be made with respect to the allocation of resources between consumption, investment and the costs of R and D .

The third part of the paper will examine the problem of controlling an R and D project charged with creating a "backstop" energy source in a growing economy faced with the eventual depletion of a finite stock of natural resources. The results of the second and third parts of the paper will depend on an existence theorem due to Hinderer [14] for discounted dynamic programming problems when constraints are present.

I. RESEARCH AND DEVELOPMENT

In this section we will discuss the innovative process emphasizing the problem of periodic re-evaluation of an R and D project and noting the differences between the uncertainties associated with basic research and the uncertainties involved in developing a new technique.

The problem of dynamic resource allocation to an R and D project under uncertainty has been studied in economics and in operations research. Most of these studies have not allowed periodic evaluation of the status of an ongoing project.[1] Recently however, Radner and Rothschild [28] and Deshmukh-Chikte [9] have considered problems where a decision maker periodically re-examines the effort devoted to a stochastic R and D-type project.

The sequential decision making approach is appropriate for the problem of planning technological change in a growing economy where it may be possible to make predictions about the next innovation, given the current state of knowledge, but next to impossible to predict the entire future stream of discoveries. As economic conditions change (say for example increases ossur in the labor force or in the aggregate capital stock) and new innovations are employed in the production process, the criteria for evaluating on-going production processes may also change. In a similar context, Koopmans [18] emphasizes that:

> "We must recognize the fact that knowledge of the extent of production possibilities, and the

> means and pace of their enlargement, is gained only through experience in their use and extension. Optimization and exploration thus have to be engaged in simultaneously, with the latter serving to guide and strengthen the former."

and recommends that models of economic growth make use of information available at the time a decision affecting the growth process is being made. In this spirit a growth model will be presented here which allows technological change as the outcome of an R and D process subject to periodic evaluation.

Sequential sampling techniques are a common feature of actual R and D projects. The search procedures present in many economic models are an outcome of the development of sequential sampling techniques. For this reason, the "search-model" approach is a natural way to consider the uncertain process of improving the current state of technical knowledge.

The periodic re-evaluation of an R and D process serves also to emphasize the <u>discrete</u> nature of technical changes. The usual justifications for considering technical change as a continuous stream are that this is a good approximation to a series of small innovations or that observed secular shifts such as increases in labor productivity are best explained in this way. It can, however, be argued that technological change occurs in large "lumps" and that the continuous secular shifts involve learning or adjustment to the last important change or marginal improvements in last innovation. In <u>Capital and Growth</u> [13], Hicks discusses an "imaginary economy" which experiences a discontinuous sequence of major innovations. While Hicks finds it unrealistic to assume that the "automobile age" is completed before the "aeroplane age" begins, he finds the discontinuity to be a relatively realistic assumption. We will consider the problem of when to introduce a single major innovation as well as examining the arrival of a number of "smaller" innovations.

The innovative process is usually considered to have several "stages". Mansfield ([25] p. 45), for example, discusses three categories: basic research, which is aimed at "the creation of new knowledge"; applied research, which is "expected to have a practical payoff"; and development, which attempts the "reduction of research findings to practice". However, Mansfield finds the distinction between basic and applied research to be rather uncertain. We will assume here that all projects have two stages, a research stage, where discoveries are made and the technical feasibility of their application is ascertained, and a development stage, where the results obtained in the research stage are made "operational" and the utility of implementing the outcome is determined.

What makes the R and D problem interesting and unique is the nature of the uncertainties associated with the innovative process. These can be divided into two categories. If we suppose that the current state of the research project is characterized by various available basic-science results, formulas, blueprints, new product ideas, production plans, etc., then the "quality" of the actual product which will result from the development process may be uncertain. Another type of uncertainty is encountered in predicting the next state of the research process. The next outcome of an experimental research project will presumably depend in some way on the current "state of knowledge". So given the current state of the research process, we may make predictions concerning the outcome of the development process and the next state of the research process. Let us try to represent this in a formal way.

Suppose that $\omega = (\omega_t)_{t \in N}$ is a sequence of real-valued random variables taking values in some set Ω and representing in some way the measured performance level corresponding to the outcomes of an experimental research process which is observed at discrete time intervals. It is assumed that the next performance level depends randomly only on the currently observed performance level and that the process "improves" over time.

A1. Suppose that (ω_t) is a submartingale and is a stationary Markov process with transition probability $p_\omega(\cdot) \equiv p(\cdot/\omega)$ which is weakly continuous in ω.

We associate with each outcome of the research process at date t, a cumulative distribution function on the uncertain outcome of the development process in the next period: $F_{\omega_t}(\cdot) \equiv F(\cdot/\omega_t)$. Let $X = (X_t)_{t \in N}$, where each X_t takes values in some set $E \subset \mathbb{R}_+$, be a sequence of random variables representing the "quality" level corresponding to the outcomes of a development process, at each date t. The value X_{t+1} is drawn from the distribution given by $F_{\omega_t}(\cdot)$ if ω_t is the current state of the research process.[2] To represent the improvements occurring in the R and D process we make the following assumption:

> A2. The distribution given by $F_\omega(\cdot)$ is stochatically increasing in ω, i.e. for $\omega' \geq \omega$, $F_{\omega'}(\cdot) \leq F_\omega(\cdot)$. Also $F_\omega(\cdot)$ is weakly continuous in ω. Let $F_\omega(0) = 0$ for all ω.

Note that the transition probability for the sequence (X_t, ω_t) is given by $q_{\omega_{t-1}}(\cdot\ ,\ \cdot) = F_{\omega_{t-1}}(\cdot)p_{\omega_{t-1}}(\cdot)$, which is weakly continuous in ω_{t-1}. Note also that X_t is stochastically independent of X_{t-1}.

At each date t the planner will be faced with a current outcome of the basic research process ω_t and a proposed innovation with "quality" level X_t selected by the development process from the distribution $F_{\omega_{t-1}}(\cdot)$. Given the expected outcome from continuing the research process and the next expected outcome of the development process as well as various economic considerations, the planner must decide whether or not to employ the innovation represented by X_t and also whether to continue to bear the costs of R and D.

II. OPTIMAL RESEARCH AND DEVELOPMENT AND PRODUCTIVE EFFICIENCY

Introduction

In this section we will consider the problems associated with periodic review and control of an R and D project in connection with resource allocation between consumption, investment and the costs of R and D, in the context of a model of economic growth. The first part of this section will focus on the problem of choosing consumption and investment while controlling an R and D project where the costs depend only on the state of research process. The second part of this section will allow the effort and costs devoted to R and D to vary while keeping the rate of capital accumulation constant. Finally, the third part will briefly examine the consequences of allowing the output of the development process to be introduced in every period.

A. R and D and Economic Growth

The R and D decision process will now be examined in the context of the standard one sector neoclassical growth model (see, for example, Burmeister and Dobell [4]). If we let Y, K, L, t denote output, capital, labor and time, respectively, the most general form of the neoclassical production function allowing disembodied technical change may be given by:

1) $$Y = F(K, L; z(t))$$

where $z(\cdot)$ is a transformation of the time parameter allowing for variable introduction dates over time for technical changes. The transformation function of the time parameter $z(\cdot)$ may be taken here as representing an efficiency parameter associated with the current technology. Assume that $F(\cdot)$ has continuous first and second partial derivatives in K, L, z. $F(\cdot)$ is also assumed to be increasing and concave in K, L and z, and homogeneous of degree one in K, L. Production requires both inputs and a positive efficiency level:

$$F(0, L, ; z) = F(K, 0; z) = F(K, L; 0) = 0$$

In the present model we will assume that while the search for a new technology is taking place $z(t) = \overline{X}$, where $\overline{X}$ is some arbitrary initial efficiency level. If the R and D process is stopped at some date T, then the current output of the development process X_t is introduced into the production process so that $z(t) = X_T$ for $t = T, T+1, \ldots$. This formulation is to emphasize the unique arrival of a major innovation as well as a certain fixity involved in the introduction of new production techniques.

We will consider the growth problem in discrete time. Letting I_t denote net investment and noting that capital depreciates at the constant rate δ we may write:

$$I_t = K_t - (1 - \delta)K_t \quad .$$

The labor force is assumed to grow at the constant rate n so that:

$$L_t = (1+n)L_{t-1} \text{ and } \beta \equiv \frac{1}{1+n} = \frac{L_{t-1}}{L_t}$$

Output is assumed to be allocated between consumption, investment and the costs of R and D while research is in progress.

We will assume for now that the costs of R and D are given in terms of the composite good and depend only upon the last outcome of the research process. The costs are given by a continuous, non-negative and non-imcreasing function of the last outcome $G(\omega)$. So, as the state of research improves, research costs will not rise. Let $G(\omega) \geq 0$, $G'(\omega) < 0$, $G''(\omega) \geq 0$, $G(0) < \infty$. The notion of state-dependent costs of observing the next state of a stochastic process is similar to many gambling problems. So, until date T:

$$2) \qquad C_{t+1} + I_{t+1} + G(\omega_t) = F(K_t, L_t; z(t))$$

We will later examine a different cost function which allows costs to vary with the level of effort devoted to R and D.

If the R and D process is terminated, research costs are no longer paid so that output is only divided between consumption and investment in the capital stock:

$$3) \qquad C_{t+1} + I_{t+1} = F(K_t, L_t; z(t))$$

$$t = T, T+1, \ldots$$

Since F is homogeneous in capital and labor, we can rewrite these constraints in per-capita terms. Letting $i_t = \frac{I_t}{L_t}$, $c_t = \frac{C_t}{L_t}$, $k_t = \frac{K_t}{L_t}$ and $g(\omega_t) = \frac{G(\omega_t)}{L_t}$, the form of the constraint before termination of the project is given by:

4) $c_{t+1} + (k_{t+1} - (1-\delta)\beta k_t) + g(\omega_t) = \beta f(k_t;\ z(t))$

After the project is completed we have:

5) $c_{t+1} + (k_{t+1} - (1-\delta)\beta k_t) = \beta f(k_t;\ z(t))$

The planner's problem is to maximize the expected value of the sum over an infinite horizon of discounted per-capita utility given a discount rate $0 < \alpha < 1$, an initial capital stock $k_o > 0$, an initial "efficiency" level $\bar{X}$, and a known initial state of the research process ω_o. Per-capita utility is assumed to be non-negative and bounded above.

At each date t, after observing the current status of the R and D project given by (ω_t, X_t), the planner may decide whether or not to continue the process. If the decision is made to stop at time T and we let $(k_T, X_T) = (k,X)$, the return from stopping the process will be given by:

6) $W(k,X) = \max\ \Sigma_{t=o}^{\infty} \alpha^t U(c_t)$

s.t. k, X given

$c_o + k_o = \beta f(k;X) + (1-\delta)\beta k$

$c_t + k_t - (1-\delta)\beta k_{t-1} = \beta f(k_{t-1}, X)$ $\quad t = 1, 2, \ldots$

$c_t \geq 0,\ k_t \geq 0$ $\quad t = 0, 1, 2, \ldots$

Note that the value of stopping depends on the level of the capital stock as well as on the current outcome of the development process. The value of stopping is given by the standard one-sector neoclassical growth model where the technology is known with certainty.

If the planner decides to continue the process at date t, he must select the desired level of capital per man for the next period noting the per-capita costs of R and D. The planner's problem is similar to an optimal stopping problem and can be written as:

$$7) \qquad \max_{\{k_t\},T} E_{\omega_o}[\Sigma_{t=1}^{T}\alpha^{t-1}U(c_t) + \alpha^{T}W(k_T,X_T)]$$

$$\text{s.t.} \qquad k_o,\ X_o,\ \omega_o \text{ given}$$

$$c_t + k_t - (1-\delta)\beta k_{t-1} + g(\omega_{t-1}) = \beta f(k_{t-1};\overline{X})$$

$$c_t \geq 0\ ,\ k_t \geq 0$$

$$t = 1, 2, \ldots .$$

In dynamic programming terms, the planner observes, at each date, a state of the system given by the level of the capital stock per man k and the state of the R and D project (ω, X) where (k,ω,X) is an element of the state space $S = R_+ \times \Omega \times E$. The planner choses a stop or continue action. If the choice is made to stop, the level of net investment per man for the next period is implicitly given by solving (6) to obtain the value of stopping. If the choice is made to continue, we then assume for technical reasons that the levels of net investment per man for the next period is uniformly bounded above by some arbitrarily large real number:

A3. Net investment per man is bounded, $i \in [0,b]$, $b \in \mathbb{R}_+$.

So the 'action space' for the planner may be given by $A = \{s,c\} \times [0,b]$. The constraints given by $c_t \geq 0$, $i_t \in [0,b]$ and the product-flow equation (4) may be rewritten in terms of a constraint correspondence from the state into the action space: ψ: $S \to A$ where ψ is given by

8) $$\psi(k,\omega,X) = \psi_1(k,\omega) \times \psi_2(k,\omega)$$

$$\psi_1(k,\omega) = \begin{cases} \{s,c\} & \text{if } \beta f(k,\overline{X}) \geq g(\omega) \\ \{s\} & \text{if } \beta f(k,\overline{X}) < g(\omega) \end{cases}$$

$$\psi_2(k,\omega) = \begin{cases} \{i \in [0,b]: \ i \leq [\beta f(k,X) - g(\omega)]\} & \text{if } \beta f(k,\overline{X}) \geq g(\\ \{0\} & \text{otherwise} . \end{cases}$$

Since ψ_1 is a compact-valued upper semi-continuous correspondence and ψ_2 is a compact-valued continuous correspondence, ψ is a compact valued upper semi-continuous correspondence (see [1])

We first establish the existence of a measurable optimal stationary strategy and an optimal value function by applying results of Hinderer [14] (p. 126). The definition of

$\bar{p}$-optimal is given in [14] (p. 81) and is taken here as describing the deterministic policy which maximizes the expected total reward over the set of deterministic policies.[3]

<u>Proposition 1</u> The optimal value function $V: \mathbb{R}_+ \times \Omega \times E \to \mathbb{R}$ is U.S.C. and satisfies the equation

9) $$V(k,\omega,X) = \max\{W(k,X),\ Q(k,\omega)\}$$

where

10) $$Q(k,\omega) = \max_{i \in \psi_2(k,\omega)} [\{U(\beta f(k,\bar{X}) - i - g(\omega)) + \alpha \int_{\Omega \times E} V(i + (1-\delta)\beta k, \omega, X')\, dF_\omega(X') p_\omega(\omega')\}]$$

Also, there exists a Borel measurable, $\bar{p}$-optimal stationary strategy

$$\pi: \mathbb{R}_+ \times \Omega \times E \to \{s,c\} \times [0,b]$$

which chooses whether to stop of continue the research project, and if the decision is made to continue, determines the level of net investment for the coming period given the current stage of the system.

<u>Proof</u> The sufficient conditions for the theorem of Hinderer are collected here

i) A is a compact metric space.

ii) ψ is U.S.C.

iii) The "law of motion' of the system $dF_\omega(\cdot)p_\omega(\cdot)$

is weakly continuous in ω .

iv) $0 < \alpha < 1$ and U is bounded.

To examine the stopping rule, it will be necessary to discuss the certainty problem given by (6). First we state the following lemma:

Lemma 1 $W(k,X)$ is non-negative, increasing in k,X and concave in X .

Proof:

i) There exist optimal policies h,σ such that

$$\bar{k}_t = h(f(\bar{k}_{t-1},X) + (1 - \delta)\bar{k}_{t-1})$$

$$\bar{c}_t = \sigma(f(k_{t-1},X) + (1 - \delta)\bar{k}_{t-1})$$

$$k_o = h(s)$$

$$c_o = \sigma(s)$$

$$k_o + c_o = s$$

such that h,σ are non-negative, continuous and increasing functions with $h(o) = \sigma(0) = 0$. (See Brock and Mirman [3]).

ii) Let $0 \le X^1 \le X^2$ be two possible efficiency parameters. The consumption plans associated with X^1 and X^2 are (c_t^1) and (c_t^2) . Choose any $\lambda \in (0,1)$. Note that the consumption plan $(\bar{c}_t)$ associated with the efficiency parameter $\bar{X} = \lambda X^1 + (1 - \lambda)X^2$ is greater than or equal to the plan $(\lambda c_t^1 + (1 - \lambda)c_t^2)$. So,

$$\lambda W(k,X^1) + (1-\lambda)W(k,X^2) = \lambda\Sigma_{t=0}^{\infty}\alpha^t U(c_t^1) + (1-\lambda)\Sigma_{t=0}^{\infty}\alpha^t U(c_t^2)$$

$$= \Sigma_{t=0}^{\infty}\alpha^t[\lambda U(c_t^1) + (1-\lambda)U(c_t^2)]$$

$$\leq \Sigma_{t=0}^{\infty}\alpha^t U(\lambda c_t^1 + (1-\lambda)c_t^2)$$

by the concavity of U.

$$\leq \Sigma_{t=0}^{\infty}\alpha^t U(\overline{c}_t) = W(k,\lambda X^1 + (1-\lambda)X^2) .$$

Similarly for k .

The optimal policy π: $S \to A$ can be decomposed into a stopping rule π_1 and an investment plan π_2 . If $\overline{\omega} = \overline{\omega}(k)$ is such that $\beta f(k,\overline{X}) = g(\overline{\omega})$ then by the assumption on the form of the constraint correspondence: for $\omega < \overline{\omega}(k)$; $\pi_1(k,\omega,X) \in \{s\}$ and for $\omega \geq \overline{\omega}(k)$; $\pi_1(k,\omega,X) \in \{s,c\}$. This is similar to certain "bankruptcy" restrictions in temporary equilibrium and labor search theory. We can characterize the stopping rule as follows:

<u>Proposition 2</u> Given $\omega \geq \overline{\omega}$, there exists a unique 'switch-point' level of quality x^* of an innovation obtained from the development process such that if $X \geq x^*$ the R + D process is stopped and the innovation is introduced and if $0 \leq X < x^*$ the R + D process is continued.

The existence of a unique $x^* = x^*(k,\omega)$ can be seen easily by noting that W is increasing in X , $W(k,0) = 0$

and that the value of continuing Q given by (10) is essentially independent from the current status of the development process (see figure 1).

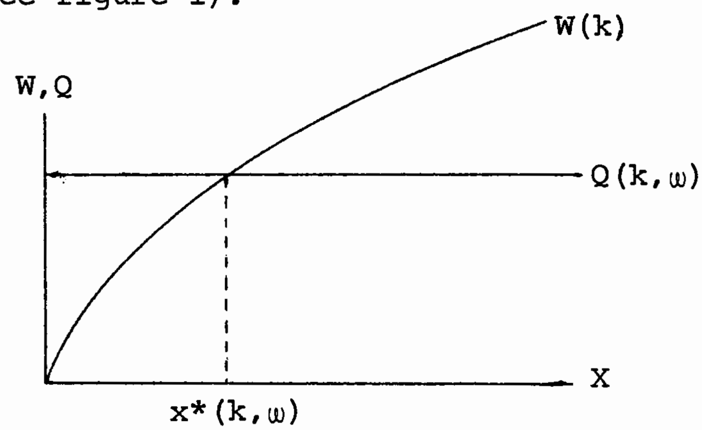

Figure 1.

Before examining the properties of x^* we turn briefly to the properties of V and Q.

<u>Proposition 3</u> V and Q are non-negative and non-decreasing in k, ω, X and are concave in k.

<u>Proof</u>: i) For $X \in [0,x^*]$, $V(k,X,\omega) = Q(k,\omega)$ which is positive and constant in X. For $X \in [x^*,\infty]$, $V(k,X,\omega) = W(k,X)$ which is non-negative and increasing in X.

ii) By a theorem of Blackwell's, there exist functions (V_n) such that $V_n \to V$ uniformly where the V_n are defined recursively by:

$$V_{n+1}(k,\omega,X) = \max\{W(k,X), Q_{n+1}(k,\omega)\}$$

where

$$Q_{n+1}(k,\omega) = \max_{i \in \psi_2(k,\omega)} [\{U(\beta f(k,\overline{X}) - g(\omega) - i)$$

(equation continued)

$$+\int_{\Omega \times E} V_n(i + (1-\delta)\beta k, \cdot, \cdot) dF_\omega(\cdot) p_\omega(\cdot)\}] .$$

Note that $V_0(k,\omega,X) = W(k,X)$ which is non-negative and increasing in k while it is constant in ω.

Suppose that the proposition is satisfied by V_n. Then for any i, $U(\beta f(k,\overline{X}) - g(\omega) - i)$ is non-negative and increasing in ω, k and concave in k since $g' < 0$ and $f_k > 0$, and $\int_{\Omega \times E} V_n(i + (1-\delta)\beta k, \omega', X') F_\omega(X') p_\omega(\omega')$ is non-negative and non-decreasing in k and ω and concave in k by the induction hypothesis. So $Q_{n+1}(k,\omega)$ and therefore V_{n+1} satisfy the proposition so that by induction V_n and Q_n satisfy the proposition for all n. By uniform convergence the proposition holds for V and Q.

Due to the concavity of V in k, the effect of accumulated capital on the optimal investment plan is indeterminate; however, it can be characterized if the capital stock is used up at the end of every period.

<u>Proposition 4</u> i) If $\frac{\partial V}{\partial k}(k,\omega,X)$ is non-decreasing in ω and X then π_2 is non-decreasing in ω.

ii) For $\delta = 1$, π_2 is non-decreasing in k.

<u>Proof</u>: It is sufficient to show that for $Q(k,\omega) = \max_i Q(k,\omega;i)$, the function $Q(k,\omega;i)$ is subadditive in the sense that

(i) For $k \geq 0$ and $0 \leq \overline{\omega}(k) \leq \omega_1 \leq \omega_2$,

$Q_1 = Q(k,\omega_2;i) - Q(k,\omega_1,i)$ is non-decreasing in i.

(ii) For $0 < k_1 < k_2$ and $\omega \geq \max(\overline{\omega}(k_1), \overline{\omega}(k_2))$

$Q_2 = Q(k_2,\omega;i) - Q(k_1,\omega;i)$ is non-decreasing in i .

(This approach follows Deshmukh and Chikte [9]).

So
$$Q_1 = [U(f(k,\overline{X}) - g(\omega_2) - i) - U(f(k,\overline{X}) - g(\omega_1) - i)]$$
$$+ [\int_\Omega\int_E V(i + (1-\delta)\beta k,\omega',X')dF\omega_2(X')p\omega_2(\omega')$$
$$- \int_\Omega\int_E V(i + (1-\delta)\beta k,\omega',X')dF\omega_1(X')p\omega_1(\omega')] .$$

Note that

$$[-U'(f(k,\overline{X}) - g(\omega_2) - i) + U'(f(k,\overline{X}) - g(\omega_1) - i)] > 0$$

since $g' < 0$. Also, by assumptions 1, 2, and since $V_k > 0$ is assumed to be non-decreasing in ω',X':

$$[\int_\Omega\int_E V_k(i + (1-\delta)\beta k,\omega',X')dF\omega_2(X')p\omega_2(\omega')$$
$$- \int_\Omega\int_E V_k(i + (1-\delta)\beta k,\omega',X')dF\omega_1(X')p\omega_1(\omega')] > 0 .$$

So π_2 is non-decreasing in ω .

For $\delta = 1$ we obtain:

$$Q_2 = [U(f(k_2,\overline{X}) - g(\omega_2) - i) - U(f(k_1,\overline{X}) - g(\omega_1) - i)] ,$$

and since

$$[-U'(\beta f(k_2,\overline{X}) - g(\omega) - i) + U'(\beta f(k_1,\overline{X}) - g(\omega) - i)] > 0$$

π_2 is also non-decreasing in k .

The switchpoint level of performance of the development process $x^* = x^*(k,\omega)$ solves the equation

(11) $$W(k,x^*) = Q(k,\omega) .$$

To examine the properties of x^* w.r.t. the capital stock and the status of the research process we will now introduce the assumption that the value function V and Q are differentiable in k and ω and that the optimal investment policy is differentiable in k. This assumption is used only to derive comparative static properties of the switchpoint x^* and to derive first-order necessary conditions for intertemporal utility maximization. (It would be interesting in a later paper to obtain properties of the value functions from more basic assumptions.) For convenience we will let the planner choose the capital stock in the next period where the optimal policy is γ so that $k_{t+1} = \gamma(k_t,\omega_t)$. So, from the existence theorem we may rewrite the problem as:

12) $$V(k_t,\omega_t,X_t) = \max\{W(k_t,X_t),\ Q(k_t,\omega_t)\}$$

where

$$Q(k_t,\omega_t) = \max_{k_{t+1}\in\psi_2(k_t,\omega_t)} \{U(\beta f(k_t,\overline{X}) + (1-\delta)\beta k_t - g(\omega_t) - k_{t+1})$$

$$+ \alpha\int_\Omega\left[\int_0^{x^*(k_{t+1},\omega_{t+1})} Q(k_{t+1},\omega_{t+1})\,dF\omega_t(X_{t+1})\right.$$

$$\left. + \int_{x^*(k_{t+1}\omega_{t+1})}^{\infty} W(k_{t+1},X_{t+1})\,dF\omega_t(X_{t+1})\right] p\omega_t(\omega_{t+1})\} .$$

Solving the maximum problem on the R.H.S. and noting (11) we obtain

13) $$U'(\beta f(k_t\overline{X}) + (1-\delta)k_t - g(\omega_t) - k_{t+1})$$

$$= \alpha\int_\Omega Q_k(k_{t+1},\omega_{t+1})F\omega_t(x^*(k_{t+1},\omega_{t+1}))p\omega_t(\omega_{t+1})$$

$$+ \alpha\int_\Omega\left[\int_{x^*(k_{t+1},\omega_{t+1})}^{\infty} W_k(k_{t+1},X_{t+1})\,dF\omega_t(X_{t+1})\right]p\omega_t(\omega_{t+1})$$

This condition must be satisfied by $k_{t+1} = \gamma(k_t,\omega_t)$. Substituting γ into the R.H.S. of (10) and again differentiating w.r.t. k_t , noting (11), we obtain:

$$14)\quad Q_k(k_t,\omega_t) = U'(\beta f(k_t,\overline{X}) + (1-\delta)\beta k_t - g(\omega_t) - \gamma(k_t,\omega_t)) \cdot [\beta fk(k_t,\overline{X}) + (1-\delta)\beta - \gamma_k(k_t,\omega_t)]$$

$$+ \gamma_k(k_t,\omega_t) \cdot \Big[\alpha\int_\Omega Q_k(\gamma(k_t,\omega_t),\omega_{t+1}) F\omega_t(x^*(\gamma(k_t,\omega_t),\omega_t)) \cdot p_{\omega_t}(\omega_{t+1})$$

$$+ \alpha\int_\Omega \Big[\int_{x^*(\gamma(k_t,\omega_t),\omega_{t+1})}^{\infty} W_k(\gamma(k_t,\omega_t),X_{t+1}) dF\omega_t(X_{t+1})\Big] p_{\omega_t}(\omega_{t+1})\Big]$$

Substituting from (13) into (14) we may write:

$$15)\quad Q_k(k_t,\omega_t) = U'(\beta f(k_t,\overline{X}) + (1-\delta)\beta k_t - g(\omega_t) - \gamma(k_t,\omega_t)) \cdot [\beta f_k(k_t,\overline{X}) + (1-\delta)\beta] .$$

Applying the same procedure to the certainty problem given by (6), and noting that $k_{t+1} = h(k_t,X)$, we also obtain (dropping the time subscripts):

$$16)\quad W_k(k,X) = U'(\beta f(k,X) + (1-\delta)\beta k - h(k,X)) \cdot [\beta f_k(k_t,\overline{X}) + (1-\delta)\beta] .$$

Suppressing the time subscripts we now substitute (15) and (16) back into (13) so that we obtain the following discrete time analog to an Euler equation:

$$17)\quad U'(\beta f(k,\overline{X}) + (1-\delta)\beta k - g(\omega) - \gamma(k,\omega))$$

$$= \alpha\int_\Omega [U'(\beta(f(\gamma(k,\omega),\overline{X}) + (1-\delta)\beta\gamma(k,\omega) - g(\cdot) - \gamma(\gamma(k,\omega),\cdot)) \cdot \beta f_k(\gamma(k,\omega),\overline{X}) + (1-\delta)\beta] \cdot [F\omega_t(x^*(\gamma(k,\omega),\cdot))] p\omega(\cdot)$$

(equation (17) continued)

$$+ \alpha \int_{\Omega}\int_{E} U'(\beta f(\gamma(k,\omega),\cdot) + (1-\delta)\beta\gamma(k,\omega) - h(\gamma(k,X),\cdot))$$
$$[\beta f_k(\gamma(k,X),\cdot) + (1-\delta)\beta] dF_{\omega}(\cdot)p_{\omega}(\cdot)$$

Noting that the consumption policy is given by $\sigma(k,X)$ in the certainty case and letting $\mu(k,\omega)$ denote the consumption plan while the R + D project is being continued (17) may be rewritten as

$$17') \quad U'(\mu(k,\omega)) = \alpha\int_{\Omega} U'(\mu(\gamma(k,\omega),\cdot)[\beta f_k(\gamma(k,\omega),\overline{X}) + (1-\delta)\beta]$$
$$\cdot F_{\omega}(x^*(\gamma(k,\omega),\cdot))p_{\omega}(\cdot)$$
$$+ \alpha \int_{\Omega}\int_{E} U'(\sigma(\gamma(k,\omega),\cdot)[\beta f_k(\gamma(k,\omega),\cdot) + (1-\delta)\beta]$$
$$dF_{\omega}(\cdot)p_{\omega}(\cdot) .$$

This is essentially a 'myopic' rule for intertemporal utility maximization.

We will now examine some of the properties of the switch-point performance level of the development process bearing in mind that the duration of the research and development process is related to the size of x^* .

(1) <u>The switchpoint increases when the status of the research process improves</u>.

From (11), $\frac{\partial x^*}{\partial \omega} = \frac{Q_{\omega}}{W_x} > 0$.

So for $\omega_2 > \omega_1$, see figure 2.

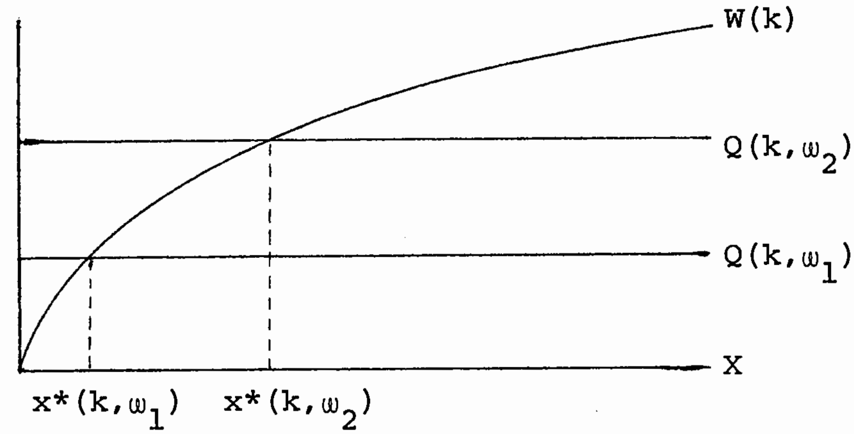

Figure 2

So as the state of the research process improves over time the minimum acceptable quality level of a developed innovation will increase.

(2) <u>The change in the switchpoint with respect to changes in the capital stock will depend on the relative marginal utilities and the relative marginal products before and after the innovation is introduced.</u>

$$\frac{\partial x^*(k,\omega)}{\partial k} = \frac{Q_k(k,\omega) - W_k(k,x^*(k,\omega))}{W_X(k,x^*(k,\omega))}$$

So $\dfrac{\partial x^*}{\partial k} \gtreqless 0$ as

$$\frac{U'(\beta f(k,\bar{X}) + (1-\delta)\beta k - g(\omega) - \gamma(k,\omega)}{U'(\beta f(k,x^*) + (1-\delta)\beta k - h(k,X))} \gtreqless \frac{f_k(k,x^*) + (1-\delta)}{f_k(k,\bar{X}) + (1-\delta)}$$

(3) <u>The switchpoint level x^* is also increasing in the original efficiency level available $\bar{X}$.</u>

Note that the value of stopping is constant in $\overline{X}$ for any given k and the value of continuing is increasing in $\overline{X}$.

(4) Note that an increase in search costs i.e. $\overline{G}(\omega) \geq G(\omega)$ will reduce Q for any ω but will not affect the value of stopping so that <u>an increase in search costs for all states will shift x^* to the left</u>.

It is interesting to note that there is also a switch-point level in terms of the performance of the research process. Since W is constant in ω and Q is non-negative and non-decreasing in ω we obtain the following (see figure 3):

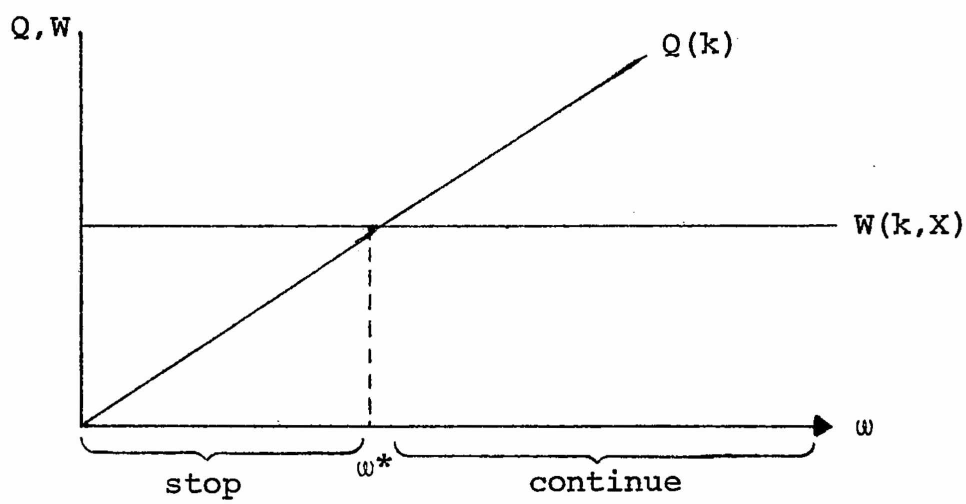

Figure 3

where ω^* is a function of k and X.

The stopping rule can be put in terms of either ω^* or x^*. The two interpretations are consistent (see figure 4).

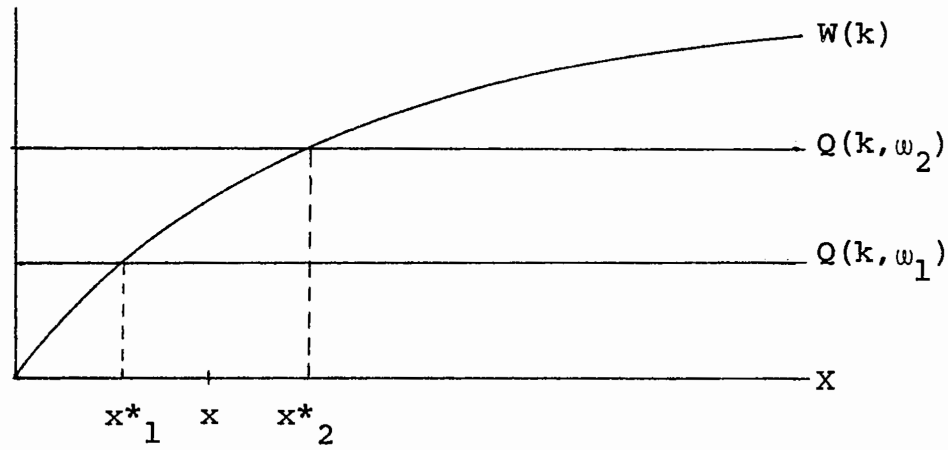

Figure 4

Suppose the current status of the development project is x. If this occurs at a low performance level of the research project ω_1 the project should be stopped since it may be unlikely to do better with a higher state of research. So if X occurs at a higher level, say ω_2, the R + D process would be continued in hopes of better results at a higher level of research. So $\omega_1 \leq \omega^*(k,X)$ and $\omega_2 > \omega^*(k,X)$. Note that $\frac{\partial \omega^*}{\partial x} > 0$, $\frac{\partial \omega^*}{\partial k} \lesseqqgtr 0$ as $\frac{\partial X^*}{\partial k} \lesseqqgtr 0$ and $\frac{\partial \omega^*}{\partial X} < 0$.

So we have shown that a stationary optimal policy exists which determines whether to stop or continue the research and development process and determines the optimal level of consumption and investment if the decision is made to continue. There is a unique switchpoint quality level for the outcome of the development process which depends on the level of the capital stock, the current status of the research process and on the initial state of technical efficiency. The investment policy is increasing in the given capital stock and is also increasing in the status level of the research process.

B. R and D with variable effort.

It is possible that by varying the level of effort devoted to the R + D process a planner may affect the expected increase in the status level of the research process. Suppose that research costs are a function of the level of effort a devoted to R and D with per-capita costs given by $g(a)$. Let $g' > 0$, $g(0) = 0$. To avoid the indeterminacies associated with multiple control variables we will assume that a constant proportion ξ of output goes to net investment. Net investment will be given by:

$$k_{t+1} - \beta(1 - \delta)k_t = \xi\beta f(k,\overline{X})$$

until the innovation is introduced, at which point a variable investment policy is allowed. Then, research cost will entail sacrifice in terms of current consumption.

18. $$c_{t+1} = (1 - \xi)\beta f(k_t,\overline{X}) - g(a_{t+1}) \ .$$

Suppose also that the status of the research process increases by independently distributed increments which can be affected by the level of effort:

19. $$\omega_{t+1}(z) - \omega_t = \eta(a_{t+1}, z)$$

where z takes values in a set Z representing 'states of the world' and has distribution $\varphi(\cdot)$. The function η is assumed to be non-negative, continuous, increasing, and differentiable in a . This approach to the performance level of the research process is similar to the models of Radner [27], and Deshmukh, Chikte [9]. Again, for technical reasons we assume that the level of effort devoted to R and D is bounded:

A4. $$a \in [0,b] , \quad b \in \mathbb{R}_+ .$$

The constraint correspondence $\psi: S \to A$ for $S = \mathbb{R}_+ \times \Omega \times E$ and $A = \{s,c\} \times [0,b]$, is given by

20) $$\psi(k,\omega,X) \equiv \psi(k) = \{a \in [0,b]: [(1-\xi)\beta f(k,\overline{X}) - g(a)] \geq 0\}$$

where ψ is continuous in k . Using the theorem of Hinderer:

<u>Proposition 5</u> The optimal value function $V: \mathbb{R}_+ \times \Omega \times E \to \mathbb{R}$ is U.S.C. and satisfies the equation:

21) $$V(k,\omega,X) = \max\{W(k,X),\ Q(k,\omega)\}$$

where

22) $$Q(k,\omega) = \max_{a \in \psi(k)} \{U((1-\xi)\beta f(k,\overline{X}) - g(a)) + \alpha \int_{Z \times E} V(\xi\beta f(k,X) + \beta(1-\delta)k, \omega + \eta(a,z), X') \cdot dF_\omega(X')\varphi(dz)\}$$

Also there exists a Borel measurable $\overline{p}$-optimal stationary strategy

$$\pi: \mathbb{R}_+ \times \Omega \times E \to \{s,c\} \times \mathbb{R}_+$$

that decides whether to stop or continue the research project and if the decision is made to continue determines the level of the capital stock for the coming period given the current state of the system.

Note that if we allow a variable investment policy after the innovation is introduced the value of stopping will be equivalent to (6).

We can decompose the optimal policy into a stopping rule π_1 and a rule for determining the level of effort π_2. As before we find that there exists a unique switchpoint level of performance of the development process x^* which will depend on the current status of the research process and on the level of the capital stock and which solves the equation

$$W(k, x^*(k,\omega)) = Q(k,\omega) .$$

To examine the properties of V, Q and the policy π_2 we made the following assumption:

<u>Assumption 5</u> $F\omega(X)$ is chosen so that for a function $\ell(\omega,X)$ convex in ω (and non-decreasing in X) the expression $\int_E \ell(\omega,X)\,dF\omega(X)$ is convex in ω.

This is used to obtain part ii of the following result:

Proposition 6

i) V, Q are non-negative and non-decreasing in ω and X and are non-negative and increasing in k .

ii) V, Q are convex in ω .

Proof: Part (i) is the same as in part A.

(ii) $V_0(k,\omega,X) = W(k,X)$ which is constant in ω .

Suppose that for some n V_n is convex in ω . Since

$$Q_{n+1}(k,\omega) = \max_{\overline{a}}\{U((1-\xi)\beta f(k,\overline{X}) - g(\overline{a})) + \int_Z\int_E V_n(\xi\beta f(k,X) + \beta(1-\delta)k, \omega + \eta(a,z), X')dF\omega(X')\varphi(dz)\}$$

So $V_n(\cdot,\omega + \eta(\overline{a},z),\cdot)$ is still convex in ω . By Assumption 3

$$\int_E V_n(\xi\beta f(k,\overline{X}) + \beta(1-\delta)k, \omega + \eta(\overline{a},z), X')dF\omega(X')$$

is convex in ω so that Q_{n+1} is convex in ω . By induction Q and V are convex in ω .

Note that since W does not depend on ω , $\frac{\partial W}{\partial \omega}$ is non-decreasing in X . If we suppose that $\frac{\partial V_n}{\partial \omega}$ is also non-decreasing in X , then the same holds for Q_{n+1} and also by induction for Q and V .

Proposition 7

i) $\overline{a} = \pi_2(k,\omega)$ is nondecreasing in k .

ii) If Assumption 5 holds then $\overline{a}$ is nondecreasing in ω .

<u>Proof</u>: Let $Q(k,\omega) = \max_{\bar{a}} \bar{Q}(k,\omega,\bar{a})$.

i) For $0 \le k_1 \le k_2$

$$\frac{\partial}{\partial \bar{a}} [Q(k_2,\omega,\bar{a}) - Q(k_1,\omega,\bar{a})]$$

$$= [-U'((1-\xi)\beta f(k_2,\bar{X}) - g(a)) + U'((1-\xi)\beta f(k_1,\bar{X}) - g(\bar{a}))]g'(\bar{a})$$

$$+ \alpha \int_Z \int_E [V_\omega(\xi\beta f(k_2,\bar{X}) + \beta(1-\delta)k_2,\omega + \eta(\bar{a},z),X')$$

$$- V_\omega(\xi\beta f(k_1,\bar{X}) + \beta(1-\delta)k_1,\omega + \eta(\bar{a},z),X')]$$

$$\cdot \eta_a(\bar{a},z)dF\omega(X')\varphi(dz)$$

≥ 0 .

ii) for $0 \le \omega_1 \le \omega_2$

$$\frac{\partial}{\partial \bar{a}} [\bar{Q}(k,\omega_2,\bar{a}) - \bar{Q}(k,\omega_1,\bar{a})]$$

$$= \alpha \int_Z \left[\int_E V\omega(\xi\beta f(k,\bar{X}) + \beta(1-\delta)k,\omega_2 + \eta(\bar{a},z),X')dF\omega_2(X') \right.$$

$$\left. - \int_E V\omega(\xi\beta f(k,\bar{X}) + \beta(1-\delta)k,\omega_1 + \eta(\bar{a},z),X')dF\omega_1(X') \right]$$

$$\cdot \eta_a(\bar{a},z)\varphi(dz)$$

$$\ge \alpha \int_Z \int_E [V\omega(\xi\beta f(k,\bar{X}) + \beta(1-\delta)k,\omega_2 + \eta(\bar{a},z),X')$$

$$- V\omega(\xi\beta f(k,\bar{X}) + \beta(1-\delta)k,\omega_1 + \eta(\bar{a},z),X')]$$

$$\cdot \eta_a(\bar{a},Z)F\omega_1(X')\varphi(dz)$$

since $F\omega(\cdot)$ is stochastically increasing in ω .

≥ 0 by the convexity of V in ω .

Similarly we obtain $\bar{a}$ increasing in β and non-increasing in δ if Assumption 1 on $F\omega(\cdot)$ holds. The result in (ii) agrees with the results of Deshmukh, Chikte [9] for the effort allocated to a simple research process. The same properties are retained for the switchpoint performance of the development process which were present in part A.

C. Innovation in each period.

Let us now return to the problem of determining the optimal investment level with state-dependent research. However we suppose that the planner must introduce the outcome of the development process in each period but has the option of discontinuing the research and development process at each stage. To insure that an inferior technique is never introduced, we assume that (X_t) is a nondecreasing sequence with independent increments $z \in \mathbb{R}_+$ where z has the distribution given by $F\omega(\cdot)$ which is defined as before. We maintain the assumption that $F\omega(\cdot)$ is stochastically increasing in ω. Noting that the constraint correspondence is already given by (8) letting $\bar{X} = X_t$ at each date t, we obtain the following proposition:

Proposition 8 There exists an U.S.C. function $V\colon \mathbb{R}_+ \times \Omega \times E \to \mathbb{R}$ given by:

23) $V(k,\omega,X) = \max\{W(k,X), Q(k,\omega,X)\}$

where

24) $$Q(k,\omega,X) = \max_{i\in\psi_2(k,\omega,X)} \{U(\beta f(k,X) - g(\omega) - i) + \alpha \int_{\mathbb{R}_+ \times E} V(i+(1-\delta)\beta k, \omega', X+z)\, dF\omega(z)\, dp_\omega(\omega')\}$$

Also there exists a Borel-measurable stationay $\bar{p}$-optimal strategy:

$$\pi: \mathbb{R}_+ \times \Omega \times E \to \{s,c\} \times [0,b]$$

with properties as given in Proposition 1.

We obtain slightly stronger properties for V and Q :

Proposition 9 i) V, Q are non-negative and increasing in k, ω, X .

ii) V, Q are concave in X .

Note that since both V and Q are concave in X the stopping rule will in general not have a switchpoint in terms of the performance level of the development process. However we may still obtain a switchpoint in terms of the status of the research process. Since ω^* solves the equation

$$W(k,X) = Q(k,\omega^*(k,X),X)$$

we obtain a unique ω^* for any given k and X such that the process is stopped for $\omega \le \omega^*$ and continued for $\omega > \omega^*$ (see figure 5). The sign of ω^* with respect to k

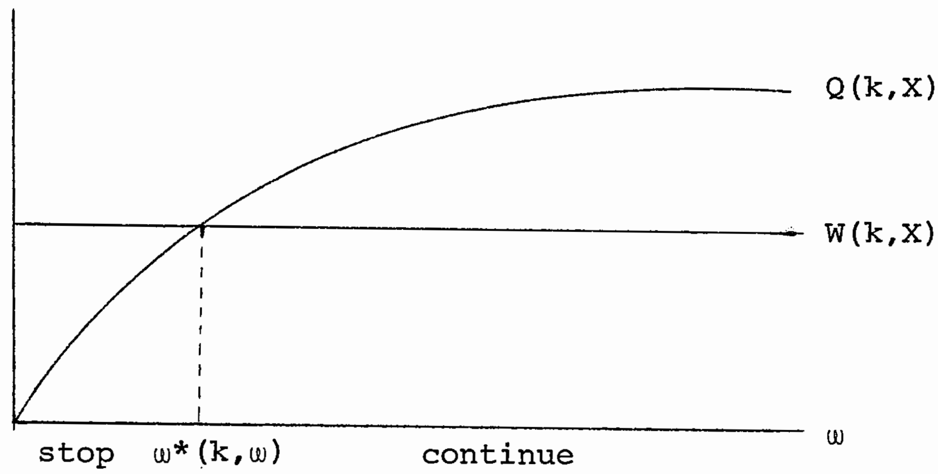

Figure 5

is determined as before while the sign of ω^* w.r.t. X is indeterminate.

The properties of the investment policy π_2 are equivalent to proposition 4. Also π_2 is increasing in the state of the development process X . The problem with innovation in each period may also be formulated in terms of a constant investment rate and a variable level of R + D effort.

III Optimal research and development of a 'backstop' energy technology

In a growing economy, the impact of a finitely available stock of a natural resource necessary to production on per-capita consumption may be lessened somewhat by capital accumulation, the discovery of new deposits of the resource or by the arrival of a 'backstop' energy technology. The problem of allocating resources between consumption, investment and the costs of searching for a new technology has been studied by Kamien and Schwarz [15] and Dasgupta, Heal and Majumdar [6]. These two papers consider a "drastic technical change" which effectively eliminates the need for the exhaustible resource in the productive process. The nature of the new technology is known from the start but its arrival date is uncertain. The conditional probability of project completion is assumed to be a known function of cumulated research effort, measured by the amount of the 'composite' commodity allocated to the R + D project. The probability distribution on the (unknown) threshold level of R + D expenditures necessary for successful project completion and the time path of R + D expenditures is used to generate a distribution on project completion dates.

While the approach outlined above is useful in deriving rules for resource allocation to an ongoing R + D project whose outcome is known but currently unavailable, the uncertainties involved seem somehow to be misplaced. What seems to be the case in practice is that many alternate techniques are available at any time while the nature of the improvements in these techniques forthcoming through R and D is generally uncertain. Manne [24] finds solar, fusion and breeder fission to be "the most plausible contenders for large-scale future supplies of energy" since they do not depend on a finite stock of fossil or nuclear fuel. It is however uncertain what future discoveries will be made and how these discoveries will affect the costs and dangers of using these energy sources.

The problem of choosing between two currently available technologies when one is low in cost but exhaustible and the other is more expensive but inexhaustible has been studied by Smith [30]. Smith has found that only the exhaustible resource should be used initially and then gradually replaced by the inexhaustible 'backstop' energy source. There is however a further reason for the delay in introducing the 'backstop' energy source. If employing the new technology will entail adjustment costs or render existing capital obsolete, the 'backstop' may not be used until more is known about the improvements expected from ongoing R + D projects.

We will approach the problem of creating an alternative to an exhaustible natural resource within the framework developed in the first part of the paper. We consider a one-sector neoclassical model of economic growth with a given initial technology. Production depends on capital, labor and the rate of utilization of a natural resource. A search is undertaken to obtain an inexhaustible 'backstop' energy source.

A drawback of the approach which will be taken is that it is difficult to characterize the properties of the time paths of consumption, investment and resource use. This problem could possibly be better handled within a continuous time formulation and in this sense, the results of [15] and [6] are more interesting. However, the analysis of the stopping rule obtained here may be useful in deriving probability distributions on the expected costs and completion dates of R + D projects.

If S_0 is the initial stock of the exhaustible natural resource and we let S_{t-1} be the remaining stock available at the beginning of period t, then $R_t = S_{t-1} - S_t$ represents the use of the natural resource in period t (or $r_t = \beta s_{t-1} - s_t$ in per-capita terms). Let $F(K,L,R)$ be the production function before project completion and let $H(K,L,R,M(X))$ be the production function after the backstop technology is introduced where $M(X)$ is the flow of services from the new energy source. In per-capita terms the constraints before and after the innovation is introduced are given by:

$$25) \quad c_t + (k_t - (1-\delta)\beta k_{t-1}) + g(\omega_{t-1}) = \beta f(k_{t-1}, s_{t-1} - s_{t/\beta})$$

$$t = 1,\ldots T$$

26) $$c_t + (k_t - (1-\delta)\beta k_{t-1}) = \beta h(k_{t-1}, s_{t-1} - s_{t/\beta}, m(X_T))$$

$$t = T+1, T+2, \ldots .$$

If we let $(k_T, s_T, X_T) = (k, s, X)$, the return from stopping the process at time T will be defined as before:

27) $$W(k,s,X) = \max \Sigma_{t=0}^{\infty} \alpha^t U(c_t)$$

$$\text{s.t. } k,s,X \text{ given}$$

$$c_0 + (k_0 - (1-\delta)\beta k) = \beta h(k, s - s_{o/\beta}, m(X)) + (1-\delta)\beta k$$

$$c_t + (k_t - (1-\delta)\beta k_{t-1}) = \beta h(k_{t-1}, s_{t-1} - s_{t/\beta}, m(X))$$

$$t = 1,2,\ldots$$

$$c_t \geq 0 \ , \ k_t \geq 0 \quad t = 0,1,\ldots .$$

The planners problem can then be written as:

28) $$\max_{\{k_t, s_t\}, T} E_{\omega_0} [\Sigma_{t=1}^{T} \alpha^{t-1} U(c_t) + \alpha^T W(k_T, s_T, X_T)]$$

$$\text{s.t. } k_0, s_0, \omega_0, X_0 \text{ given}$$

$$c_t + k_t - (1-\delta)\beta k_{t-1} + g(\omega_{t-1}) = \beta f(k_{t-1}, s_{t-1} - s_{t/}$$

$$c_t \geq 0 \ , \ k_t \geq 0$$

$$t = 1,2,\ldots$$

The constraint correspondence will be upper-semicontinuous and is defined by:

29) $$\psi(k,s,\omega,X) = \psi_1(k,s,\omega) \times \psi_2(k,s,\omega)$$

$$\psi_1(k,s,\omega) = \begin{cases} \{s,c\} & \text{if } [\beta f(k,s) - g(\omega)] \geq 0 \\ \{s\} & \text{otherwise.} \end{cases}$$

$$\psi_2(k,s,\omega) = \begin{cases} \left\{ (i,s') \in [0,b] \times [0,s_0]: i \leq [\beta f(k,s-s'/\beta)-g(\omega)], \text{ and } [\beta f(k,s-s'/\beta)-g(\omega)] \geq 0 \right\} \\ \qquad \text{for } [\beta f(k,s) - g(\omega)] \geq 0 \\ \{0\} \times \{s\} \quad \text{otherwise.} \end{cases}$$

Applying the theorem of Hinderer [14],

we again obtain existence for the given problem.

<u>Proposition 10</u> The optimal value function $V: \mathbb{R}^2_+ \times \Omega \times E \to \mathbb{R}$ is U.S.C. and satisfies the equation:

30) $$V(k,s,\omega,X) = \max\{W(k,s,X), Q(k,s,\omega)\}$$

where

31) $$Q(k,s,\omega) = \max_{(i,s')\in\psi_2(k,s,\omega)} \{U(\beta f(k,s-s'/\beta) - g(\omega) - i) + \alpha \int_{\Omega\times E} V(i+(1-\delta)\beta k, s', \omega', X')\, dF_\omega(X')\, dp_\omega(\omega')\}$$

Also there exists a Borel measurable $\overline{p}$-optimal stationary strategy

$$\pi: \mathbb{R}^2_+ \times \Omega \to E \to \{s,c\} \times [0,b] \times [0,s_0]$$

which decides whether to stop or continue the research project and if the decision is made to continue determines the net investment level and the remaining stock of the natural resource for the coming period given the current state of the system. We may characterize the value functions:

Proposition 11 The functions Q and V are non-negative and non-decreasing in k, s, ω and X.

If we assume that the investment plan i is fixed we may characterize the properties of the resource depletion strategy:

Proposition 12 The resource depletion strategy $s' = \pi_3(k,s,\omega)$ is

i) non-decreasing in s

ii) non-decreasing in k for $\delta = 1$

iii) non-decreasing in ω if $\frac{\partial V}{\partial k}(k,s,\omega,X)$ is non-decreasing in ω and X.

The proof is similar to proposition 4.

As before we obtain a unique switchpoint in terms of the status of the development process. The properties of this switchpoint will be examined after a statement of the Euler necessary conditions for this problem. From the certainty problem as stated in (27) we obtain

32) $$W_k(k,s,X) = U'(\zeta(k,s,X)) \cdot \beta[hk(k,s-\frac{1}{\beta}\sigma(k,s,X),m(X))+(1-\delta)]$$

33) $$W_s(k,s,X) = U'(\zeta(k,s,X)) \cdot \beta h_s(k,s-\frac{1}{\beta}\sigma(k,s,X),m(X))$$

where ζ and σ denote the optimal consumption and resource use policies. Let θ denote the investment policy for the certainty problem. The problem stated in (27) has been considered elsewhere:

i) for the case where $h_s = 0$ by Dasgupta and Heal [5] who show that the model resembles the standard neoclassical growth model and

ii) for the case where $h(k,s)$ is a Cobb-Douglass production by Stiglitz [31].

Note that the policies $\bar{k} = \theta(k,s,X)$ and $\bar{s} = \sigma(k,s,X)$ (where $\bar{k}$ and $\bar{s}$ denote the values of the capital and resource stocks in the next period) satisfy the basic efficiency condition:

$$34) \quad h_k(\bar{k},\bar{s}-\tfrac{1}{\beta}\sigma(\bar{k},\bar{s},X),m(X)) + (1-\delta) = \frac{h_s(\bar{k},\bar{s}-\sigma(\bar{k},\bar{s},X),m(X))}{\beta h_s(k,s-\frac{1}{\beta}\bar{s},m(X))}$$

(see for example Stiglitz [31]).

Solving the maximization problem on the R.H.S. of (31) and letting the optimal policies be given by

$$\bar{i} = \gamma(k,s,\omega)$$

$$\bar{s} = \lambda(k,s,\omega)$$

$$\bar{c} = \zeta(k,s,\omega)$$

and also letting $\bar{k} = \bar{i} + (1-\delta)\beta k$ we obtain:

$$35) \qquad Q_k(k,s,\omega) = U'(\zeta(k,s,\omega))\beta[f_k(k,s-\tfrac{1}{\beta}\lambda(k,s,\omega)) + (1-\delta)]$$

and

$$36) \qquad Q_s(k,s,\omega) = U'(\zeta(k,s,\omega))\beta f_s(k,s-\tfrac{1}{\beta}\lambda(k,s,\omega)) \;.$$

Also, noting the existence of the switchpoint $x^* = x^*(k,s,\omega)$ we may restate the necessary conditions as

$$37) \quad U'(\zeta(k,s,\omega)) = \alpha \int_{\Omega} \int_{0}^{x^*(\bar{k},\bar{s},\omega)} Q_k(\bar{k},\bar{s},\omega')\, dF_{\omega}(X')\, p_{\omega}(\omega') + \alpha \int_{\Omega} \int_{x^*(\bar{k},\bar{s},\omega')}^{\infty} W_k(\bar{k},\bar{s},X')\, dF_{\omega}(X')\, p_{\omega}(\omega')$$

$$38) \quad [U'(\zeta(k,s,\omega)) \cdot f_s(k,s-\tfrac{1}{\beta}\lambda)] = \alpha \int_{\Omega} \int_{0}^{x^*(\bar{k},\bar{s},\omega')} Q_s(\bar{k},\bar{s},\omega')\, dF_{\omega}(X')\, p_{\omega}(\omega') + \alpha \int_{\Omega} \int_{x^*(\bar{k},\bar{s},\omega')}^{\infty} W_s(\bar{k},\bar{s},X')\, dF_{\omega}(X')\, p_{\omega}(\omega')$$

Substituting from (32) and (35) into (37) and also from (33) and (36) into (38) we obtain the following 'myopic' rules for intertemporal utility maximization:

$$39) \quad U'(\zeta(\bar{k},\bar{s},\omega)) = \alpha \int_{\Omega} U'(\zeta(\bar{k},\bar{s},\omega')) \cdot \beta[f_k(\bar{k},\bar{s}-\tfrac{1}{\beta}\lambda(\bar{k},\bar{s},\omega')) + (1-\delta)] \cdot F_{\omega}(x^*(\bar{k},\bar{s},\omega'))\, p_{\omega}(\omega')$$

$$+ \alpha \int_{\Omega} \int_{x^*(\bar{k},\bar{s},\omega')}^{\infty} U'(\rho(k,s,X'))\, \beta[h_k(k,s-\tfrac{1}{\beta}\sigma(k,s,X')), \bar{m}(X') + (1$$

$$\cdot\, dF_{\omega}(x^*(k,s,\omega'))\, p_{\omega}(\omega')$$

$$40) \quad U'(\zeta(\bar{k},\bar{s},\omega)) = \left\{ \alpha \int_{\Omega} U'(\zeta(\bar{k},\bar{s},\omega'))\beta f_s(\bar{k},\bar{s}-\frac{1}{\beta}\lambda(k,s,\omega')) \cdot F_{\omega}(x^*(k,s,\omega'))p_{\omega}(\omega') + \alpha \int_{\Omega}\int_{x^*(k,s,\omega)} U'(\zeta(k,s,X'))\ \beta h_s(k,s-\frac{1}{\beta}\sigma(k,s,X'),m(X')) \cdot dF_{\omega}(x^*(k,s,\omega'))p_{\omega}(\omega') \right\} \cdot \left\{ \frac{1}{f_s(k,s-\frac{1}{\beta}\lambda(k,s,\omega))} \right\}$$

We now turn to the switchpoint level of performance of the development process given by $x^* = x^*(k,s,\omega)$ noting that x^* solves the equation:

$$W(k,s,x^*(k,s,\omega)) = Q(k,s,\omega)$$

i) The switchpoint increases when the status of the research process improves.

$$\frac{\partial x^*}{\partial \omega} = \frac{Q\omega}{W_x} > 0 \ .$$

ii) The change in the switchpoint with respect to changes in the capital stock and the available stock of the exhaustible resource will depend on the relative marginal utilities and the relative marginal products before and after the innovation is introduced.

a) $\frac{\partial x^*}{\partial k} \gtreqless 0$ as

$$\frac{U'(\zeta(k,s,\omega))}{U'(\zeta(k,s,X))} \gtreqless \frac{h_k(h,s-\frac{1}{\beta}\sigma(k,s,X),m(X)) + (1-\delta)}{f_k(k,s-\frac{1}{\beta}\lambda(k,s,\omega)) + (1-\delta)}$$

b) $\frac{\partial x^*}{\partial s} \gtreqless 0$ as

$$\frac{U'(\zeta(k,s,\omega))}{U'(\zeta(k,s,X))} \gtreqless \frac{h_s(k,s-\frac{1}{\beta}\sigma(k,s,X),m(X))}{f_s(k,s-\frac{1}{\beta}\lambda(k,s,\omega))}$$

Two special cases may be of interest at this point. We first consider the case where the introduction of the innovation will result in obsolescence of the capital stock so that the rewards from stopping will be independent of the current level of capital, (we assume here that after the innovation, capital is no longer essential to production)[4]. Then, the minimum acceptable quality of the developed innovation will always rise with increases in the capital stock since

$$\frac{\partial x^*}{\partial k} = \frac{Q_k(k,s,\omega)}{W_x(0,s,x^*(k,s,\omega))} > 0 .$$

This is what might be expected since it will take a better innovation to justify the destruction of a larger capital stock. Otherwise the sign of x^* in k may not be so clear since a higher capital stock increases the gains from stopping as well as the gains from continuing. The possibility that a portion of the aggregate capital stock may become obsolete when an innovation is introduced is considered by Hicks [13] (p. 300), who states that "every technical improvement implies a loss of capital". The case considered by Hicks where consumption rises slightly, and the marginal product of capital is greatly increased (due to the innovation as well as the fall in the capital stock) has an ambiguous effect on x^* .

Another special case involves the possibility that the exhaustible natural resource will have no economic value after the innovation is introduced. If we assume here that $h_s = 0$, the sign of $\frac{\partial x^*}{\partial s}$ will be clearly positive, implying that as the stock of resources falls over time the switch-point will also fall. So, given a constant capital stock, a lower quality innovation will become acceptable as the finite stock of the natural resource is depleted.

IV Conclusion

We have considered a number of approaches to technological change and economic growth emphasizing the essential separation of basic research from the development process and the periodic reevaluation of the successive innovations. The level of the capital stock and the remaining stock of an exhaustible natural resource were shown to have an effect on the process of periodic reevaluation through the stopping rule. Also the uncertainties associated with research and development were shown to have an effect on the level of investment and on the amount of effort devoted to the R + D process. The emphasis on periodic review and the derivation of simple stopping rules with respect to 'switchpoint' levels of performance of the development process should allow practical application for economic planning.

The approach followed here should be used to derive probability distributions on the duration of the search for an innovation and the total amount of expenditures which will be required. Lucas [22] finds the uncertain project completion time to be "the most critical of the many uncertainties involved in the problem" (p. 696). Finding the expected completion time from more fundamental uncertainties concerning the outcome of R + D projects may yield interesting results.

A shortcoming of the present approach is that it is difficult to characterize several policy functions at once. Better results may be achieved perhaps with a continuous time formulation.

The next step which needs to be taken is to construct a model of the competitive market, using the approach to R + D presented here, to observe the effects of market structure and intensity of competition on the timing and quality of innovations. It would also be interesting to study the effects of market entry and technological externalities on the optimal level of R + D expenditures when firms can periodically review the outcomes of their research and development projects.

FOOTNOTES

1. This point was made for operations research by Deshmukh-Chikte [9]. See also Scherer [29], Kamien and Schwartz [16] and Loury [21].

2. This approach is similar to two models which employ finite state spaces. First, a model of Derman [8] is concerned with optimal maintenance policies. Secondly, a model of Lippman and McCall [20] deals with job search in a dynamic economy. What Lippman and McCall add to the discussion is the critical separation of the state of the economy from the state faced by the individual searcher (the best wage offer). This implies that the reservation wage is based on the state of the economy and is independent of the current wage offer.

3. This theorem is based on fundamental results of Maitra [23], Blackwell [2], Dubins and Savage [10], Strauch [32] and Denardo [7]. The theorem provides sufficient conditions for the existence of stationary optimal (measurable) plans for problems such as those recently considered by Mirrlees [26], Levhari-Srinivasan [19], Hahn [11] and Brock-Mirman [3] if a bounded action space is an acceptable assumption. For a discussion of the meaning of $\overline{p}$-optimality see Hinderer [14], pp. 9-12.

   Hinderer gives the following definition:

   Definition The plan $f^* \in \Delta$ is called $\overline{p}$-optimal (i.e. optimal in the mean with respect to p) if

$$G_f = \sup_{f \in \Delta} G_f \equiv G \ .$$

G is called the maximal expected total reward. Δ is the set of deterministic plans.

4. The complete destruction of the capital stock is assumed by Dasgupta and Heal [5] and Kamien and Schwartz [15]. They also assume that the economic value of the remaining stock of natural resources falls to zero after the innovation is introduced. Dasgupta and Heal find it very likely that "if fusion reactors ever become a commercial proposition ... then power stations generating electricity from fossil fuels will be rapidly phased out". (p. 21).

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