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A SHORT NOTE ON A LINEAR EQUATION ASSOCIATED WITH CERTAIN ECONOMIC RESPONSE SYSTEMS[+]

by

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A Short Note on a Linear Equation Associated with Certain Economic Response Systems

We consider three problems:

Problem 1: Find the sign of $\frac{dp_i}{d\alpha}$ for the system of the equations [1] (the equations represent demand and supply in a service differentiated transport sector):

$$p_i(q_i,\alpha) = R_1(\Sigma q_k)$$

$$p_i(q_j) = R_j(\Sigma q_k) \qquad j \neq i$$

$$MC_k^h(q_k^h) = p_k(q_k) \qquad k=1,\ldots,n$$

$$q_k = \sum_h q_h^k \qquad h=1,\ldots,H_k$$

where suitable conditions are given for $\frac{\partial p_k}{\partial q_k}$, R_k' and $\frac{\partial p_i}{\partial \alpha}$, and there are H_k firms providing mode k service with n modes.

Problem 2: Does a solution exist (and what is it) for the following system of first order conditions [4*]:

$$\varphi_{ih}(-2+2\gamma_i+\frac{1}{n}) + \bar{\varphi}_h(1-2\gamma_i) = -\Pi_{ih}(1-\frac{1}{n})$$

where n is the number of agents, γ_i is the i^{th} agents share of surplus from trade, Π_{ih} is the agents actual marginal rate of substitution between good h and the numeraire good while φ_{ih} is the announced marginal rate substitution ($\bar{\varphi}_h = \frac{1}{n}\sum_i \varphi_{ih}$)?

* A related problem is in [3].

Problem 3: Given the n-1 demands ($m_j, j \neq 1$) for a public good y ($y = \Sigma m_j$), consider a consumer who desires to minimize his Groves-Ledyard tax [2] C_i:

$$C_i = \alpha_i qy + \frac{\gamma}{2}\left[\frac{n-1}{n}(m_i - \hat{\mu}_i)^2 - \frac{1}{n-2}\sum_{j \neq i}(m_j - \hat{\mu}_i)^2\right]$$

where q is the price of the good, $\gamma > 0$ is a coefficient set by the planning, α_i is the consumer's share of the cost of the good and $\hat{\mu}_i = \frac{1}{n-1}\sum_{j \neq i} m_j$. If on the other hand <u>all</u> the m_i are to be found, is there a solution?

What these problems all have in common is that their solution depends on a system of linear equations that is considered in the following theorem:

<u>Theorem</u>: The system of equations $(R-BC)x = d$ with $|R| = 0$, $C = FEP$, all matrices $n \times n$, E the matrix of ones and x,d are $n \times 1$ vectors, has the solution:

$$x = R^{-1}\left[d + \frac{e'PR^{-1}d}{1-e'PR^{-1}BFe} BFe\right]$$

(where e is an $n \times 1$ vector of ones) on the condition that $e'PR^{-1}BFe \neq 1$.

Proof:

$$(R-BC)x = d$$

$$\Rightarrow Rx = B(x+d)$$

$$\Rightarrow x = R^{-1}BCx + R^{-1}d$$

$$= R^{-1}BFEPx + R^{-1}d$$

$$= R^{-1}BFe \cdot e'Px + R^{-1}d$$

$$= (e'Px)\cdot(R^{-1}BFe) + R^{-1}d$$

$$\Rightarrow e'Px = e'Px \cdot e'PR^{-1}BFe + e'PR^{-1}d$$

$$\Rightarrow e'Px = \frac{e'PR^{-1}d}{1-e'PR^{-1}BFe}$$

$$\Rightarrow x = R^{-1}d + \frac{e'PR^{-1}d}{1-e'PR^{-1}BFe} \cdot R^{-1}BFe \qquad \blacksquare$$

In the first problem the comparitive statics conditions can be written as a system of (n-1) equations:

$$(I - MT)dp = d$$

where

$$M = \begin{bmatrix} R_2' & \cdots & R_2' \\ \vdots & & \vdots \\ R_n' & \cdots & R_n' \end{bmatrix} = \begin{bmatrix} R_2' & & & 0 \\ & R_3' & & \\ & 0 & \ddots & \\ & & & R_n' \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ \vdots & \cdots & 1 \end{bmatrix}$$

$$= CE$$

$$T = \begin{bmatrix} k_1r_2 & & 0 \\ & k_1r_3 & \\ 0 & & \ddots \\ & & & k_1r_n \end{bmatrix} = F$$

and where $k_1 = \dfrac{\partial p_1/\partial q_1}{\partial p_1/\partial q_1 - R_1'}$ and $r_j = \sum_h [1/(MC_j^h)']$

$$d = \frac{-\partial p_1/\partial \alpha}{\partial p_1/\partial q_1 - R_1'} \cdot \begin{bmatrix} R_2' \\ \vdots \\ R_n' \end{bmatrix}$$

and finally $dp = \begin{bmatrix} dp_2/d\alpha \\ \vdots \\ dp_n/d\alpha \end{bmatrix} = x$

In this case (I-MT) has an inverse (i.e. the conditions of the theorem are met).

In the second problem if we let B = P = I, and R and C be diagonal matrices with

$$r_{ii} = -2 + 2\gamma_i + \frac{1}{n}$$

and

$$c_{ii} = (1 - 2\gamma_i)/n$$

and if $\varphi_h = (\varphi_{1h},\ldots,\varphi_{nh})'$ is the $n \times 1$ solution vector while $d = (-\Pi_{1h}/(1-\frac{1}{n}),\ldots,-\Pi_{nh}/(1-\frac{1}{n}))'$ then we can represent problem two as:

$$(R - CE)\varphi_h = d \; .$$

Here again we find the conditions of the theorem are met and the announced marginal rates of substitution (φ_h) can be expressed in terms of the actual marginal value of substitution (d).

Finally, in the third problem we find that the condition that $e'PR^{-1}BFe \neq 1$ is not met. The system of equations is

$$(nI - E)m = - \frac{qn}{\gamma} \alpha \ .$$

Letting $R = nI$, $B = P = F = I$ we see that $e'PR^{-1}BFe = 1$.

REFERENCES

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