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STRATEGIC BEHAVIOUR IN PLANNING PROCEDURES WITH PRIVATE GOODS *

by

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Abstract: This paper studies the consequences of a certain type of misrepresentation of preferences by the consumers on the properties of a dynamic adjustment process in an economy with only private goods.

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I. INTRODUCTION

Both in economies with public goods and in economies with only private goods, it has been proved that there are no mechanisms selecting Pareto optima that are both incentive compatible and individually rational (Hurwicz [1972]; Roberts [1977]). This means that there must always arise situations when an economic agent, who is asked to give information and who has the no-trade option will find that it is possible to misrepresent the information and be strictly better off. The information he is asked to give is about his preferences or perhaps his costs; the crucial point is that this information is private, the other agents cannot check the accuracy of this consumer's announcements. The "no-trade" option means that if the outcome of the mechanisms is harmful to the consumer he can choose to stay at the initial position.

On the other hand, the literature on public goods contains studies in which specific assumptions made on the behaviour of the agents allow positive results on incentives to be obtained. Groves and Ledyard [1977] and Green and Laffont [1975] have defined mechanisms that are incentive compatible. Roberts [1977] and Schoumaker [1976] have studied the consequences of misrepresentation in dynamic adjustment processes when consumers choose their announcements to maximize their change in utility at each instant in the procedure. Though consumers typically misrepresent, these procedures still lead the economy to a Pareto optimal state.

This paper studies the properties of a dynamic procedure, of the type introduced by Malinvaud [1970] for an economy with only private goods when consumers choose their messages to maximize their increase in utility at each instant in the procedure. Each iteration is studied as a non-cooperative game where the consumer's strategy choice is his message and his payoff is his change in utility; we shall study the Nash eqilibria of those games.

The next section describes the procedure, as Tulkens and Zamir [1976] have defined it, and discusses the assumptions we shall make on the behavior of the consumers. The way consumers misrepresent and the effect of this misrepresentation on the properties of the procedure is analysed in two cases. In the first, which is a special version of the procedure due to Tulkens and Zamir, misrepresentation does not change the structure of the procedure but accelerates its convergence to an efficient state (Section III). In the second, the form of the procedure is altered by misrepresentation but it still converges to an optimum state though more slowly than when consumers report truthfully (Sections IV and V). It is also shown that the equilibrium misrepresentations at any instant are stable: given an arbitrary set of announcements, successive revisions of their announcement according to a specific rule ensures convergence to the equilibrium messages.

In Section VI these results are compared to those obtained by Roberts in the case of public goods.

II. THE PROCEDURE AND THE LOCAL GAME

The economy studied here consists of H private goods, indexed $h = 1,\ldots,H$; good 1 is taken as the numeraire. There are n consumers indexed by $i = 1 \ldots n$; x_{ih} denotes the quantity of good h available to consumer i. Each consumer has a strictly quasi concave, differentiable utility function $U_i(x_{i1},x_{ih},\ldots,x_{iH})$, which is strictly increasing in the numeraire good, and a vector of initial endowments denoted $\omega_i = (\omega_{i1},\ldots,\omega_{iH})$. An allocation is a $(n \times H)$-tuple $(x_{11},\ldots,x_{ih},\ldots,x_{nH})$.

The planning procedure is the continuous time process in Tulkens and Zamir [1976]. Consumers are asked to report their (true) marginal rates of substitution (π_{ih}) between each good $h = 2 \ldots H$ and the numeraire.

At each instant in the procedure the following changes in allocation take place for each consumer

(i) for each good other than the numeraire, consumer i's holding of that good changes proportionally to the difference between his marginal rate of substitution (π_{ih}) and the average marginal rate of substitution $(\bar{\pi}_h)$.

(ii) for the numeraire good the change is composed of two parts: (1) consumer i pays for the increase in his holding of goods and is paid for the decreases, at a price equal to his marginal rate of substitution; (2) he receives a fraction of the surplus that is generated by this system of payments.

Formally: (a dot above a variable indicates a time derivative $\dot{x} = dx/dt$)

$$\dot{x}_{ih} = a(\pi_{ih} - \bar{\pi}_h) \quad h = 2 \ldots H \; ; \; i = 1 \ldots n$$

$$\dot{x}_{i1} = - \Sigma_{h\neq 1}\pi_{ih}\,\dot{x}_{ih} + \gamma_i a\, \Sigma_j \Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2 \quad i = 1 \ldots n$$

where $\bar{\pi}_h = \frac{1}{n}\Sigma_j \pi_{jh}$ is the average marginal rate of substitution,

$\pi_{ih} \equiv \frac{\delta U_i / \delta x_{ih}}{\delta U_i / \delta x_{i1}}$ is consumer i's marginal rate of substitution between good h and the numeraire good, 1,

$\Sigma_j \Sigma_{h \neq i} (\pi_{jh} - \bar{\pi}_h)^2$ is the surplus

$\gamma_i \geq 0$ is the coefficient of redistricution of the surplus, and

$\Sigma_j \gamma_j = 1.$

Tulkens and Zamir show that if consumers report their preferences. correctly, this procedure converges from any initial allocation to a Pareto optimum. It is easy to see that along the path of this procedure each consumer's utility increases monotonically.

We shall drop the assumptions that the marginal rates of substitution, on which the process is based, are the correct ones and assume instead that consumers behave in a non-cooperative competitive way: at every instant each consumer is supposed to choose his announced marginal rates of substitution (Ψ_{ih}, $h = 2 \ldots H$), given the announcements of the other consumers, to maximise his increase in utility ($\dot{U}_i = dU_i/dt$) determined by the process at that instant in the procedure.

Thus his problem is to maximize, via his choice of $\Psi_i = (\Psi_{i2}, \ldots, \Psi_{iH})$ the function:

$$
\begin{aligned}
\dot{U}_i &= v_i(\Psi_1, \ldots, \Psi_n) \\
&= \frac{\delta U_i}{\delta x_{i1}} [-\Sigma_{h \neq 1} \Psi_{ih} \dot{x}_{ih} + \gamma_i \, a \, \Sigma_j \Sigma_{h \neq 1} (\Psi_{jh} - \bar{\Psi}_h)^2] \\
&\quad + \Sigma_{h \neq 1} \frac{\delta U_i}{\delta x_{ih}} a (\Psi_{ih} - \bar{\Psi}_h)
\end{aligned}
$$

We shall be studying the non-cooperative local game where the consumers' announced marginal rates of substitution are their strategies and their increase in utility at that instant is their payoff. The equilibrium concept we shall use in this game is the Nash equilibrium: a set of announcements $\Psi_1^*,\ldots,\Psi_n^*$ defined by the property that, for each i, if the other consumers announce Ψ_j^*, $j \neq i$, consumer i maximizes his increase in utility at that instant by announcing $\Psi_i^* = (\Psi_{i2}^*,\ldots,\Psi_{iH}^*)$. We shall express these equilibrium announcements as functions of the consumers' true marginal rates of substitution and see how this misrepresentation affects the procedure.

III. BEST REPLY STRATEGIES AND NASH EQUILIBRIUM IN THE μ PROCESS

(a) The Procedure

Tulkens and Zamir study the choice of the γ_i's: how do consumers decide upon the redistribution of the surplus. They formalize the idea of the selection of the redistribution coefficients by looking at it as a cooperative game and study various solution concepts, one of them being the core. They exhibit a particular core allocation, called the μ-imputation, which leads to redifining the procedure in the following way.

$$\dot{x}_{ih} = a(\pi_{ih} - \bar{\pi}_h) \quad h = 2 \ldots H, \; i = 1 \ldots n$$

$$\dot{x}_{i1} = - \Sigma_{h\neq 1}\pi_{ih}\dot{x}_{ih} + a\, \Sigma_{h\neq 1}(\pi_{ih} - \bar{\pi}_h)^2$$

This corresponds to specifying γ_i as $\dfrac{\Sigma_{h\neq 1}(\pi_{ih} - \bar{\pi}_h)^2}{\Sigma_j\Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2}$

In this case consumer i receives a share of the surplus which is equal to his contribution to the total surplus. His announcement affects the value of his γ_i and he knows this.

We shall see that in this case misrepresentation does not alter the structure of the procedure but accelerates its convergence.

With misrepresentation the procedure becomes

$$\dot{x}_{ih} = a(\Psi_{ih} - \bar{\Psi}_h) \quad h = 2,\ldots,H \qquad i = 1 \ldots n$$

$$\dot{x}_{i1} = - \Sigma_{h\neq 1}\Psi_{ih}\,\dot{x}_{ih} + a\, \Sigma_{h\neq 1}(\Psi_{ih} - \bar{\Psi}_h)^2$$

this last equation can be rewritten as

$$\dot{x}_{i1} = a\ \Sigma_{h\neq 1}[(-\ \Psi_{ih} + \Psi_{ih} - \bar{\Psi}_h)(\Psi_{ih} - \bar{\Psi}_h)]$$

$$= a\ \Sigma_{h\neq 1}(-\ \bar{\Psi}_h)(\Psi_{ih} - \bar{\Psi}_h)$$

$$= -\ a\ \Sigma_{h\neq 1}\bar{\Psi}_h(\Psi_{ih} - \bar{\Psi}_h)$$

(b) Best Reply Strategies

Consumer i is going to choose his Ψ_{ih}'s to maximize:

$$\dot{U}_i = v_i(\Psi_1,\ldots,\Psi_n)$$

$$= \Sigma_h \frac{\delta U_i}{\delta x_{ih}}\ \dot{x}_{ih}$$

$$= \frac{\delta U_i}{\delta x_{i1}}[\Sigma_{h\neq 1}\pi_{ih}\ a(\Psi_{ih} - \bar{\Psi}_h) - a\ \Sigma_{h\neq 1}\bar{\Psi}_h(\Psi_{ih} - \bar{\Psi}_h)]$$

$$= \frac{\delta U_i}{\delta x_{i1}} a\ \Sigma_{h\neq 1}(\Psi_{ih} - \bar{\Psi}_h)(\pi_{ih} - \bar{\Psi}_h)$$

Maximizing this expression with respect to $\{\Psi_{ih}\}_{h\neq 1}$ we have:

$$\frac{\delta U_i}{\delta x_{i1}}[a(1 - \frac{1}{n})(\pi_{ih} - \bar{\Psi}_h) + a(\Psi_{ih} - \bar{\Psi}_h)(-\ \frac{1}{n})] = 0$$

$$(1 - \frac{1}{n})\pi_{ih} - \bar{\Psi}_h(1 - \frac{1}{n} - \frac{1}{n}) - \Psi_{ih}\frac{1}{n} = 0$$

$$(1 - \frac{1}{n})\pi_{ih} - \bar{\Psi}_h(1 - \frac{2}{n}) - \Psi_{ih}\frac{1}{n} = 0$$

$$(n-1)\pi_{ih} - \bar{\Psi}_h(n-2) - \Psi_{ih} = 0 \qquad (1)$$

This is the best reply. On the other hand if every consumer chooses this best reply strategy one obtains the remarkable property that $\bar{\Psi}_h = \bar{\pi}_h$. Indeed summing (1) over i and dividing by n we have

$$(n-1)\bar{\pi}_h - (n-2)\bar{\Psi}_h - \bar{\Psi}_h = 0$$

therefore

$$\bar{\Psi}_h = \bar{\pi}_h. \tag{2}$$

(c) Nash Equilibrium

A Nash equilibrium is a set of strategies Ψ^*_{ih} $i = 1 \ldots h,\ h = 2 \ldots H$ such that every consumer i's best reply strategy, if the other consumers have announced Ψ^*_{jh}, is to announce Ψ^*_{ih}. (1) and (2) allow us to express the Nash equilibrium values of Ψ_{ih} as functions of the true marginal rates of substitution

$$\Psi_{ih} = (n-1)\pi_{ih} - (n-2)\bar{\pi}_h$$

$$= \lambda\pi_{ih} + (1-\lambda)\bar{\pi}_h \qquad \lambda = n-1 \geq 1$$

Consumer i's Nash eqilibrium strategy is thus, for each h, a combination of his true marginal rate of substitution and of the average of the (true) marginal rates of substitution. More specifically, if consumer i is buying good $h(\pi_{ih} > \bar{\pi}_h)$, he overreports his marginal rate of substitution $(\Psi_{ih} > \pi_{ih})$. On the other hand, if he is selling the good $(\pi_{ih} < \bar{\pi}_h)$ he underreports his marginal rate of substitution $(\Psi_{ih} < \pi_{ih})$. This result is somewhat surprising but can be understood in the following way: take for example the case of overreporting when the consumer is buying the good; the excess payment will be returned to him as surplus so he will not lose from overreporting.

(d) The Process

We can now replace the Ψ_{ih}'s by their values, in terms of the true rates, in the procedure. The procedure with misrepresentation becomes:

$$\dot{x}_{ih} = a\ (\Psi_{ih} - \bar{\Psi}_h) = a\ ((n-1)\ \pi_{ih} - (n-2)\ \bar{\pi}_h - \bar{\pi}_h)$$

$$= a\ (n-1)(\pi_{ih} - \bar{\pi}_h)$$

$$= a\ \lambda\ (\pi_{ih} - \bar{\pi}_h) \qquad \lambda = n-1 \geq 1$$

$$\dot{x}_{i1} = -a\ \Sigma_{h\neq 1}\bar{\Psi}_h(\Psi_{ih} - \bar{\Psi}_h)$$

$$= -a\ \Sigma_{h\neq 1}\ \bar{\pi}_h\ ((n-1)\ \pi_{ih} - (n-2)\ \bar{\pi}_h - \bar{\pi}_h)$$

$$= -a\ \Sigma_{h\neq 1}\ \bar{\pi}_h\ (\pi_{ih} - \bar{\pi}_h)(n-1)$$

$$= -\ \Sigma_{h\neq 1}\ \pi_{ih}\ \dot{x}_{ih} + \lambda\ a\ \Sigma_{h=1}(\pi_{ih} - \bar{\pi}_h)^2$$

Thus the only effect of misrepresentation is that the speed of adjustment increases from a to $\lambda a = a(n-1)$: the procedure converges more rapidly. The proof of convergence of Tulkens and Zamir is directly applicable here since it does not depend upon the speed of adjustment. Thus despite misrepresentation the procedure still computes Pareto optima.

(e) Stability of the Nash Equilibrium

The next question is whether or not this Nash equilibrium at each instant is stable: starting from any given set of messages, if each consumer i adjusts his announcement according to some rule do their strategy choices converge to a Nash equilibrium? This requires defining another dynamic process to describe the adjustment of the Ψ_{ih}.

The first step is to define an adjustment rule for the announced marginal rates of substitution: To do this let us rewrite (1) explicitly in terms of Ψ_{ih}:

$$(n-1)\pi_{ih} - \overline{\Psi}_h(n-2) - \Psi_{ih} = 0$$

$$(n-1)\ \pi_{ih} - \Sigma_{j\neq i}\ \Psi_{jh}\ \frac{n-2}{n} - \frac{n-2}{n}\ \Psi_{ih} - \Psi_{ih} = 0$$

$$\Psi_{ih}\ [1+ \frac{n-2}{n}] = (n-1)\ \pi_{ih} - \Sigma_{j\neq i}\Psi_{jh}\ \frac{n-2}{n}$$

$$\Psi_{ih} = \frac{n}{2(n-1)}\ [(n-1)\ \pi_{ih} - \frac{n-2}{n}\ \ \Sigma_{j\neq i}\Psi_{jh}]$$

We shall assume that consumer i adjusts his announced marginal rate of substitution to reduce the discrepancy between his best reply choice and his message. Since we are in a continuous time process, we shall specify a differential equation:

$$\dot{\Psi}_{ih} = k_i\{\frac{n}{2(n-1)}\ [(n-1)\pi_{ih} - \frac{n-2}{n}\ \Sigma_{j\neq i}\Psi_{jh}] - \Psi_{ih}\} \qquad h = 2 \ldots H \tag{3}$$

This system has the same convergence properties as the homogeneous system:

$$\dot{\Psi}_{ih} = k_i[\ \frac{-(n-2)}{2(n-1)}\ \Sigma_{j\neq i}\Psi_{jh} - \Psi_{ih}] \qquad h = 2 \ldots H \tag{4}$$

or in matrix notation

$\dot{\Psi} = KM\Psi$ where K is a diagonal matrix with element k_i on the diagonal

and M is such that $h_{ij} = -1$ if $i = j$

$= \frac{-(n-2)}{2(n-1)}$ if $i \neq j$

The convergence of this system to a Nash equilibrium can be proved by showing that the eigenvalues of the Matrix KM have negative real parts. (See for example Golomb and Shanks (1965).) This is done in Appendix 1. Consequently whatever the initial announcements of the consumers, if they adjust their messages according to (3), their announced marginal rates of substitution will converge to their Nash equilibrium values.

IV. BEST REPLY STRATEGIES AND NASH EQULIBRIUM WITH FIXED DISTRIBUTION PROFILE

In Section III the redistribution coefficients γ_i were functions of the consumers messages. Here we shall analyse the procedure when the γ_i are fixed a priori and satisfy: $0 < \gamma_i < 1$ for all i, $\Sigma_i \gamma_i = 1$.

(a) Best Reply Strategies

Consumer i will choose Ψ_{ih}, $h = 2,\ldots,H$ to maximize

$$v_i(\Psi_1,\ldots,\Psi_n) = \dot{U}_i$$

$$= \Sigma_h \frac{\delta U_i}{\delta x_{ih}} \dot{x}_{ih}$$

$$= \Sigma_{h\neq 1} \frac{\delta U_i}{\delta x_{ih}} a(\Psi_{ih} - \overline{\Psi}_h) + \frac{\delta U_i}{\delta x_{i1}} (-\Sigma_{h\neq 1}\Psi_{ih} \dot{x}_{ih} + \gamma_i a \Sigma_j \Sigma_{h\neq 1}(\Psi_{jh} - \overline{\Psi}_h)^2)$$

$$= \Sigma_{h\neq 1} \frac{\delta U_i}{\delta x_{ih}} a(\Psi_{ih} - \overline{\Psi}_h) + \frac{\delta U_i}{\delta x_{i1}} (-\Sigma_{h\neq 1} \Psi_{ih} a(\Psi_{ih} - \overline{\Psi}_h) + \gamma_i a \Sigma_j \Sigma_{h\neq 1}(\Psi_{jh} - \overline{\Psi}_h)^2)$$

The maximization yields:

$$\frac{\delta v_i}{\delta \Psi_{ih}} = \frac{\delta U_i}{\delta x_{ih}} a(1 - \frac{1}{n}) + \frac{\delta U_i}{\delta x_{i1}} [-a(\Psi_{ih} - \bar{\Psi}_h) - \Psi_{ih}\, a(1 - \frac{1}{n})$$

$$+ \gamma_i\, 2\, a\, \Sigma_j(\Psi_{jh} - \bar{\Psi}_h)(- \frac{1}{n})$$

$$+ \gamma_i\, 2\, a\, (\Psi_{ih} - \bar{\Psi})] = 0 \qquad h = 2 \dots H$$

Dividing by a and by $\frac{\delta U_i}{\delta x_{i1}}$ and taking into account the fact that

$\Sigma_j(\Psi_{jh} - \bar{\Psi}_h) = 0$ this equation becomes:

$$\pi_{ih}(1 - \frac{1}{n}) - (\Psi_{ih} - \bar{\Psi}_h) - \Psi_{ih}(1 - \frac{1}{n}) + 2\, \gamma_i(\Psi_{ih} - \bar{\Psi}_h) = 0 \qquad (5)$$

$$\pi_{ih}(1 - \frac{1}{n}) - (\Psi_{ih} - \frac{1}{n}\Psi_{ih} - \frac{1}{n}\Sigma_{j\neq i}\Psi_{jh}) - \Psi_{ih}(1 - \frac{1}{n})$$

$$+ 2\gamma_i(\Psi_{ih} - \frac{1}{n}\Psi_{ih} - \frac{1}{n}\Sigma_{j\neq i}\Psi_{jh}) = 0$$

$$\pi_{ih}(1 - \frac{1}{n}) + \Psi_{ih}(1 - \frac{1}{n})(- 1 - 1 + 2\gamma_i) + \frac{1}{n}\Sigma_{j\neq i}\Psi_{jh}(1 - 2\gamma_i) = 0$$

$$\Psi_{ih}(1 - \frac{1}{n})(2 - 2\gamma_i) = \pi_{ih}(1 - \frac{1}{n}) + \frac{1}{n}\Sigma_{j\neq i}\Psi_{jh}(1 - 2\gamma_i)$$

$$\Psi_{ih} = \frac{1}{2-2\gamma_i}\pi_{ih} + \frac{1-2\gamma_i}{(n-1)(2-2\gamma_i)}\Sigma_{j\neq i}\Psi_{jh}$$

$$= \frac{1}{2-2\gamma_i}\pi_{ih} + (1 - \frac{1}{2-2\gamma_i})\frac{1}{n-1}\Sigma_{j\neq i}\Psi_{jh}$$

Given the other consumers' announcements, consumer i, to maximize his increase in utility at that instant t should announce a marginal rate of substitution that is a combination of his true marginal rate of substitution and of the average of the other consumers' announced marginal rates of substitution.

If $\gamma_i = \frac{1}{2}$ then to announce the truth $(\Psi_{ih} = \pi_{ih})$ is the best strategy of consumer i regardless of what the others have announced, i.e. announcing the truth is a dominant strategy.

(b) Nash Equilibrium

We are now going to compute the Nash equilibrium strategies of the consumers, as functions of their true marginal rate of substitution. Another way of developing (5) yields:

$$\Psi_{ih}(-1 - 1 + \frac{1}{n} + 2\gamma_i) + \bar{\Psi}(1 - 2\gamma_i) + \pi_{ih}(1 - \frac{1}{n}) = 0$$

$$\Psi_{ih}(-2 + 2\gamma_i + \frac{1}{n}) + \bar{\Psi}_h(1 - 2\gamma_i) = -\pi_{ih}(1 - \frac{1}{n}) \qquad i = 1,\ldots,n$$

For each h we have a system of equations which can be written in matrix notation:

$$[\Lambda - \Delta e e']\Psi_h = -(1 - \frac{1}{n})\pi_h$$

where Λ is a diagonal matrix with $\lambda_i = (-2 + 2\gamma_i + \frac{1}{n})$ and Δ is a diagonal matrix with $\delta_i = \frac{-1+2\gamma_i}{n}$; e is a vector of ones and Ψ_h and π_h are vectors of the Ψ_{ih} and π_{ih} respectively.

The matrix $[\Lambda - \Delta e e']$ can be inverted if for all $i = 1 \ldots n$ $\gamma_i < 1 - \frac{1}{2n}$. We can compute the announced marginal rates of substitution (the computations are in Appendix 2)

$$\Psi_{ih} = \hat{\pi}_{ih} \frac{1-n}{2n(\gamma_i - 1)+1} + \frac{\Sigma_j \pi_{jh} \dfrac{1-n}{2n(\gamma_j - 1)+1}}{1 - \Sigma_j \dfrac{2\gamma_j - 1}{2n(\gamma_j - 1)}} \; \frac{2\gamma_i - 1}{2n(\gamma_i - 1)+1} \tag{6}$$

This defines the values of the Nash equilibrium strategies as a function (so the equilibrium is unique) of the distribution profile γ, the true marginal rates of substitution and the number of consumers. Consumer i's Nash equilibrium strategy thus appears as a complicated weighted average of his own marginal rate of substitution and of some average of the other consumer's marginal rates of substitution

In the next section we shall make a particular simplifying assumption on the γ_i's and study the properties of the procedure thus defined.

V EQUAL REDISTRIBUTION COEFFICIENTS

If each consumer receives the same share of the surplus ($\gamma_i = \frac{1}{n}$) expression (4) is simplified.

$$2n(\gamma_i - 1) + 1 = 2n(\frac{1}{n} - 1) + 1 = 2 - 2n + 1 = 3 - 2n$$

$$2\gamma_i - 1 = \frac{2}{n} - 1 = \frac{2-n}{n}$$

Equation (4) thus becomes:

$$\Psi_{ih} = \pi_{ih} \frac{1-n}{3-2n} + \frac{\frac{1-n}{3-2n} \Sigma_j \pi_{jh}}{1-n \frac{2-n}{n} \frac{1}{3-2n}} \cdot \frac{(2-n)/n}{3-2n}$$

$$= \pi_{ih} \frac{1-n}{3-2n} + \frac{\frac{1-n}{3-2n} \Sigma_j \pi_{jh}}{\frac{3-2n-2+n}{3-2n}} \frac{(2-n)/n}{3-2n}$$

$$= \pi_{ih} \frac{1-n}{3-2n} + \bar{\pi}_h \frac{2-n}{3-2n}$$

$$= \lambda \pi_{ih} + (1-\lambda) \bar{\pi} \qquad \text{where } \lambda = \frac{1-n}{3-2n} \quad 0 < \lambda < 1$$

Notice that as in Section III we get:

$$\bar{\Psi}_h = \frac{1}{n} \Sigma_i \Psi_{ih} = \bar{\pi}_h$$

Consumer i's message is thus a convex combination of his true marginal rate of substitution and of the average one. When the number of consumers is small, consumer i puts more weight on his own rate (when $N = 3$, λ is equal to 2/3) but as the number of consumers grows he puts more and more weight on the average marginal rate of substitution and at the limit he weighs both equally ($n \to \infty, \lambda \to \frac{1}{2}$). $\lambda = 1$ when there are two consumers and then it is a dominant strategy for the consumer to report truthfully. This result has already been noted. (see section IV)

By taking the convex combination, the consumer is making an announcement that is closer to the average one than his true marginal rate of substitution. If he is selling the good $(\pi_{ih} < \bar{\pi}_h)$ he will overreport his price $(\pi_{ih} < \Psi_{ih} < \bar{\Psi}_h = \bar{\pi}_h)$; if on the other hand he is buying the good $(\pi_{ih} > \bar{\pi}_h)$ he will underreport his price $(\pi_{ih} > \Psi_{ih} > \bar{\Psi}_h = \bar{\pi})$. Such behavior has obvious intuitive appeal.

We will now write down the revised procedure and check its feasibility, monotonicity and convergence to a Pareto optimum.

1. The Revised Procedure.

The procedure defines the changes in allocation that will happen in terms of the announced marginal rates of substitution. These have been expressed in terms of the true marginal rates of substitution. We shall now try to express the procedure in terms of the true marginal rates of substitution.

$$x_{ih} = a(\Psi_{ih} - \bar{\Psi}_{ih}) = a(\lambda\,\pi_{ih} + (1-\lambda)\,\bar{\pi}_h - \bar{\pi}_h)$$

$$= a\,\lambda\,(\pi_{ih} - \bar{\pi}_h) \tag{7}$$

These changes in allocation are thus of the same structure as when consumers report truthfully but the speed of adjustment is reduced. If there are few consumers λ is large. When n goes to infinity λ goes to 1/2.

$$\dot{x}_{i1} = -\Sigma_{h\neq 1} \Psi_{ih} \dot{x}_{ih} + \frac{1}{n} a \Sigma_j \Sigma_{h\neq 1} (\Psi_{jh} - \bar{\Psi}_h)^2$$

$$= -\Sigma_{h\neq 1} (\lambda \pi_{ih} + \bar{\pi}_h(1-\lambda)) \; a \lambda (\pi_{ih} - \bar{\pi}_h)$$

$$+ \frac{1}{n} a \Sigma_j \Sigma_{h\neq 1} (\lambda(\pi_{jh} - \bar{\pi}_h))^2$$

$$= -\Sigma_{h\neq 1} \pi_{ih} \; a \lambda (\pi_{ih} - \bar{\pi}_h) + a (1-\lambda) \lambda \Sigma_{h\neq 1}(\pi_{ih} - \bar{\pi}_h)^2$$

$$+ \frac{1}{n} a \Sigma_j \Sigma_{h\neq 1} \lambda(\pi_{jh} - \bar{\pi}_h)^2$$

$$- \frac{1}{n} a \Sigma_j \Sigma_{h\neq 1} (1-\lambda) \lambda (\pi_{jh} - \bar{\pi}_h)^2$$

$$\dot{x}_{i1} = -a \lambda \Sigma_{h\neq 1} \pi_{ih}(\pi_{ih} - \bar{\pi}_h) + a \frac{1}{n} \lambda \Sigma_j \Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2$$

$$+ \lambda(1-\lambda) \; a \; [\Sigma_{h\neq 1}(\pi_{ih} - \bar{\pi}_h)^2 - \frac{1}{n} \Sigma_j \Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2] \qquad (8)$$

When consumers misrepresent their preferences, the change in numeraire, expressed in terms of the true marginal rates of substitution, can be decomposed in three parts:

(i) the first is the payment for the changes in the levels of the other goods. This term is similar to the one in the procedure when consumers report truthfully their preferences except for the speed of adjustment instead of a we have $a\lambda$ which is a slower speed of adjustment.

(ii) the second part is again of the same type as the second element of the change in numeraire good when consumers report correctly: it is the redistribution of the surplus. Here too the only difference is in the speed of adjustment.

(iii) in addition to these two terms, when consumers misrepresent their preferences there is an additional quadratic term: consumers receive or pay a transfer that is proportional to the difference between their contribution to the variance $(\Sigma_{h\neq 1}(\pi_{ih} - \bar{\pi}_h)^2)$ and the average variance $(\frac{1}{n}\Sigma_j\Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2)$. If consumer i is "more" different than the average consumer, he receives a transfer from the other consumers.

2. Feasibility.

This is easily confirmed

$\Sigma_i \dot{x}_{ih} = 0$ obviously for each $h = 2,\ldots,H$

$$\Sigma_i \dot{x}_{i1} = - a\lambda\Sigma_{h\neq 1}[\Sigma_i \pi_{ih}^2 - n\pi_h^2 - \Sigma_j \pi_{jh}^2 - n\bar{\pi}_h^2 + 2n\bar{\pi}_h^2]$$

$$+ \lambda(1-\lambda)\, a\, [\Sigma_{h\neq 1}\Sigma_i(\pi_{ih} - \bar{\pi}_h)^2 - \Sigma_j\Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2]$$

$$= 0$$

3. Monotonicity.

We want to determine the sign of $\dot{U}_i$. In the procedure where all consumers report their preferences correctly we know that utilities are non-decreasing. In the present case,

$$\dot{U}_i = \Sigma_h \frac{\delta U_i}{\delta x_{ih}} \dot{x}_{ih}$$

$$= \frac{\delta U_i}{\delta x_{i1}}[\Sigma_{h\neq 1}\pi_{ih}\lambda a(\pi_{ih} - \bar{\pi}_h) - a\lambda\Sigma_{h\neq 1}\pi_{ih}(\pi_{ih} - \bar{\pi}_h)$$

$$+ \frac{1}{n}\lambda\Sigma_j\Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2$$

$$+ \lambda(1-\lambda)\ a\ [\Sigma_{h\neq 1}(\pi_{ih} - \bar{\pi}_h)^2 - \frac{1}{n}\Sigma_j\Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2]$$

$$= \frac{\delta U_i}{\delta x_{i1}}[\lambda(1-\lambda)\ a\ \Sigma_{h\neq 1}(\pi_{ih} - \bar{\pi}_h)^2 + \frac{1}{n}\lambda^2\ \Sigma_j\Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2]$$

$$\geq 0.$$

In the procedure where everybody reports his preferences correctly

$$\dot{U}_i = \frac{\delta U_i}{\delta x_{i1}}\ \frac{1}{n}\ a\ \Sigma_j\ \Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2$$

Can these two magnitudes be compared? If consumer i is the average consumer ($\pi_{ih} = \bar{\pi}_h \ \forall\ h$) then he clearly gains less in utility than he would have if everyone had reported his preferences truthfully. If he is different from the average this loss will be in part compensated for by the first term

$$\lambda(1-\lambda)\ a\ \Sigma_{h\neq 1}(\pi_{ih} - \bar{\pi}_h)^2$$

4. Convergence.

The system of equations that defines the procedure with misrepresentation is, as we have seen:

$$\dot{x}_{ih} = a\ \lambda\ (\pi_{ih} - \bar{\pi}_h) \qquad h = 2,\ldots,H.$$

$$\dot{x}_{ih} = -\ a\ \lambda\ \Sigma_{h\neq 1}\pi_{ih}(\pi_{ih} - \bar{\pi}_h) + \frac{1}{n}\lambda\ a\ \Sigma_j\ \Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2$$

$$+ \lambda(1-\lambda)\ a\ [\Sigma_{h\neq 1}(\pi_{ih} - \bar{\pi}_h)^2 - \frac{1}{n}\Sigma_j\ \Sigma_{h\neq 1}(\pi_{jh} - \bar{\pi}_h)^2]\ .$$

This can be interpreted as the procedure of Tulkens-Zamir but with a different surplus. Their surplus is:

$$a \frac{1}{n} \Sigma_j \Sigma_{h \neq 1} (\pi_{jh} - \bar{\pi}_h)^2$$

Ours is for the ith consumer

$$a \lambda \frac{1}{n} \Sigma_j \Sigma_{h \neq 1} (\pi_{jh} - \bar{\pi}_h)^2$$

$$+ \lambda(1-\lambda) \; a \; [\Sigma_{h \neq 1} (\pi_{ih} - \bar{\pi}_h)^2 - \frac{1}{n} \Sigma_j \Sigma_{h \neq 1} (\pi_{jh} - \bar{\pi}_h)^2]$$

If we can show that this new surplus has the same properties as theirs, their proof of convergence will be applicable to our procedure.

What has to be shown is that the surplus is Lipschitzian on any compact set. To show this, Tulkens and Zamir use the Lipschitzian property of the marginal rates of substitution. Our surplus is a function only of the marginal rates of substitution so the same line of reasoning as in their paper can be used.

In this case too, one may ask whether or not the Nash equilibrium that has been defined is stable. It is straightforward to apply the analysis of Appendix 1 to this case and show that indeed the equilibrium is stable.

VI. CONCLUSION

This paper has studied the consequences of dropping the assumption that consumers will reveal their preferences correctly in the context of planning procedures for the allocation of private goods. Instead of truthful revelation we have assumed that the consumers choose their announced marginal rates of substitution to maximize their increase in utility at that instant along the procedure. Because of the complexity of the general case we have considered two interesting special cases, one where each consumer, as his share of the surplus, receives the amount he has contributed to it (and recognizes this fact) and the other where all consumers are treated equally, in terms of their share of the surplus. In both instances consumers will typically misrepresent their preferences, but the direction of misrepresentation differs between the two.

We have shown that this misrepresentation does not affect the procedure and more specifically the procedure still converges to a Pareto optimal allocation which all prefer to the initial allocation.

These results are similar to that of Roberts [1977] who studied a procedure of the same type for public goods. In his study the redistribution coefficients are given any fixed value strictly between zero and one and the procedure with misrepresentation is the same as the one with correct revelation except for the speed of adjustment which is smaller. As we have seen the case with private goods only is more complex and the results are more varied. However when the coefficients of redistribution are fixed we also have the result of slower convergence.

APPENDIX 1

$$\dot{\psi} = KM\psi \qquad K = \begin{pmatrix} k_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \; k_n \end{pmatrix} \qquad k_i > 0 \;\forall\, i$$

$$M = (m-1)\,I - mee' \text{ where } e' = (1 \;\ldots\; 1)$$

<u>Lemma 1</u>

M is symetric negative definite

<u>Proof</u>

a) $Me = [m(1-n) - 1]e$ since $e'e = n$

$\lambda_1 = m(1-n) - 1 = -\frac{n}{2}$ characteristic root

$q_1 = e$ characteristic vector

b) Let x be such that $x'e = 0$

$Mx = (m-1)x$

the dimension of the space of x's such that $x'e = 0$ is $(n-1)$.

Therefore $\lambda_2 = \ldots = \lambda_n = m - 1 = \frac{-n}{2(n-1)} < 0$

Q.E.D.

<u>Lemma 2</u>

The roots of KM = roots of $K^{\frac{1}{2}} M K^{\frac{1}{2}}$

($\Rightarrow$ 0 $K^{\frac{1}{2}} M K^{\frac{1}{2}}$ is negative definite)

<u>Proof</u>

$$|\lambda I - KM| = |K|^{\frac{1}{2}} |\lambda I - KM| |K|^{\frac{1}{2}}$$
$$= |\lambda I - K^{\frac{1}{2}} M K^{\frac{1}{2}}|$$

Q.E.D.

APPENDIX 2

Λ is a diagonal matrix $\lambda_i = -2 + 2\gamma_i + \frac{1}{n}$

Δ is a diagonal matrix $\delta_i = \frac{-1 + 2\gamma_i}{n}$

We have the system:

$$[\Lambda - \Delta e e']\psi_h = -(1 - \frac{1}{n})\pi_h \qquad e' = (1 \dots 1)$$

$$[\Lambda - \Delta e e']^{-1} = \Lambda^{-1} + \Lambda^{-1}\Delta e(1 - e'\Lambda^{-1}\Delta e)^{-1} e'\Lambda^{-1}$$

(partitioned matrix formula, see for example Theil.)

Therefore:

$$\psi_h = -(1 - \frac{1}{n})\ [\Lambda^{-1}\pi_h + \frac{e'\Lambda^{-1}\pi_h}{1 - e'\Lambda^{-1}\Delta e}\ \Lambda^{-1}\Delta e]$$

and $e'\Lambda^{-1}\pi_h = \Sigma_i \frac{\pi_{ih}}{\lambda_i} = \Sigma_i \frac{\pi_{ih}}{-2 + 2\gamma_i + \frac{1}{n}}$

$$1 - e'\Lambda^{-1}\Delta e = 1 - \Sigma_i \frac{\delta_i}{\lambda_i} = 1 - \Sigma_i \frac{-1 + 2\gamma_i}{2n(\gamma_i - 1) + 1}$$

we can write:

$$\psi_{ih} = -(1 - \frac{1}{n}) \left[\frac{\pi_{ih}}{-2 + 2\gamma_i + \frac{1}{n}} + \frac{\Sigma_j \frac{\pi_{jh}}{-2 + 2\gamma_j + \frac{1}{n}}}{1 - \Sigma_j \frac{-1 + 2\gamma_j}{2n(\gamma_j - 1) + 1}} \quad \frac{-1 + 2\gamma_i}{2n(\gamma_i - 1) + 1} \right]$$

The condition $\gamma_i \neq 1 - \frac{1}{2n}$ guarantees that denominators do not vanish.

REFERENCES

(1) Golomb, M. and M. Shanks, Elements of Ordinary Difference Equations, Second Edition, McGraw Hill Company, (1965).

(2) Green, J. and J. J. Laffond, Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods, 1975, forthcoming in Econometrica.

(3) Groves, T. and J. Ledyard, Optimal Allocation of Public Goods: A Solution to the "Free-Rider" Problem, Econometrica, 45 (4), 783-803, (1977).

(4) Hurwicz, L. On Informationally Decentralized Systems, in Decision and Organization: A Volume in Honor of Jacob Marschak, ed. by C. McGuire and R. Radner. North Holland, Amsterdam 1972.

(5) Malinvaud, E., Procedures for the Determination of a Program of Collective Consumption, European Economic Review, 2, 187-217, (1970-1971).

(6) Roberts, J., Incentives in Planning Procedures for the Provision of Public Goods, Center for Mathematical Studies in Economics and Management Science, Northwestern University, Discussion Paper 264, 1977, forthcoming in The Review of Economic Studies.

(7) Schoumaker, F., Best Replay Strategies in the Champsaur-Dreze-Henry Procedure, Center for Mathematical Studies in Economics and Management Science, Northwestern University, Discussion Paper 262, 1976.

(8) Theil, H. , Principles of Economics, John Wiley and Sons, (1971).

(9) Tulkens, H. and S. Zamir, Local Games in Dynamic Exchange Processes. Center for Operations Research and Econometrics, Universite Catholique de Louvain, Discussion Paper 7606, 1976, forthcoming in The Review of Economic Studies.