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A TIME SERIES APPROACH TO THE COMPUTATION OF
EFFICIENT PORTFOLIOS FROM HISTORIC DATA

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ABSTRACT

This note describes a time-series approach to the problem of determining an efficient set of portfolios from historic data. The procedure is shown to give results which are equivalent to the Markowitz portfolio model, while being computationally simpler than both the Markowitz and Sharpe 'diagonal' models.

A Time Series Approach to the Computation of
Efficient Portfolios From Historic Data

Since its introduction by Markowitz in 1952 [4], the mean variance portfolio model has been widely used by investors and researchers. The objective of portfolio analysis is the determination of an 'efficient set' of portfolios, i.e. portfolios which have the largest possible expected return for a given standard deviation of return and the smallest possible standard deviation of return for a given expected return. In 1963 Sharpe [5] introduced a simplified portfolio model--the 'diagonal' or 'single-index' model--which greatly reduced the computational requirements of the Markowitz model. In addition to its usefulness in investment analysis, this work has been the basis for extensive empirical research into the nature of capital markets. (Sharpe [6], Lintner [3], Fama [2]).

This note describes a computational procedure for determining the efficient set of portfolios from historic data. The procedure is shown to give results which are equivalent to the full Markowitz portfolio model. The procedure is simpler computationally than both the Markowitz and Sharpe models and does not suffer from the approximating assumptions of the latter. The underlying idea is to compute a new time series which is a weighted sum of the historic time series and which has minimum variance. This process provides some interesting insights into the assumptions underlying empirical research in this area.

We assume we have T periods of returns data for n investment possibilities. The historical return from investment i in time period t is given

by r_{it} and the average return by $\bar{r}_i = \frac{1}{T} \sum_{t=1}^T r_{it}$. The historical variances and covariances are given by:

$$\sigma_{ij} = \frac{1}{T} \sum_{t=1}^T (r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j) \quad i=1,2,\dots,n; j=1,2,\dots,n$$

Let w_i represent the proportion of funds invested in security i in portfolio, p (usually $w_i \geq 0$ although negative values corresponding to short-sales or leveraged portfolios are possible). Let e_p be a 'required' return from the portfolio. In its simplest form the Markowitz formulation of the portfolio problem is to find weights $w_i \geq 0, i=1,2,\dots,n$ to minimize the portfolio variance, v_p :

$$\min_w v_p = \frac{1}{T} \sum_i \sum_j w_i w_j \sigma_{ij} \quad (1a)$$

subject to

$$\sum_{i=1}^n w_i = 1 \quad (1b)$$

$$\sum_{i=1}^n w_i \bar{r}_i = e_p \quad (1c)$$

This problem is often augmented by additional inequality constraints. It can be solved by quadratic programming and the efficient frontier of portfolios can be found by varying e_p parametrically.

The Markowitz formulation requires the prior computation of n average returns and $\frac{n(n+1)}{2}$ variance and covariance terms. The Sharpe 'diagonal' model (also associated with Lintner [3]) was partly motivated by a desire to reduce computational requirements. In this model it is assumed that

individual security returns are related only through their relations with one or more indices of general business activity. It is assumed that the individual covariances between securities are zero. As a result of the last assumption the Sharpe-Lintner models produce solutions which are only approximately efficient (Cohen and Pogue [1]).

The approach adopted here is to find weights $\alpha_i \geq 0$, $i=1,2,\dots,n$ which 'smooth' the combined time series around the required average return, e_p , by minimizing the sum of the squared deviations:

$$\min_{\alpha} v_p = \frac{1}{T} \sum_{t=1}^T (e_p - \sum_{i=1}^n \alpha_i r_{it})^2 \quad (2a)$$

subject to

$$\sum \alpha_i = 1 \quad (2b)$$

Note that (2) is a simple constrained least-squares problem.

THEOREM:

Let w^* solve (1) and α^* solve (2). Then $w^* = \alpha^*$.

Proof:

We convert problem (1) into problem(2) by expanding the objective function (1a) and using (1c):

$$\begin{aligned} v_p &= \frac{1}{T} \sum_i \sum_j w_i w_j \sum_t (r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j) \\ &= \frac{1}{T} \sum_t \sum_i \sum_j (w_i r_{it} - w_i \bar{r}_i)(w_j r_{jt} - w_j \bar{r}_j) \\ &= \frac{1}{T} \sum_t \sum_i (w_i r_{it} - w_i \bar{r}_i) \sum_j (w_j r_{jt} - w_j \bar{r}_j) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{T} \sum_t \left[\sum_i w_i \bar{r}_i - \sum_i w_i r_{it} \right] \left[\sum_j w_j \bar{r}_j - \sum_j w_j r_{jt} \right] \\ &= \frac{1}{T} \sum_t \left(\sum_i w_i \bar{r}_i - \sum_i w_i r_{it} \right)^2 \\ &= \frac{1}{T} \sum_t \left(e_p - \sum_i w_i r_{it} \right)^2 \end{aligned}$$

Thus (1) has the same form as (2).

Note that any number of additional constraints can be added to (both) (1) and (2) and the equivalence will still hold. Also, that the variance of the optimal portfolio is given directly by the optimal value of the objective function in (2). The efficient set of portfolios can be traced out by varying e_p over the desired range.

Problem (2) involves the minimization of a quadratic function of n weights as is the case in both the Markowitz and Sharpe-Lintner models. However, it works directly with the original time series data and avoids the calculation of the covariances required by the Markowitz model and the regression coefficients ('betas') required by the Sharpe-Lintner models. Thus the time series smoothing approach has a considerable advantage over the other approaches for certain kinds of empirical work.

This formulation also provides a different and very intuitive interpretation of the process of computing optimal portfolios from historic data--namely that the Markowitz formulation computes that linear combination of time series which is the closest fit (in the least-squares sense) to the constant rate of return, e_p . This is basically an outcome

of the one time period decision framework underlying the Markowitz model. Note that it might be useful in some applications to regard e_p as a function of time. Thus, if returns have been trending upwards over an extended period, e_p might be replaced by a trend function in the portfolio selection model (2). Finally, although the discussion to date has been in terms of empirical studies, the time series model can also be used in normative work. This would be accomplished by forecasting the individual time series over the relevant time horizon to generate the data for (2).

References

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