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THE DESIGN OF MECHANISMS FOR EFFICIENT ALLOCATION OF PUBLIC GOODS

by

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1. Introduction

The purpose of this paper is to present a systematic method for constructing and classifying individually incentive compatible mechanisms that lead to Pareto Optimal allocations of public and private goods at each Nash noncooperative equilibrium in messages and furthermore balance the budget at each such allocation. The equilibrium concept and setup that we use is the same as Groves and Ledyard [1], [3].

Our paper is motivated by the desire to have a systematic method of constructing "Groves Ledyard type" mechanisms. The papers by Groves and Ledyard present a very clever set of quadratic tax functions that turn out to be individually incentive compatible and balance the budget. But how Groves and Ledyard come upon these particular functions remains mysterious after reading their articles. It is hoped that the treatment presented here will enable the reader to routinely construct individually incentive compatible mechanisms that balance the budget in a variety of different situations and, furthermore, tailor the mechanisms to achieve different objectives that are specific to the particular situation at hand.

The paper is organized as follows. Section 1 contains the introduction. Section 2 derives two conditions on tax functions that must be satisfied if the tax functions are to lead to a Pareto Optimal provision of pure public goods and if the budget is to be balanced in the sense that the amount allocated to the construction of each public good is equal to the amount spent on the construction of each public good under these tax functions. The efficiency condition amounts to the requirement that the sum of marginal taxes across individuals must add up to the pure public good price for each public good where all prices are denominated in terms of some pure private good as numeraire. Neither the efficiency condition or the budget balance condition involves subjective information such as utility functions or individual incomes. Of course, the conditions only guarantee Pareto Optimality. They do not say any thing about the desirability of the resulting distribution of utility income. The Groves-Ledyard quadratic tax functions are shown to satisfy the two conditions.

In Section 3 the method is applied to designing tax functions on individuals that belong to a fixed exogenously given coalition structure that lead to Pareto Optimal balanced budget allocations in equilibrium even though members of each coalition collude in sending their messages to the government. The coalition structure is assumed fixed, however, and the problem is treated only to illustrate a use of the design method and it is not intended as a serious treatment of the "coalition problem".

Section 4 derives conditions on the matrices of the tax functions that lead to Pareto Optimality and budget balance. For example, in the case of N agents, one pure private good with price p, one pure public good with price q consider tax functions of the form

$$C_h(q,m) = \alpha_h q \cdot (\sum_{j=1}^{N} m_j) + m^t A_m^h m$$

where $\sum_{h=1}^{N} \alpha_h = 1$, m_h denotes a real number that is the proposed increment by h to the total quantity of public good, and $m \equiv (m_1, \ldots, m_n)$. Here $m^t A_m^h m$ denotes the quadratic form with NXN matrix A^h. It is shown that if

$$(*) \qquad \sum_{h=1}^{N} A_{hj}^h = 0, \; j = 1, 2, \ldots, N$$

then the first order conditions for Pareto Optimal allocation obtain at any equilibrium solution $\{\bar{x}_h\}_{h=1}^N, \bar{m}$ to the noncooperative game defined by

(1.1) $$\underset{x_h, m_h}{\text{maximize}} \quad U_h(x_h, \sum_{j=1}^N m_j)$$

(1.2) $$\text{s.t.} \quad p\, x_h + C_h(q,m) \leqq w_h$$

where x_h denotes private good consumption by h and w_h denotes the income of h.

Furthermore, if

(**) $$\sum_{h=1}^N A_{ij}^h = 0, \qquad i,j = 1, 2, \ldots, N$$

then budget balance obtains. Also if

(***) $$A_{hh}^h > 0, \; h = 1, 2, \ldots, N$$

holds then convexity of C_h in m_h holds and each $\bar{x}_h$, $\bar{m}_h$ that solves the first order conditions of optimality will, indeed, be a maximum of utility subject to the budget constraint (1.2) for each h = 1, 2, ..., N.

It is straightforward to show that the Groves-Ledyard tax functions:

$$C_h(q,m) \equiv \alpha_h\, q(\sum_{j=1}^N m_j) + \frac{\gamma}{2}\{\frac{N-1}{N}(m_h - \hat{\mu}_h)^2 - \hat{\sigma}_h^2\}$$

$$\hat{\mu}_h \equiv \frac{1}{N-1} \sum_{i \neq h} m_i, \quad \hat{\sigma}_h^2 \equiv \frac{1}{N-2} \sum_{j \neq h} (m_j - \hat{\mu}_h)^2$$

satisfy (*) - (***).

In Section 5 we attack the general problem of constructing consumption allocation functions and tax functions on consumption externalities that lead to Pareto Optimality and budget balance. The problem is solved by the usual (to students of public goods) device of creating an artificial pure public good "the consumption of good r by agent h", and putting the allocation function

$$f_{hr}(m) = \sum_{i=1}^{N} \sum_{j=1}^{N} m^{i}_{jr}$$

where

$$m^{i}_{jr}$$

denotes the proposed increment by agent i to the consumption of r by agent j.

Thus, we are back to the pure public goods case. The necessary conditions for this case are rearranged into a useful form for incentive design in Section 5.

In Section 6 an abstract theory of optimal incentive design is presented that includes all of the examples contained in the other sections of the paper. It is based upon some work of Smale [13], [14] which presents first order conditions for Pareto Optima with constraints on the state space in an abstract framework. In this section we present a rather abstract condition on gradients of tax functions that guarantee that Smale's first order conditions for Pareto Optima are satisfied at all non-cooperative equilibria.

Finally, section 7 briefly discusses optimal incentive design when public capital goods are introduced and capital markets are imperfect in the sense that individuals cannot borrow against their future incomes.

2. A Simple Method

In order to see the ideas more clearly, we shall consider the simplest imaginable model. Let there be N individuals, where individual h chooses his private goods bundle $x_h \in R^L_+$ and his "message" $m_h \in R^K$ to solve

$$\text{(2.1)} \qquad \text{maximize } U_h(x_h, \sum_{j=1}^{N} m_j)$$

$$\text{s.t. } \; p \cdot x_h + C_h(q, m_1, \ldots, m_N) \leqq w_h$$

where U_h, C_h, p, q, w_h denote utility function, tax function, private goods price vector, public goods price vector, and income, respectively.

Note that $G \equiv \sum_{j=1}^{N} m_j$ enters the utility function of each h. G is the amount of public goods provided if the message vector is $(m_1, \ldots, m_N)$. Thus, for this model, think of m_h as the incremental amount of public good proposed by h.

Groves and Ledyard, operating in a much more general context than (2.1), must solve two problems. First, they must design, given their equilibrium concept, which is non-cooperative equilibrium in $(m_1, \ldots, m_N)$, a set of C_h so that the equilibrium demands generated by (2.1) for each p, q, w_h satisfy the Samuelson-Lindahl first order conditions for a Pareto Optimal allocation:

$$\text{(2.2)} \qquad \sum_{h=1}^{N} U_{hk}/U_{h\ell} = q_k/p_\ell$$

(equality[1/] holds because m_{hk} is allowed to be negative, h = 1, 2, ..., N) where U_{hk}, $U_{h\ell}$ are short for marginal utility of h, with respect to public good k and private good ℓ. Here q_k, p_ℓ denote price of public good k and private good ℓ. A set of C_h that satisfy (2.2) will ensure that demanders will not "under-reveal" their preferences for public goods.

Second, Groves and Ledyard must further restrict the C_h, the utility functions, the initial endowments, and the production sets, so that a general equilibrium exists.

In this section, I am interested only in the design of the C_h so that the Samuelson-Lindahl condition (2.2) and budget balance (to be defined below) holds for each p, q, w_h. Nothing will be done on the existence of general equilibrium in this paper.

Definition 1.1: Given p, q; $w_1, \ldots, w_N$; $C_1, \ldots, C_N$ a demand vector $(\bar{x}_1, \ldots, \bar{x}_N; \bar{m}_1, \ldots, \bar{m}_N)$ is any vector $z \equiv (x_1, \ldots, x_N; m_1, \ldots, m_N)$ that satisfies: For each h = 1, 2, ..., N

$$(2.3) \qquad U_h(\bar{x}_h, \sum_{j \neq h}^{N} \bar{m}_j + \bar{m}_h) \geqq U_h(x_h, \sum_{j \neq h}^{N} \bar{m}_j + m_h)$$

for all x_h, m_h that satisfy the budget constraint

$$p \cdot x_h + C_h\ (q, m_h, \hat{\bar{m}}_h) \leqq w_h,\ x_h \geqq 0,$$

Here $(m_h, \hat{\bar{m}}_h)$ denotes $(\bar{m}_1, \ldots, \bar{m}_{h-1}, m_h, \bar{m}_{h+1}, \ldots, \bar{m}_N)$.

The equilibrium concept is just a standard non-cooperative equilibrium à la Cournot-Nash. Groves and Ledyard also need the C_h to satisfy the Budget Balance Condition: For each p, q, $w_1, \ldots, w_N$

(2.4) $$q\cdot(\sum_{j=1}^{N} \bar{m}_j) = \sum_{j=1}^{N} C_j(q, \bar{m}).$$

I.e., the taxes C_h collected must sum up to the expenditure $q\cdot(\sum_{j=1}^{N} \bar{m}_j)$ upon goods in order that the government budget be balanced.

We may now state

Basic Problem: Design the C_h so that for each p, q, $w_1, \ldots, w_N$ equations (2.2) and (2.4) are satisfied.

The basic problem is important because, at the very least, the tax structure designed to provide public goods should generate an efficient allocation. The tax structure may be further manipulated to achieve desirable social objectives such as income redistribution, but certainly efficiency is a basic property.

Theorem 2.1 Assume that $\bar{x}_h >> 0$ for each h, for each p, q, $w_1, \ldots, w_N$. In order that the C_h solve the Basic Problem, it is sufficient that for each demand vector $(\bar{x}_1, \ldots, \bar{x}_N; \bar{m}_1, \ldots, \bar{m}_N)$

(2.5) $$\sum_{h=1}^{N} C_{hk} = q_k, \; k = 1, 2, \ldots, K$$

(2.6) $$\sum_{h=1}^{N} C_h = q \cdot (\sum_{j=1}^{N} m_j).$$

Here $C_{hk} = \partial C_h/\partial m_{hk}$.

Proof: Write down the necessary conditions for a solution to (2.3):

(2.7) $$U_{h\ell} \leqq \lambda_h p_\ell \; (=\lambda_h p_\ell, \text{ if } x_{h\ell} > 0)$$

(2.8) $$U_{hk} = \lambda_h C_{hk}$$

where λ_h is the marginal utility of income to h, $C_{hk} = \partial C_h/\partial m_{hk}$, and m_{hk} is the k^{th} component of the message vector m_h. Since m_{hk} is allowed to vary over all of R, therefore (2.8) holds with equality. Take the quotient of (2.8) to (2.7), use the assumption that $\bar{x}_h >> 0$, and sum over h to get:

$$(2.9) \qquad \sum_{h=1}^{N} U_{hk}/U_{h\ell} = \sum_{h=1}^{N} C_{hk}/p_\ell .$$

From (2.9), we see that if we require

$$(2.10) \qquad \sum_{h=1}^{N} C_{hk} = q_k$$

for each q, $(m_1, \ldots, m_N)$, then the Samuelson-Lindahl condition (2.2) will be satisfied. But (2.10) is just (2.5).

Equation (2.6) is just the budget balance condition. This ends the proof.

Theorem 1 gives a useful method of search for tax functions C_h that generate Pareto Optimal allocations. Note especially that (2.9) and (2.10) do not depend on utility functions.

Let us show how (2.5) and (2.6) lead naturally to the Groves-Ledyard tax functions. Consider tax functions of the form

$$(2.11) \qquad C_h = \alpha_h q \cdot (\sum_{j=1}^{N} m_j) + D_h (m_1, \ldots, m_N)$$

where $\alpha_h > 0$, $\sum_{h=1}^{N} \alpha_h = 1$. One might think of α_h as the fraction of the total government budget imputed to h. Obviously, if we put $D_h \equiv 0$ for all h, then (2.5) and (2.6) will be satisfied. But Groves-Ledyard point out that $D_h = 0$ leads to non-existence problems for general equilibrium [1, p. 36]. Therefore, we must look for non-trivial D_h.

From (2.11), (2.5) and (2.6), the D_h must satisfy:

$$(2.12) \qquad \sum_{h=1}^{N} D_{hk} = 0, \; k = 1, 2, \ldots, K$$

$$(2.13) \qquad \sum_{h=1}^{N} D_h = 0.$$

Here D_{hk} denotes $\partial D_h/\partial m_{hk}$. Groves and Ledyard choose, e.g.,

$$(2.14) \qquad D_h = \frac{\gamma}{2}\left[\frac{N-1}{N}(m_h - \hat{\mu}_h)^2 - \hat{\sigma}_h^{\,2}\right] \cdot e$$

where e denotes the K dimensional vector (1, 1, ..., 1) and "·" is scalar product where $\gamma > 0$ is arbitrary and

$$(2.15) \qquad \hat{\mu}_h = \frac{1}{N-1}\sum_{i\neq h} m_i,\ \hat{\sigma}_h^{\,2} = \frac{1}{2(N-1)(N-2)}\sum_{i\neq j}\sum_{j\neq h}(m_i - m_j)^2$$

$$= \frac{1}{N-2}\sum_{j\neq h}(m_j - \hat{\mu}_h)^2.$$

It is straightforward to verify by computation that the Groves-Ledyard D_h satisfy (2.12) and (2.13). Equations (2.12) and (2.13) generate the Groves-Ledyard tax rules in a "natural" way. Viz., positing D_h of quadratic form and proceeding to find the coefficients implied by (2.12) and (2.13). If one further adds the "equity" requirement that h and j be treated equally in some sense, the coefficients of D_h are restricted even further. Note that the Groves-Ledyard D_h have an equal treatment character--i.e., D_h is the <u>same function</u> of m_h, $\hat{\mu}_h$, $\hat{\sigma}_h^{\,2}$ independent of h.

What happens if a group of individuals form a coalition and collaborate in sending their messages? Quite clearly such a coalition can make itself better[2/] off if the other players play non-cooperatively. This is so because each member of the coalition would internalize the external effect of his message on the utilities and on the tax bills of his fellows. This suggests

<u>Design Problem with Coalitions</u>: Design tax functions on coalitions as well as on individuals, so that a Pareto Optimal allocation of public and private goods results.

Since for Pareto Optimality to obtain, the Samuelson-Lindahl first order necessary condition (2.2) and budget balance condition (2.4) must hold, therefore, we must consider the problem of designing tax functions on coalitions and on individuals so that (2.2) and (2.4) hold. Obviously, this design problem will depend upon the game theoretic equilibrium concept used.

3. The Design Problem Under Coalition Formation

There are many ways to formulate this problem. Since my main interest is in the lobbying problem, as discussed in Brock and Magee [2], we shall look at equilibrium concepts that have a mixed cooperative and non-cooperative nature. Let us first look at a situation where the coalitions that are able to police their members are exogenously given.

Let $S_1, \ldots, S_I, S_{I+1}$ be a partition of $\{1,2,\ldots,N\}$ into non-overlapping subsets such that

$$\bigcup_{i=1}^{I+1} S_i = \{1,2,\ldots,N\}.$$

Here $\bigcup_{i=1}^{I+1} S_i$ denotes the set theoretic union of the sets S_i. The players in set S_i, $i = 1,2,\ldots,I$ are assumed to collaborate in sending their messages and deciding on their private goods consumption, whereas the players in the last set, S_{I+1} do not cooperate at all. Call $S_1, \ldots, S_I, S_{I+1}$ a coalition structure.

We have to say something about how the members of S_i cooperate. Intuitively, we want each player in S_i to internalize his impact on the other players into his own decision when he chooses his m_h. Given the messages of the players outside of S_i, the players of S_i can make themselves better off if they cooperate in sending the m_i. Coalition S_i chooses $\{x_h\}_{h \in S_j}, \{m_h\}_{h \in S_j}$ to solve

$$(3.1) \qquad \text{"Pareto Optimize"} \quad \{U_s(x_s, \sum_{j=1}^{N} m_j)\}_{s \in S_i}$$

$$\text{s.t.} \sum_{s \in S_i} [p \cdot x_s + \alpha_s q \cdot (\sum_{j=1}^{N} m_j) + D_s] \leqq \sum_{s \in S_i} w_s.$$

Definition 3.1:[3/] Given p, q, $w_1, \ldots, w_N$, a non-cooperative equilibrium relative to the coalition structure $S_1, S_2, \ldots, S_I, S_{I+1}$ is a vector $\bar{x}_1, \ldots, \bar{x}_N, \bar{m}_1, \ldots, \bar{m}_N$ such that for each coalition S_i, $i = 1,2,\ldots,I$, $\{\bar{x}_s\}_{s \in S}$, $\{\bar{m}_s\}_{s \in S_i}$ solves (3.1) for $s \in S_i$, with $x_j = \bar{x}_j$, $m_j = \bar{m}_j$, $j \notin S_i$ for each i. For $h \in S_{I+1}$, $\bar{x}_h$, $\bar{m}_h$

$$(3.2) \qquad \text{maximize } U_h(x_h, \sum_{j \neq h} \bar{m}_j + m_h)$$

$$\text{s.t. } p \cdot x_h + \alpha_h q \cdot (\sum_{j \neq h} \bar{m}_j + m_h) + D_h (m_h, \hat{\bar{m}}_h) \leqq w_h$$

The idea of the definition is that given the strategies of the other players, the players of S_i pick their strategies to componentwise maximize the set of utility functions of the players of S_i. Coalitions S_i, S_j are not allowed to cooperate and each player of S_{I+1} plays non-cooperatively. We hasten to add that the coalition structure is fixed in this definition.

Think of $S_1, \ldots, S_I$ as "lobbies" that for some exogenously given reason are able to "police" their members well enough so that each S_i member takes into account, in some way, the impact of his message on his S_i-fellows' utility levels. Such "pressure groups" are cited as a cause of inefficiency of the economic system. This is so because they are "concentrated", whereas the rest of the economy is "diffuse".

For the rather special concept of pressure group equilibrium outlined above, we will show that $\{D_h\}_{h=1}^N$ may be constructed so that a non-cooperative equilibrium of Definition 3.1 (D.3.1) type will satisfy the Samuelson-Lindahl condition (2.2) and the budget balance condition (2.4).

<u>Theorem 3.1</u> Assume $x_h \gg 0$ for all h and $U_{s\ell} \neq 0$ for all s, ℓ. In order that a non-cooperative equilibrium of type D.3.1 satisfy (2.2) and (2.4), it is sufficient that for each $(m_1, m_2, \ldots, m_N)$ the following hold

$$(3.3) \qquad \sum_{i=1}^{I} |S_i|^{-1} \left(\sum_{s \in S_i} \sum_{s_0 \in S_i} D_{sm_{s_0 k}} \right) + \sum_{h \in S_{I+1}} D_{hm_{hk}} = 0, \ k=1,2,\ldots,K$$

$$(3.4) \qquad \sum_{h=1}^{N} D_h = 0.$$

Here $D_{sm_{s_0 k}}$ denotes $\partial D_s / \partial m_{s_0 k}$ and $|S_i|$ denotes the number of elements in S_i.

Proof: Write out the necessary conditions for an equilibrium of type D.3.1. It is well known and is discussed in more detail in Section 6 below that one may generate Pareto Optima by finding the solutions to the first order conditions gotten by maximizing a weighted sum of utilities where each U_h receives nonnegative weight λ_h. In this spirit consider the following problem for each coalition S_i: Choose $\{x_h\}_{h \in S_i}$, $\{m_h\}_{h \in S_i}$ to solve the first order necessary conditions for

$$(3.1)' \qquad \text{maximize} \sum_{s \in S_i} \lambda_s U_s(x_s, \sum_{j=1}^{N} m_j)$$

$$\text{s.t.} \qquad \sum_{s \in S_i} [p \cdot x_s + \alpha_s q \cdot (\sum_{j=1}^{N} m_j) + D_s] \leq \sum_{s \in S_i} w_s.$$

The reason we always have to add the qualifying phrase "solve the first order necessary conditions" instead of just "maximize" is because the U_h are not necessarily concave and as is pointed out in Section 6 below under weak sufficient conditions may be found so that any solution to the first order conditions of the above problem are Pareto Optima. However, some of these Pareto Optima may not maximize the above weighted sum of utilities. With the above qualifications in mind let us solve (3.1)' and continue on with the proof. Let Λ_{S_i} denote the "marginal utility of income" to S_i. From (3.1)', for each ℓ, k, $s_0 \in S_i$

$$(3.5) \qquad \lambda_{s_0} U_{s_0\ell} \leqq \Lambda_{S_i} p_\ell \; (= \Lambda_{S_i} p_\ell, \text{ if } x_{s_0\ell} > 0)$$

$$(3.6) \qquad \sum_{s\in S_i} \lambda_s U_{sm_{s_0k}} = \Lambda_{S_i} \left[\sum_{s\in S_i} \alpha_s q_k + \sum_{s\in S_i} D_{sm_{s_0k}} \right]$$

Substitute (3.5) into (3.6) to get (when $x_{s\ell}>0$)

$$(3.7) \qquad \sum_{s\in S_i} [\Lambda_{S_i} p_\ell U_{s\ell}^{-1}] U_{sm_{s_0k}} = \Lambda_{S_i} \left[\sum_{s\in S_i} \alpha_s q_k + \sum_{s\in S_i} D_{sm_{s_0k}} \right]$$

But, if $\Lambda_{S_i}>0$ which we assume (a very mild requirement!), this is equivalent to

$$(3.8) \qquad \sum_{s\in S_i} [U_{sm_{s_0k}}/U_{s\ell}] = \left\{ \sum_{s\in S_i} \alpha_s q_k + \sum_{s\in S_i} D_{sm_{s_0k}} \right\} p_\ell^{-1}.$$

Here $D_{sm_{s_0k}}$ denotes $\partial D_s/\partial m_{s_0k}$. For $h \in S_{I+1}$, we get as in Section 2,

$$(3.9) \qquad U_{hk}/U_{h\ell} = \alpha_h q_k + D_{hm_{hk}}.$$

Sum (3.8) over $s_0 \in S_i$ and use $U_{sm_{s_0k}} = U_{sG_k}$, where $G_k \equiv \sum_{j=1} m_{jk}$ to get (3.10) and (3.11).

$$(3.10) \qquad \sum_{s_0 \in S_i} \sum_{s \in S_i} [U_{sm_{s_0 k}} / U_{s\ell}] = \{|S_i|(\sum_{s \in S_i} \alpha_s q_k) + \sum_{s_0 \in S_i} \sum_{s \in S_i} D_{sms_0 k}\} p_\ell^{-1}$$

Here $|S_i|$ denotes the number of elements of S_i.

$$(3.11) \qquad \text{L.H.S. } (3.10) = \sum_{s \in S_i} |S_i| \, (U_{s\, G_k} / U_{s\ell}).$$

Substitute (3.11) into (3.10) and simplify to get

$$(3.12) \qquad \sum_{s \in S_i} (U_{s\, G_k} / U_{s\ell}) = \{ \sum_{s \in S_i} \alpha_s \, q_k + |S_i|^{-1} \sum_{s, s_o \in S_i} \sum D_{s\, m_{s_o k}} \}.$$

Note that (3.12) is "part" of the Samuelson-Lindahl condition (2.2). Note also that (3.12) is independent of the weights λ_t. Now, sum (3.12) over $i = 1,2,\ldots,I$ and add the result to the sum of (3.9) over $h \in S_{I+1}$ to get

$$(3.13) \qquad \sum_{j=1}^{N} (U_{j\, G_k} / U_{j\ell}) = q_k / p_\ell + \{ \sum_{i=1}^{I} |S_i|^{-1} (\sum_{s, s_o \in S_i} \sum D_{s\, m_{s_o k}} + \sum_{h \in S_{I+1}} D_{h\, m_{hk}} \} / p_\ell .$$

Thus for (2.2) to hold, we need

$$(3.14) \qquad 0 = \sum_{i=1}^{I} |S_i|^{-1} (\sum_{s, s_o \in S_i} \sum D_{s\, m_{s_o k}}) + \sum_{h \in S_{I+1}} D_{h\, m_{hk}} .$$

But (3.14) is just (3.3). Equation (3.4) is just the budget balance condition. This ends the proof.

It is important to notice that both conditions (3.3) and (3.4) are independent of "subjective" information, such as utility functions and "welfare weights" λ_t.

Let us use conditions (3.3) and (3.4) to work up an example of Groves-Ledyard tax functions for coalitions.

Theorem 3.2 Let $C = I + |S_{I+1}|$ equal the number of coalitions, where unit coalitions, $\{j\}$, $j \in S_{I+1}$ are counted as one coalition. Let

$$(3.15) \quad \mu_i = \frac{1}{|S_i|} \sum_{h \in S_i} m_h, \text{ for } i = 1,2,\ldots,I$$

$$\mu_i = m_i, \; i \in S_{I+1}$$

$$(3.16) \quad \hat{\mu}_i = \frac{1}{C-1} \sum_{j \neq i}^{C} \mu_j$$

$$(3.17) \quad \hat{\sigma}_i^2 = \frac{1}{C-2} \sum_{j \neq i}^{C} (\mu_j - \hat{\mu}_i)^2.$$

Let $C > 2$, $\gamma > 0$, $\{\alpha_h\}_{h=1}^{N}$ be given. Put

$$C_i = \alpha_i \, q \cdot (\sum_{h=1}^{N} m_h) + D_i, \quad i = 1,2,\ldots,N$$

where

$$(3.18) \quad D_i = \frac{\gamma}{2} \{ \frac{C-1}{C} (\mu_i - \hat{\mu}_i)^2 - \hat{\sigma}_i^2 \}, \text{ e}$$

Then $\{D_i\}_{i=1}^{N}$ is a set of functions that satisfy (3.3) and (3.4) for all vectors $(m_1, \ldots, m_N)$.

Proof: Let $(z)_k$ denote the k^{th} component of vector z. Calculate

$$(3.19) \quad D_{s\, m_{s_o} k} = \gamma \, (\frac{C-1}{C}) \, (\mu_i - \hat{\mu}_i)_k \; |S_i|^{-1}, \; s \in S_i.$$

Here $(x)_h$ denotes the k^{th} component of vector x. Sum (3.19) over s, s_o to get

$$\gamma \, (\frac{C-1}{C}) \, (\mu_i - \hat{\mu}_i)_k \; |S_i| = \sum_{s \in S_i} \sum_{s_o \in S_i} D_{s\, m_{s_o} k}$$

In calculating L.H.S. (3.3), use the fact that $|S_j| = 1$ for $j \in S_{I+1}$ to get

$$(3.20) \quad \gamma \, (\frac{C-1}{C}) \sum_{i=1}^{C} (\mu_i - \hat{\mu}_i)_k = \text{L.H.S. (3.3)}.$$

Our problem reduces to: Show that for any sequence of vectors in R^K, denoted by $\mu_1, \mu_2, \ldots, \mu_C$, that

$$(3.21) \qquad \sum_{i=1}^{C} (\mu_i - \hat{\mu}_i)_k = 0.$$

But (3.21) is obvious from the definition

$$\hat{\mu}_{ik} = \frac{1}{C-1} \sum_{h \neq i} \mu_{hk}$$

of $\hat{\mu}_{ik}$.

It remains to check (3.4). We must show:

$$(3.22) \qquad \frac{C-1}{C} \sum_{i=1}^{C} (\mu_i - \hat{\mu}_i)^2 - \sum_{i=1}^{C} \hat{\sigma}_i^{\,2} = 0.$$

Now L.H.S. (3.22) is a K dimensional vector, and we are to show that each component is 0. But that problem is just equivalent to: Given C numbers $\mu_1, \ldots, \mu_C$, form $\hat{\mu}_i$, $\hat{\sigma}_i^{\,2}$, then show that (3.22) holds. And this is exactly what Groves and Ledyard [8 , p.28] prove. In fact, if we think of each coalition as being an individual then proving that (3.21) and (3.22) hold is identical to the Groves and Ledyard proof. This ends the proof of Theorem 3.2.

The above treatment of coalitions is not very interesting for the following reason. It will not necessarily be in the interest for each s to remain in his exogenously given S_i. In other words, some other coalition may be able to improve upon the allocation supplied by a noncooperative equilibrium of type D.3.1. What is really needed for a satisfactory resolution of the "coalition problem" is a set of tax functions $\{C_h\}$ so that allocations that "are equilibrium" (in some interesting sense of that much-used word) are Pareto Optimal.

One way out would be to impose huge taxes on any coalition that attempted to form. Thus, no coalition would ever find it in its self interest to form. Thus, the problem would collapse to the no coalition case and we've solved that one. But this sort of thing is not very interesting from a practical point of view. Bennett and Cohn show, in a related context, that there is no mechanism that is immune to manipulation by colluding agents.

It is worth mentioning that coalitional agreements are not costless to enforce in the "real world". Hence, each coalition has a "free rider" problem of its own to solve. There is not always some[4/] government agency to enforce the agreement. So, a "Groves Ledyard" mechanism cannot be enforced by the coalition against its own members.

Hence, the question of coalitions is unsettled. We offer our exercise only as an application of our general method of constructing mechanisms to a subproblem that may be useful whenever someone presents a solution to the coalition problem.

Turn now to

4. Examples of Quadratic Incentive Design Using (3.3) and (3.4)

Let us show how to use (3.3) and (3.4) to systematically search for quadratic D_h that assure Pareto Optimal allocation of public goods. Return to the case $I = N$, and $|S_i| = 1$ of an equilibrium of type 3.1. This is just the Groves-Ledyard case of where each coalition contains just one member. For the sake of simplicity, let us design quadratic D_h for one public good only. Put

$$(4.1) \qquad D_h\,(m_1, \ldots, m_N) = m^T A^h m \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} A^h_{ij}\, m_i\, m_j$$

where m^T denotes the transpose of the column vector m, and A^h is the matrix $[A^h_{ij}]$. Equation (4.1) just states that D_h is quadratic. The linear terms are already embodied in $\alpha_h\, q\, (\sum_{j=1}^{N} m_j)$ - the share of the budget imputed to h. We want to use (3.3) and (3.4) in order to classify the matrices $\{A^h\}^N$.

that correspond to Pareto Optimal quadratic tax structures.

In the case I=N and $|S_i|=1$, i=1,2,...,N, record (3.3) and (3.4) for convenience. Equations (3.3) and (3.4) become, for this case,

(4.2) $$\sum_{h=1}^{N} D_{hk} = 0, \text{ for}^{5/} \text{ all } (m_1, \ldots, m_N)$$

(4.3) $$\sum_{h=1}^{N} D_h = 0, \text{ for all } (m_1, \ldots, m_N).$$

Here, as usual, $D_{hk} = \partial D_h/\partial m_{hk}$. Apply (4.2) and (4.3) to (4.1) in the case of one public good to get

(4.4) $$\sum_{h=1}^{N} D_{hk} = \sum_{h=1}^{N} \frac{\partial}{\partial m_h} (m^T A^h m) = 0.$$

(4.5) $$\sum_h D_h = \sum_h m^T A^h m = m^T (\sum_h A^h) m = 0.$$

Since (4.4) and (4.5) must hold for all vectors m, therefore it immediately follows that[6/]

(4.6) $$\sum_{h=1}^{N} A^h_{hj} = 0, \; j = 1,2,\ldots,N$$

(4.7) $$\sum_{h=1}^{N} A^h = 0$$

must hold.

If, in addition, differential convexity of $m^T A^h m$ in m is desired (so that the constraint set defined by the budget constraint and the tax function

$$C_h = \alpha_h q \cdot (\sum_{j=1}^{N} m_j) + D_h$$

is a convex set), then

problem

$$\text{Maximize} \sum_{t \in S_i} \lambda_t U_t$$

$$\text{s.t.} \sum_{t \in S_i} p\, x_t + \sum_{t \in S_i} C_t \leq \sum_{t \in S_i} w_t \; : \; \mu_{S_i}$$

over $\{m_t\}_{t \in S_i}$, $\{x_t\}_{t \in S_i}$.

5/ In detail

$$\frac{\partial}{\partial m_h} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} A^h_{ij} m_i m_j \right) = \frac{\partial}{\partial m_h} \left(\sum_{j=1}^{N} A^h_{hj} m_h m_j \right)$$

$$+ \frac{\partial}{\partial m_h} \left(\sum_{i \neq h} \sum_{j=1}^{N} A^h_{ij} m_i m_j \right) = 2 A^h_{hh} m_h + \sum_{j \neq h} A^h_{hj} m_j$$

$$+ \sum_{i \neq h} A^h_{ih} m_i = 2 \sum_{j=1}^{N} A^h_{hj} m_j .$$

If $\sum_{h=1}^{N} \sum_{j=1}^{N} A^h_{hj} m_j = 0$ for all $(m_1, \ldots, m_N)$,

then $\sum_{h=1}^{N} \sum_{j=1}^{N} A^h_{hj} m_j = \sum_{j=1}^{N} \left(\sum_{h=1}^{N} A^h_{hj} \right) m_j = 0$

implies $\sum_{h=1}^{N} A^h_{hj} = 0$, $j = 1,2,\ldots,N$.

Equation (4.6) requires

$$(4.13)\qquad -(A^2_{1j} + \dots + A^N_{1j}) + A^2_{2j} + \dots + A^N_{Nj} = 0,\ j = 1,2,\dots,N.$$

Apply (4.8) to get

$$(4.14)\qquad -(A^2_{11} + \dots + A^N_{11}) > 0,\quad A^2_{22} > 0,\ \dots,\ A^N_{NN} > 0$$

for the restriction implied by convexity of D_h in m_h.

Obviously, any selection of matrices $A^2, \dots, A^N$ that satisfy (4.13) and (4.14) will satisfy (4.6) - (4.8), with A^1 defined by (4.12). To give an example of a solution to (4.13) and (4.14) for N = 3, let $x \in R$, $x > 0$. Put

$$(4.15)\qquad A^2 = \begin{bmatrix} -x & -x & x \\ & x & -x \\ & & y \end{bmatrix},\quad A^3 = \begin{bmatrix} -x & x & -x \\ & y & -x \\ & & x \end{bmatrix}.$$

Here y is arbitrary and the lower half of each matrix is defined by symmetry. It is obvious that (4.15) is a solution to (4.12) and (4.13) for N = 3.

We leave to the reader the straightforward job of proving that solutions to (4.13) and (4.14) exist for N > 3, and extending the above analysis to K public goods.

It is instructive to see how the restrictions on the $\{D_h\}$ needed to insure the Samuelson-Lindahl conditions under the fixed coalition structure $\{S_1, S_2, \dots, S_I; S_{I+1}\}$ translate into requirements on the sequence of matrices $\{A^h\}_{h=1}^N$. The relevant equations are (3.3) and (3.4). Obviously, (3.4) is just

$$(4.16)\qquad \sum_{h=1}^{N} A^h = 0,$$

so this is the same as the non-cooperative case. This is to be expected since (4.16) is just the budget balance condition, and that has nothing to do with coalitions.

Equation (3.3) is a different matter. It is a straightforward exercise to verify the fact that

$$(4.17)\quad \sum_{i=1}^{I} |S_i|^{-1} \left(\sum_{s \in S_i} \sum_{s_o \in S_i} A^s_{s_o j} \right) + \sum_{h \in S_{I+1}} A^h_{hj} = 0$$

is necessary and sufficient for (3.3) to hold.

5. Externalities in Consumption

The design of Groves-Ledyard type mechanisms to achieve an efficient allocation in the face of consumption externalities may be facilitated by our methods. Furthermore, the minimum dimension of the message space required for efficient allocation may be systematically explored as a function of the externality pattern among individuals. This is especially important in designing mechanisms to internalize "local" externalities such as lawn-mower noise when only a neighboring set of people are affected by the emitter. We illustrate these ideas by means of an example.

Consider man h's problem: Choose $x_h \in R^J_+$ to

$$(5.1)\qquad \text{maximize } U_h(x_1,\ldots,x_N) \quad \text{s.t. } P \cdot x_h \leq w_h$$

where each $x_i \in R^J_+$. Here everyone else's consumption affects h, h=1,2,...,N. First we find[8/] necessary conditions for Pareto Optimality by solving the first order conditions to

$$(5.2) \qquad \text{Maximize} \sum_{h=1}^{N} \lambda_h U_h(x_1, \ldots, x_n)$$

$$\text{s.t.} \sum_{h=1}^{N} P\cdot x_h \leqq \sum_{h=1}^{N} w_h.$$

Here $\lambda_1 \geqq 0, \ldots, \lambda_N \geqq 0$ is a set of nonnegative utility weights. Form the Lagrangian

$$(5.3) \qquad L = \sum_{h=1}^{N} \lambda_h U_h(x_1,\ldots,x_N) + \Lambda \left(\sum_{h=1}^{N} w_h - \sum_{h=1}^{N} P\cdot x_h\right).$$

First order necessary conditions for an interior maximum are:

$$(5.4) \qquad \frac{\partial L}{\partial x_{ir}} = 0 = \sum_{h=1}^{N} \lambda_h U_{hx_{ir}} - P_r\Lambda = 0, \; i=1,2,\ldots,N \quad r=1,2,\ldots,J.$$

Let us eliminate the multipliers $\{\lambda_h\}_{h=1}^{N}$, and Λ from (5.4) and write it in a form anologous to (2.2) which we derived for the case of L pure private goods and K pure public goods. Put r=s and solve (5.4) for $\{\lambda_h\}_{h=1}^{N}$ in terms of Λ, P_s, and the matrix $[U_{hx_{is}}]$. We get, writing matters in matrix notation

$$(5.5) \qquad \lambda^t[U_{hx_{is}}] = P_s\Lambda e^t$$

where "t" denotes transpose and e denotes the column vector with $e_i=1$, $i=1,2,\ldots,N$.

Solving (5.5) (assuming $[U_{hx_{is}}]^{-1}$ exists) for the column vector λ gives us

$$(5.6) \qquad \lambda = [U^t_{hx_{is}}]^{-1} P_s\Lambda e.$$

Inserting the solution for λ from (5.6) into (5.4) gives us

$$(5.7) \qquad e^t P_s \Lambda [U^t_{hx_{is}}]^{-1} [U_{hx_{ir}}] = P_r \Lambda e^t, \; r=1,2,\ldots,J.$$

Rewrite this as

$$(5.8) \qquad e^t [U^t_{hx_{is}}]^{-1} [U_{hx_{ir}}] = P_r P_s^{-1} e^t, \; r=1,2,\ldots,J.$$

Notice that in the case where "1" is a pure private good and "2" is a pure public good then

$$U_{hx_{i1}} = U_{hx_{h1}} \delta_{hi}, \; h=1,2,\ldots,N, \; i=1,2,\ldots,N$$

where

$$\delta_{hi} \equiv 1, \; h=i, \; \delta_{hi} \equiv 0, \; h \neq i.$$

Also, in this case, we must have

$$(5.9) \qquad U_{hx_{i2}} = U_{hx_{j2}}, \; i, j = 1, 2, \ldots, N$$

since

$$U_h(x_1, x_2, \ldots, x_N) = U_h(x_{h1}, \sum_{j=1}^{N} x_{j2}, x_3, \ldots, x_N)$$

in the case when "1" is a pure private good and "2" is a pure public good. Hence, in this case, (5.8) collapses to

$$(5.10)\qquad \{e^t[U^t_{hx_{h1}}\delta_{hi}]^{-1}[U_{hx_{i2}}]\}_j = \{P_2P_1^{-1}\ e^t\}_j = \sum_{h=1}^{N} U^{-1}_{hx_{j1}}\ U_{hx_{j2}}$$

$$= \sum_{h=1}^{N} U^{-1}_{hx_{h1}}\ U_{hx_{h2}}.$$

Here $\{a\}_j$ denotes jth component of column vector a. Notice that (5.9) was used in obtaining the right hand side of (5.10). But (5.10) is just (2.2) for the case when "1" is a pure private good and "2" is a pure public good. It should be clear now that (2.2) is a special case of the general efficiency condition (5.8).

It should be mentioned that the assumption that $[U_{hx_{is}}]^{-1}$ exists as a severe restriction for good s. For example if good s is a pure public good then $[U_{hx_{is}}]^{-1}$ will not exist. This is so because

$$U_{hx_{is}} = U_{hx_{js}} \qquad i,j = 1,\ 2,\ \ldots,\ N.$$

Hence, all the columns of $[U_{hx_{is}}]$ are identical when s is a pure public good.

In fact, when all of the goods are pure public goods it is not possible to reduce the Pareto Optimum necessary conditions (5.4) to a form like (5.8) because $[U_{hx_{is}}]^{-1}$ does not exist for any s. It is desirable to reduce the Pareto Optimum necessary conditions to such a form in order to use the simple design procedure for informationally decentralized tax functions that will be discussed later in this section.

If there is at least one pure private good s that everyone desires, then (5.4) may be expressed in the form (5.8). This is so because

$$U_{hx_{is}} = U_{hx_{hs}} \delta_{hi}$$

reduces to a diagonal matrix with diagonal element $U_{hx_{hs}} > 0$, $h=1,2,\ldots,N$ in this case.

Even if the matrices $[U_{hx_{is}}]$ are all singular there may still be informationally decentralized individually incentive compatible procedures. For example, if there are only two pure public goods and three individuals then (5.4) becomes, putting $U_{hx_{is}} = U_{hG_s}$

$$\lambda_1 U_{1G_1} + \lambda_2 U_{2G_1} + \lambda_3 U_{3G_1} = P_1 \Lambda$$

$$\lambda_1 U_{1G_2} + \lambda_2 U_{2G_2} + \lambda_3 U_{3G_2} = P_2 \Lambda.$$

It may be possible to create allocation functions f_{is} and message spaces M_i to achieve the above relationship but that will require different methods which are developed in Section 6 of this paper.

Can we achieve (5.8) with an informationally decentralized incentive mechanism like that presented in Section 2? To attack this question we set up an N player noncooperative game as in Section 2.

Consider h's problem: Choose $m_h \in M_h$ to

$$\text{(5.11)} \quad \begin{aligned} &\text{maximize } U_h(y_1,\ldots,y_r) \\ &\text{s.t. } C_h(P; m_1,\ldots,m_N) \leqq w_h. \\ &y_i = f_i(m_1,\ldots,m_N), \qquad i=1, 2, \ldots, N. \end{aligned}$$

Here C_h is not the same function as in previous sections. The functions f_i

are called allocation functions by Groves and Ledyard. The function f_{ir} gives for example, the amount of commodity r allocated by the "Government" to consumption by i as a function of the message vector $(m_1, \ldots, m_N)$. Equilibrium is just a standard non-cooperative equilibrium to the N player game defined by (5.11).

At first, motivated by Groves and Ledyard's treatment of pure private goods and pure public goods, I tried to get by with

$$(5.12) \qquad f_{ir}(m_1,\ldots,m_N) \equiv \sum_{h=1}^{N} m_{ir}^h, \text{ where}$$

$$(5.13) \qquad m_{ir}^h \equiv m_r^h, \; i=1, 2, \ldots, N, \; r=1, 2, \ldots, J, \; h=1, 2, \ldots, N.$$

In other words (5.13) means the incremental amount proposed to the Government by h that i be allowed to consume of r should be independent of i. By hindsight it is obvious that the NJ dimensional message space implied by (5.13) is not "large enough" to attain the necessary condition for Pareto Optimal allocation viz. equation (5.8). We will become more specific below.

Our task is to design the allocation functions f, construct message spaces M_h, and design C_h so that a noncooperative equilibrium to the N player game defined by (5.11) satisfies the necessary condition for Pareto Optimal allocation viz. (5.8) above.

To do this, write down the necessary conditions for a noncooperative equilibrium: for each h, $m_{ir}^h \in R^1$ solves

$$(5.14) \qquad 0 = \frac{\partial L_h}{\partial m_{ir}^h} = \frac{\partial U_h}{\partial f_{ir}} \frac{\partial f_{ir}}{\partial m_{ir}^h} - \Lambda_h \frac{\partial C_h}{\partial m_{ir}^h} .$$

Note that (5.14) is not an inequality since negative messages m_{ir}^h are allowed as well as positive ones. Here we are taking M_h to be the space of all NJ dimensional matrices $[m_{ir}^h]$, h=1,2,...,N. Let us put $f_{ir}(m_1,\ldots,m_N) \equiv \sum_{h=1}^{N} m_{ir}^h$ and see how far we can go with this specification of f. Hence, (5.14) becomes

(5.15) $$0 = U_{hx_{ir}} - \Lambda_h C_{hm_{ir}^h}$$

where subscripts denote the obvious partial derivations. Put r=s in (5.15) and solve for the diagonal matrix $[\Lambda_h]$:

(5.16) $$[U_{hx_{is}}^t][C_{hm_{is}^h}^t]^{-1} = [\Lambda_h].$$

(Construct the C_h so that the required inverse matrix exists for each s.) From (5.16) and (5.15) we get

(5.17) $$[U_{hx_{is}}^t]^{-1}[U_{hx_{ir}}] = [C_{hm_{is}^h}^t]^{-1}[C_{hm_{ir}^h}].$$

Premultiply (5.17) by e^t,

(5.18) $$e^t\,[U_{hx_{is}}^t]^{-1}[U_{hx_{ir}}] = e^t[C_{hm_{is}^h}^t]^{-1}[C_{hm_{ir}^h}].$$

Comparing (5.8) with the necessary condition (5.18) for Pareto Optimality allows us to uncover:

Design Rule for Pareto Optimality:

$$(5.19)\quad e^t[C^t_{hm^h_{is}}]^{-1}[C_{hm^h_{ir}}] = P_r P_s^{-1}\, e^t,\ r=1,2,\ldots,N,\ s=1,2,\ldots,N.$$

Remark: (5.19) is assumed to hold only for s such that $[U^t_{hx_{is}}]^{-1}$ exists. This inverse exists if s is a pure private good provided that each $U_{hx_{hs}} \neq 0$. Notice that (5.19) does not guarantee budget balance.

In order to derive a condition for budget balance, we need to isolate the expenditure on good r. Specialize each C_h to the form

$$(5.20)\quad C_h(P,m) = \sum_{r=1}^{J} C_{hr}(P_r,\ m^1_{.r},\ \ldots,\ m^N_{.r}).$$

Here

$$m^h_{.r}$$

denotes the N dimensional vector with ith component m^h_{ir}. The amount budgeted by h for good r is C_{hr}. The total amount X_r allocated by the government to the production of r with the allocation functions

$$f_{ir} \equiv \sum_{h=1}^{N} m^h_{ir}$$

is

$$(5.21)\quad X_r = \sum_{i=1}^{N} P_r \cdot f_{ir} = \sum_{i=1}^{N} P_r\Big(\sum_{h=1}^{N} m^h_{ir}\Big).$$

The total amount T_r allocated by consumers h=1,2,...,N to the consumption of r is given by

$$T_r = \sum_{h=1}^{N} C_{hr}.$$

Thus budget balance requires the

Budget Balance Condition:

$$(5.22) \qquad P_r \cdot \left(\sum_{i=1}^{N} \sum_{h=1}^{N} m_{ir}^{h}\right) = \sum_{h=1}^{N} C_{hr}(P_r, m_{\cdot r}^{1}, \ldots, m_{\cdot r}^{N})$$

for all $r = 1,2,\ldots,J$; for all noncooperative equilibrium messages m.

Conditions (5.19) and (5.22) constitute a fairly concrete design procedure for constructing $\{C_{hr}\}_{h,r}$ that are individually incentive compatible and balance the budget at a Nash noncooperative equilibrium in messages. The reader may use (5.19) and (5.22) to design quadratic $\{C_{hr}\}$ corresponding to our analysis of Section 4. Furthermore, if the externality network, i.e., the sets $E(r) = \{(h,i) | \frac{\partial U_h}{\partial X_{ir}} \neq 0\}$ are known to the designer then the dimensions of the message spaces required can be economized upon. In the above analysis we assumed $E(r) = \{(h,i) | h=1,2,\ldots,N, i=1,2,\ldots,N\}$, $r=1,2,\ldots,J$. In many practical applications E(r) will be a much smaller set of "externality pairs" and the dimensions of the corresponding message spaces may be reduced.

This is a good place to point out that we have not solved the problem of finding if optimal incentive mechanisms are to be designed for the AHM model with production externalities as well as consumption externalities such that a general equilibrium is Pareto Optimal. This is a subject that is important but beyond the scope of this article. It is hoped that the methods presented here will prove useful in attacking this more general and more interesting problem.

We turn now to an abstract formulation of the incentive design problem.

6. The Design of Individually Incentive Compatible Mechanisms in An Abstract Setting.

It is worthwhile to look at an abstract formulation that contains all of the examples considered in this paper in order that the common unifying structure be exposed. This section will build on Smale's "Global Analysis and Economics V: Pareto Theory with Constraints", [p. 213-221]. First we will outline Smale's setup and state his first order necessary and sufficient condition that a local Pareto point must satisfy. Secondly, we will define the incentive design problem in this setup and then we will derive a sufficient condition on individual tax functions so that Smale's first order conditions for a local Pareto Point obtain at a Nash noncooperative equilibrium in message space. Notation used will be Smale's where possible.

Consider the following problem: 'Pareto Optimize' real C^2 functions $U_1,\ldots,U_N$ defined on an open set $W \subseteq R^\ell$ subject to constraints given by contributions of the form $g_\beta(x) \geq 0, \beta=1,2,\ldots,n$. We say that $x \in \theta$ if there is no open interval (a,b) and no curve ρ: $(a,b) \to W$ passing through x that satisfies the constraints and strictly increases all the U_i, $i=1,2,\ldots,N$. θ is the set of local Pareto Optima. Smale shows that $x \in \theta$ implies there exist nonnegative multipliers $\{\lambda_i\}_{i=1}^N$, $\{\mu_\beta\}_{\beta \in B_x}$ not all zero such that

$$(*) \qquad \sum_{i=1}^{N} \lambda_i \, DU_i(x) + \sum_{\beta \in B_x} \mu_\beta \, D_\beta g(x) = 0$$

where

$$B_x = \{\beta | g_\beta(x) = 0\}.$$

Also he shows that if $\{D_\beta g(x)\}_{\beta \in B_x}$ is a linearly independent family of gradients, and if a nonnegative, not all zero, set of multipliers $\{\lambda_i\}_{i=1}^N$ $\{\mu_\beta\}_{\beta \in B_x}$ exists such that (*) holds then $x \in \theta$. Hence (*) is necessary and sufficient for $x \in \theta$. Interpret (*) as: there is no open half-space that contains all of the gradients $\{DU_i(x)\}_{i=1}^N$, $\{D_\beta g(x)\}_{\beta \in B_x}$.

Smale's theorem may be applied to obtain first order necessary and sufficient conditions for a local Pareto Optimum in the general Arrow-Hahn-McKenzie (AHM) model presented in Chapter Six of Arrow and Hahn's General Competitive Analysis.

The General Nash Noncooperative Equilibrium Incentive Design Problem for the AHM model (NEIDP), may now be defined as: Characterize tax functions $\{C_j\}_{j=1}^N$ on consumers and firms and design message spaces for consumers and firms together with allocation functions that allocate consumption vectors to consumption and production vectors to producers such that a Nash noncooperative equilibrium in messages is Pareto Optimal. I put the words "Nash noncooperative equilibrium" in the definition in order to emphasize that the same problem may be studied with a different game theoretic solution concept. It is important to realize, as Roberts [12] points out, that under some game theoretic setups, no individually incentive compatible mechanisms will exist.

We will not attack the problem in its full generality here. Rather, we will consider a sub-problem described below.

In Smale's setup let

$$x \equiv (x_{11}, x_{12}, \ldots, x_{1j}, x_{21}, \ldots, x_{2j}, \ldots, x_{N1}, \ldots, x_{NJ})$$

denote the state vector of the system where x_{ir} denotes the consumption of good r by person i. Let the income of person i be w_i and let the price of good r be P_r. Then consider the problem "Pareto Optimize" $(U_1(x), \ldots, U_N(x))$

$$(6.1) \qquad \text{s.t.} \quad g(x) \geqq 0: \sum_{i=1}^{N} w_i - \sum_{i=1}^{N} P \cdot x_i \geqq 0.$$

Problem (6.1) describes Pareto Optimal "demand vectors". Notice that the whole state vector enters each utility function as an argument. Thus, general consumption externalities are covered by (6.1). Smale's (*) becomes, for this special case: $x \in \theta$ if there exist a nonzero vector $(\lambda_1, \ldots, \lambda_N, \mu) \geqq 0$ such that

$$(6.2) \qquad \sum_{i=1}^{N} \lambda_i \, DU_i(x) + \mu \, Q = 0$$

where

$$Q_{ir} = - P_r, \; i=1,2,\ldots,N, \; r=1,2,\ldots,J.$$

To move toward a precise definition of the NEIDP, let consumer h solve

$$(6.3) \qquad \underset{m^h \in M_h}{\text{maximize}} \; U_h(f(m)) \qquad \text{s.t.} \quad C_h(P,m) \leqq w_h$$

where $m \equiv (m^1, \ldots, m^N) \in M_1 x \; \ldots \; x M_N$, and $f_{ir}(m)$ is the amount of good r allocated by the "government" to consumption by i as a function of the message vector as received by the government, and $C_h(P,m)$ denotes the tax levied on h as a function of the price vector P and the community message vector m. The first order part of the NEIDP is to find f, $\{M_h\}_{h=1}^{N}$, $\{C_h\}_{h=1}^{N}$ such that at each Nash noncooperative equilibrium $\bar{x}=f(\bar{m})$, in messages for the N player game defined by (6.3), we have (6.2) satisfied at $\bar{x}$ for some nonzero $(\lambda_1, \ldots, \lambda_N, \mu) \geqq 0$.

The budget balance part of the NEIDP requires that the amount spent on each good r equal the amount allocated to r at each Nash noncooperative equilibrium $\bar{x}=f(\bar{m})$. In order to pose this nicely and to say something more specific about characterizing $\{C_h\}$ that solve the NEIDP we specialize still further. Put $M_h=R^{NJ}$

$$f(m) = \sum_{h=1}^{N} m_{ir}^h, \quad m_{ir}^h \in R, \quad C_h(P,m) = \sum_h C_{hr}(P,m^{\cdot}_{\cdot r})$$

where the (h,i) component of the N^2 vector $m^{\cdot}_{\cdot r}$ is m_{ir}^h.

First order necessary conditions for a Nash noncooperative equilibrium $\bar{x}=f(\bar{m})$ are: There are numbers $\Lambda_h \geqq 0$, h=1,2,...,N such that at $\bar{x}$, $\bar{m}$ we have

$$(6.4) \qquad D_hU_h = \Lambda_h D_h C_h, \quad h=1,2,\ldots,N.$$

Here the (i,r) component of the symbol D_hC_h is

$$\frac{\partial C_h}{\partial m_{ir}^h},$$

and Λ_h is the marginal utility of income to h. Equations (6.4) were derived on the assumption that equilibrium $x_{ir}>0$ for all i,r. If inequalities $x_{ir} \geqq 0$ are effective at Nash equilibrium then an extra multiplier will appear in (6.4). We cavalierly assume such boundary problems away. It is beyond the scope of this article to extend the theory to the case of boundary equilibria. However, the presence of the extra multipliers at boundaries should make the generalization interesting and nontrivial. Plug (6.4) into (6.2) to get

$$(6.5)\qquad \sum_{i=1}^{N} \lambda_i\ (\Lambda_i D_i C_i) + \mu\ Q = 0.$$

Now if we assume all $\Lambda_i>0$ at Nash equilibrium (a weak restriction) then we may state

Proposition 1: Assume $\Lambda_h>0$ for all h, at all Nash equilibria. If $\{C_h\}_{h=1}^{N}$ are such that each Nash equilibrium $\bar{x}=f(\bar{m})$ of the noncooperative game (6.3) is Pareto Optimal then it is necessary that there exist $(\lambda_1(\bar{m}),\ldots,\lambda_N(\bar{m}),\mu(\bar{m}))\geqq 0$ such that

$$(6.6)\qquad \sum_{i=1}^{N} \lambda_i(\bar{m})\ D_i C_i(P,\bar{m}) + \mu(\bar{m})Q = 0,$$

Proof: Let $\bar{m}$ be a Nash equilibrium. Then (6.4) must hold for some $(\Lambda_1,\ldots,\Lambda_N)\geqq 0$. Now $\bar{x}=f(\bar{m})$ is a Pareto Optimum. Therefore, there is $(\lambda_1',\ldots,\lambda_N',\mu')\geqq 0$ such that (6.2) holds at $\bar{x}$. Insert (6.4) into (6.2). Q.E.D.

Condition (6.6) says, geometrically, that for noncooperative equilibrium to be Pareto Optimal, it is necessary to design the $\{C_h\}_{h=1}$ so that at each Nash equilibrium to (6.3) the vectors

$$D_1C_1(P,\bar{m}),\ D_2C_2(P,\bar{m}),\ldots,D_NC_N(P,\bar{m}),Q$$

do not lie in the same open half space.

Remark: It should be noted that additional restrictions need to be added to the cost functions in the case

$$\frac{\partial U_h}{\partial x_{jr}} = 0 \text{ for some } h,j,r.$$

I.E., in case

$$\frac{\partial U_h}{\partial x_{jr}} = 0$$

the cost function C_h must be restricted so that

$$\frac{\partial C_h}{\partial m^h_{jr}} = 0$$

in order that it be possible for

$$\frac{\partial U_h}{\partial m^h_{jr}} = \Lambda_h \frac{\partial C_h}{\partial m^h_{jr}}$$

to hold.

All of the examples that we have treated in the previous sections of this paper are special cases of (6.6).

The budget balance (BB) condition is: For each good r, for each Nash equilibrium $\bar{x}=f(\bar{m})$,

$$(6.7) \qquad \sum_{h=1}^{N} C_{hr}(P,\bar{m}_{\cdot r}) = P_r \cdot \sum_{h=1}^{N} \sum_{i=1}^{N} \bar{m}^h_{ir}.$$

Conditions (6.6) and (6.7) constitute first order necessary conditions for a solution to the NEIDP for the sub problem (6.1). The words "first order" are to be emphasized since Smale develops second order conditions for $x\notin\theta$ as well and these are quite different than the first order conditions. Development of second order solutions to the NEIDP is beyond the scope of this paper.

Conditions (6.6) and (6.7), while general, are not very useful in their present form since it is difficult to tell, a priori, what $\bar{x},\bar{m}$ are Nash equilibrium. Thus we turn to,

<u>Basic Problem</u>: Classify the $\{C_h\}_{h=1}^N$ that satisfy (6.6) and (6.7) for all m and all $P\geqq0$.

Since we want the $\{C_h\}$ to generate Pareto Optima at Nash equilibria generated by a wide class of utility functions, goods prices, and income distributions it is natural to study the <u>basic problem</u> defined above. In order that local constrained <u>maxima</u> and not other types of critical points are generated by the $\{C_h\}$ we will also require: for all h

$$(6.8) \qquad [\frac{\partial C_h}{\partial m_{ir}^h}]$$

is a positive definite matrix for each m, P. Notice that (6.6) just says that for all $P\geqq0$, and all m we have

$$(6.9) \qquad C^+(D_1C_1(P,m),\ldots,D_NC_N(P,m)) \subseteq C^+(Q).$$

Here

$$C^+(a_1,\ldots,a_N)=\{z|\text{ there is } \lambda_1\geqq0,\ldots,\lambda_N\geqq0 \text{ such that } z=\sum_{i=1}^N \lambda_i a_i\}$$

denotes the convex cone generated by the vectors $a_1,\ldots,a_N$. We now have a geometric criterion that must be satisfied by the $\{C_h\}_{h=1}^N$ in order that Pareto Optimality obtain at Nash equilibrium.

Criterion (6.9) is still not very useful from the practical point of view in the case that the Center knows the externality pattern amongst individuals. In order to see why consider the case where all goods are pure private goods and assume that the Center knows that all goods are pure private goods. Here, in order to minimize bureaucratic cost, the Center orders each h to send messages m^h_{hr} and to send no m^h_{ir}, $i \neq h$. The cost functions and budget balance condition, for the pure private goods case become

$$C_{hr}(P,m) = C_{hr}(P_r, m^h_{hr}) \tag{6.11}$$

and

$$\sum_{h=1}^{N} C_{hr}(P_r, m^h_{hr}) = P_r\left(\sum_{h=1}^{N} m^h_{hr}\right) \tag{6.12}$$

The allocation functions are

$$f_{hr}(m) = m^h_{hr} \tag{6.13}$$

and the utility functions are of the form

$$U_h(x) = U_h(f(m)) = U_h(f_{h1}(m), \ldots, f_{hJ}(m)). \tag{6.14}$$

Criterion (6.9) is not practically useful in this case since it does not take into account the natural implicit restriction on the functional form of each C_h due to the character of pure private goods. Of course, (6.11)-(6.13) is not the only mechanism for efficiently allocating pure private goods but it seems unreasonable to bear the extra bureaucratic cost of communicating and processing messages of the form m^h_{ir}, $i \neq h$ and using the allocation mechanism

$$f_{hr}(m) = \sum_{j=1}^{N} m^j_{hr}$$

when the utility functions are of the form (6.14)--i.e., there are no external effects.

Analogous remarks are pertinent to the case when each good is either a pure public good or a pure private good, and the Center knows this. For pure public goods it is natural to impose the requirement

$$(6.15) \qquad m^j_{hr} = m^j_{h'r}, \text{ for all } h,h',$$

for all j and for all pure public goods r. Equation (6.15) captures the anonymity of a pure public good--the external effect on j caused by the consumption of h is independent of who h is. A natural allocation rule for pure public good r is

$$(6.16) \qquad f_{hr}(m) = \sum_{j=1}^{N} m^j_{hr} = \sum_{j=1}^{N} m^j_{jr}$$

since

$$m^j_{jr} = m^j_{hr}$$

by (6.15)

Natural implicit restrictions such as (6.15) are not taken into account by (6.9). Notice, however, that (6.4) implies

(6.17) $$\Lambda_h \frac{\partial C_h}{\partial m^h_{ir}} = \frac{\partial U_h}{\partial m^h_{ir}} = \frac{\partial U_h}{\partial f_{ir}} \frac{\partial f_{ir}}{\partial m^h_{ir}} = 0, \text{ for } i \neq h, \text{ r pure private and}$$

(6.18) $$\Lambda_h \frac{\partial C_h}{\partial m^h_{ir}} = \frac{\partial U_h}{\partial m^h_{ir}} = \frac{\partial U_h}{\partial f_{ir}} \frac{\partial f_{ir}}{\partial m^h_{ir}} = \frac{\partial U_h}{\partial y_r} \text{ for r pure public}$$

where for each pure public good r

(6.19) $$y_r \equiv f_{ir}(m) \qquad i=1,2,\ldots,N.$$

Hence, if $\Lambda_h > 0$ then

(6.20) $$\frac{\partial C_h}{\partial m^h_{ir}} = 0, \text{ for } i \neq h, \text{ r pure private}$$

(6.21) $$\frac{\partial C_h}{\partial m^h_{ir}} = \frac{\partial C_h}{\partial m^h_{hr}}, \text{ for all i, r pure public.}$$

The restrictions (6.20), (6.21) are not captured by (6.9). The criterion (6.9) is most appropriate for cases when the "Office of Efficient Allocation" does not know the externality pattern amongst the agents of the economy. Notice that in all events (6.9) is only necessary for efficient allocation. We have <u>not</u> proved that it is sufficient in any sense of that word.

Let us show that for pure public good k when there is at least one pure private good "n" and all goods are either pure private or pure public the condition

(6.22) $$\sum_{h=1}^{N} \frac{\partial C_{hk}}{\partial m^h_{hk}} (P_k, m) = P_k$$

derived in Section Two of this paper is a special case of requirement (6.9) when (6.20) and (6.21) are assumed to be satisfied and $C_{hn}(P,m) = P_n m_{hn}^h$ for pure private goods n.

Put

$$P \equiv (p_1,\ldots,p_L,\ q_1,\ldots,q_k),\ K + L = J.$$

Assume that the first L goods are private goods, the last K goods are public goods so that

$$U_h(x) \equiv U_h(x_{11},x_{12},\ldots,x_{1J},\ x_{1J},x_{21},\ldots,x_{2J},\ldots,x_{N1},\ldots,x_{NJ})$$

is of the form

$$U_h(x) = U_h(x_{j1},\ldots,x_{hL},\ \sum_{i=1}^{N} x_{i,L+1},\ldots,\sum_{i=1}^{N} x_{iJ}),\ h=1,2,\ldots,N.$$

Consider for private good n the necessary condition (6.9): For all m,P there exists nonzero $(\lambda_1,\ldots,\lambda_N,\mu)\geqq 0$, such that,

$$(6.22)\qquad \sum_{h=1}^{N} \lambda_h\ D_h C_h = \mu\ Q.$$

We get for the (i,n) component of (6.22), using (6.20)

$$(6.23)\qquad \sum_{h=1}^{N} \lambda_h \frac{\partial C_{h1}}{\partial m_{i1}^h} = \mu P_n = \lambda_i \frac{\partial C_{in}}{\partial m_{in}^i},\quad i=1,2,\ldots,N.$$

For k pure public we get from (6.22) using (6.21)

$$(6.24)\qquad \sum_{h=1}^{N} \lambda_h \frac{\partial C_h}{\partial m^h_{ik}} = \mu P_k = \sum_{h=1}^{N} \lambda_h \frac{\partial C_{hk}}{\partial m^h_{hk}}, \qquad i=1,2,\ldots,N.$$

Notice that if the Center uses the natural specification,

$$(6.25)\qquad C_{in}(P,m) = P_n m^i_{in}$$

for private goods n then

$$(6.26)\qquad \frac{\partial C_{in}}{\partial m^i_{in}} = P_n$$

and (6.23) implies for $P_n>0$

$$(6.27)\qquad \mu = \lambda_i, \quad i=1,2,\ldots,N.$$

Insert (6.27) into (6.24) to get

$$(6.28)\qquad \mu P_k = \sum_{h=1}^{N} \mu \frac{\partial C_{hk}}{\partial m^h_{hk}} .$$

Now $\mu>0$ since for all i $\mu=\lambda_i$ and $(\lambda_1,\ldots,\lambda_N,\mu)\neq 0$. Hence cancelling μ from (6.27) we get

$$(6.29)\qquad P_k = \sum_{h=1}^{N} \frac{\partial C_{hk}}{\partial m^h_{hk}}$$

which is (6.22)

We leave it to the reader to explore the meaning of (6.9) for more general externality patterns than pure public and pure private goods.

Notice that the results are much more specific if the Center is assumed to know the externality pattern because then it is reasonable to impose additional restrictions on the C_{hr} such as (6.20), (6.21), and (6.25). These restrictions together with (6.8), (6.9) and budget balance are useful tools to narrow down the close of C_{hr} that lead to Pareto Optimal allocation at Nash equilibrium for arbitrary utility functions $U_1,\ldots,U_N$, goods prices $P_1,\ldots,P_J$ and income distributions $w_1,\ldots,w_N$.

7. The Design of Mechanisms for Efficient Accumulation of Public Capital

The methods outlined above may be used to design tax functions over time that will lead to efficient allocation of public capital. When capital is introduced into the model we must specify how time is to be introduced. At the abstract level if capital markets are perfect so that individuals may borrow against their future income then by the usual procedure we may reduce an intertemporal model of capital accumulation and current consumption to a static model by dating the goods where the "new budget constraint" is just that the present value of the expenditure stream must not be larger than the present value of the income stream. The static methods developed above may be applied directly and the only interesting question that remains is the intertemporal structure of the class of C_h that generate Pareto Optima in Nash equilibrium.

In the case that capital markets are not perfect a second best concept of Pareto Optimality, i.e., a component wise maximum of the vector of utility functions subject to institutional constraints must be formulated and tax functions must be designed (if possible) in such a way to attain a second best Pareto Optimum. Such an approach is needed for problems where, to take a well known example, institutional restraints prevent perfect capitalization of future wage income. We believe that a fruitful way to approach this problem is to use Smale's work on characterizing Pareto Optima with constraints and follow the analysis outlined in Section Six above. Due to space limitations, we must refer this problem to future research and move on.

Let us quickly illustrate the designs of $\{C_h\}$ for public capital allocation under perfect capital markets by working through an example in continuous time. Let there be one public capital good, one private good and revert back to the notation of Sections Two-Four above. Assume each individual lives T periods and gets utility from $x_h(t)$, $f(m(t)) \equiv \sum_{j=1}^{N} m_j(t)$, $F(M(t)) \equiv \sum_{j=1}^{N} M_{j0} + \sum_{j=1}^{N} \int_0^t m_j(s)ds$ where $x_h(t)$ denotes consumption of the private good at time t, $f(m(t))$ denotes the allocation of investment in the public capital good at time t as a function of the sum of the proposed increments to public investment by each agent j, and $F(M(t))$ denotes the allocation of the stock of public capital at time t which is assumed to be the sum of capital held by the agents at time 0, $\sum_{j=1}^{N} M_{jo}$, plus the sum of past proposed investments by the members of the community up to time t.

Put

$$g(t) \equiv \sum_{j=1}^{N} m_j(t), \; G(t) \equiv \sum_{j=1}^{N} M_{jo} + \sum_{j=1}^{N} \int_0^t m_j(s)ds .$$

Notice that

$$(7.1) \qquad \dot{M}_j(t) = m_j(t), \; j=1,2,\ldots,N$$

where "·" denotes time derivative.

Consider the problem of finding first order conditions to

(7.2) $$\text{maximize} \sum_{h=1}^{N} \lambda_h \int_0^T U_h(x_h(t), \sum_{j=1}^{N} m_j(t), \sum_{j=1}^{N} M_j(t), t)dt$$

(7.3) $$\text{s.t.} \sum_{h=1}^{N} [p(t) \cdot x_h(t) + \dot{A}_h(t) + q(t)m_h(t)] \leqq \sum_{h=1}^{N} [w_h(t) + r(t)A_h(t)]$$

(7.4) $$A_h(0) = A_{h0},\ A_h(T) \geqq 0,\ x_h(t) \geqq 0,\ m_h(t) \in R.$$

(7.5) $$\dot{M}_h(t) = m_h(t),\ M_h(0) = M_{h0},\ h=1,2,\ldots,N.$$

Problem (7.2) says that each consumer gets a sum of instantaneous utility over his lifetime [0,T] when the instantaneous utility function U_h depends on private consumption flow, public investment flow and the stock of public capital. The budget constraint for agent h is

$$p(t)\ x_h(t) + \dot{A}_h(t) + q(t)\ m_h(t) \leqq w_h(t) + r(t)\ A_h(t)$$

when $A_h(t)$ denotes assets at time t. Finding necessary conditions for a Pareto Optimum is done by finding first order conditions for the optimal control problem (7.2) subject to constraints (7.3)-(7.5). The controls are $\dot{A}_h(\cdot)$, $m_h(\cdot)$ and the states are $A_h(\cdot)$, $M_h(\cdot)$. The controls are assumed to be drawn from the set of piecewise continuous controls.

It is straightforward to show that the set of optima to (7.2) subject to the constraints (7.3)-(7.5) is the same as the set of optima to (7.2) under the constraints (7.3'), (7.4), and (7.5) where (7.3') is given by

$$(7.3')\quad \int_0^T \sum_{h=1}^N (p(t)x_h(t) + q(t)\, m_h(t))]e^{-\int_0^t r(s)ds}\, dt$$

$$\leqq \sum_{h=1}^N A_{h0} + \int_0^T [\sum_{h=1}^N w_h(t)]e^{-\int_0^t r(s)ds}\, dt.$$

This familiar reduction requires some sort of nonsatiation, e.g., $\frac{\partial U_h}{\partial x_h} > 0$, h=1,2,...,N so that (7.3), (7.3') will hold with equality at optimum for each t. More will be said about this reduction technique later. Let $\mu_h(t)$ denote the costate variable for $M_h(t)$ and let Λ denote the shadow price of constraint (7.5). Then assuming that optimal $x_h(t)>0$ for all t standard optimal control theory given us the necessary conditions for an optimum of (7.2) subject to (7.3'), (7.4), and (7.5) which are listed below. Put $r(s)\equiv r$ to ease notation. For all h, all t,

$$(7.6)\quad \lambda_h\, U_{hx_h}(t) - p(t)\, e^{-rt}\Lambda = 0$$

$$(7.7)\quad \sum_{j=1}^N \lambda_j U_{jg}(t) + \mu_h(t) - q(t)\, e^{-rt}\Lambda = 0,$$

$$(7.8)\quad \dot{\mu}_h(t) = -\sum_{j=1}^N \lambda_j\, U_{jG}(t),$$

$$(7.9)\quad \mu_h(T)\, M_h(T) = 0.$$

Where subscripted U's denote the obvious partial derivatives evaluated along the optimum path at time t.

Assuming $M_h(T)\neq 0$ for all h, equations (7.6)-(7.9) may be rearranged by integrating (7.8) backwards, using (7.9), then using (7.6) and (7.7) to read, for all t, it is necessary that

$$(7.10)\qquad \sum_{j=1}^{N} (U_{jg}(t)/U_{jx_j}(t)) + \sum_{j=1}^{N} \int_t^T (U_{jG}(s)/U_{jx_j}(s))\, e^{r(t-s)}ds = q(t)/p(t)$$

must hold. Equation (7.10) is intuitive because when there is no utility from stocks of public capital, $U_{jG}\equiv 0$ for all j and (7.10) collapses, as expected, to the static or "flow utility" case analyzed in sections Two-Four above. Hence, the difference between stocks and flows is highlighted by (7.10). Turn now to the formulation of the incentive design problem. It will follow that given in the previous sections of this paper.

Write down the obvious problem facing agent h

$$(7.11)\qquad \underset{m_h(\cdot),\dot{A}_h(\cdot)}{\text{maximize}} \int_0^T U_h(x_h(t), \sum_{j=1}^{N} m_j(t), \sum_{j=1}^{N} M_{j0} + \sum_{j=1}^{N} M_j(t), t)dt$$

$$\text{s.t.}\quad (7.12)\quad p(t)\, x_h(t) + \dot{A}_h(t) + C_h(q(t), m(t), M(t), t) \leqq w_h(t)$$

$$(7.13)\quad A_h(0) = A_{h0},\ A_h(t) \geqq 0$$

$$(7.14)\quad \dot{M}_h(t) = m_h(t).$$

Reduce (7.12) as was done to get (7.3')

$$(7.12')\qquad \int_0^T (p(t)\, x_h(t) + C_h(q(t),m(t),M(t),t))\, e^{-rt}dt \leqq \int_0^T w_h(t)\, e^{-rt}dt + A_{h0}.$$

The equilibrium concept will be intertemporal Nash noncooperative equilibrium, i.e., equilibrium $m_h(\cdot)$, $\dot{A}_h(\cdot)$ maximize (7.11) over the set of piecewise continuous functions of time subject to (7.13), (7.14) and (7.12')--<u>given</u> the functions $\{m_j(\cdot),\dot{A}_j(\cdot)\}_{j\neq h}$ chosen by the other players. Such an equilibrium is called an open loop equilibrium.

It has been criticized by Kydland [10] for not being the appropriate concept for economics. We shall analyze it here anyway and refer the fascinating problem of incentive design for Kydland's "feedback equilibrium" concept to future research.

Let $\gamma_h(t)$ denote the costate variable for $M_h(t)$ and Λ_h denote the shadow price of constraint (7.12') along an open loop equilibrium $\{m_h(\cdot),\dot{A}_h(\cdot)\}_{h=1}^{N}$. Since $m_h(\cdot),\dot{A}_h(\cdot)$ must maximize (7.11) subject to (7.13) (7.14) and (7.12') standard application of optimal control theory leads to the necessary conditions (assuming $x_h(t)>0$, all t),

$$(7.15) \qquad U_{hx_h}(t) - p(t)\, e^{-rt}\, \Lambda_h = 0,$$

$$(7.16) \qquad U_{hg}(t) + \gamma_h(t) - C_{hm_h}(t)\, e^{-rt}\, \Lambda_h = 0,$$

$$(7.17) \qquad \dot{\gamma}_h(t) = -U_{hG}(t) + \Lambda_h C_{hM_h}\, e^{-rt}$$

$$(7.18) \qquad \gamma_h(T)M_h(T) = 0.$$

Assuming that $M_h(T)\neq 0$ so that $\gamma_h(T)=0$ we may integrate (7.17) and manipulate (7.15), (7.16) in order to eliminate Λ_h and obtain, finally,

$$(7.19)\qquad (U_{jg}(t)/U_{jx_j}(t) + \int_t^T (U_{jg}(s)/U_{jx_j}(s))e^{r(t-s)}\,ds$$

$$= C_{hm_h}(t)/p(t) + (\int_t^T C_{hM_h}(t)e^{r(t-s)}\,ds)/p(t).$$

Obviously to attain (7.10) we need only require that the $C_h(\cdot)$ satisfy

$$(7.20)\qquad \sum_{h=1}^{N} C_{hm_h}(t) + \sum_{h=1}^{N}\int_t^T C_{hM_h}(t)e^{r(t-s)}ds = q(t)$$

for each t. Quite clearly (7.20) can be achieved by tax functions that are independent of M_h and are independent of time t. In other words, any member of the class of tax functions constructed in Sections Two-Four will satisfy (7.20). It is somewhat surprising that taxes at line t need only be made a time <u>independent</u> function of proposed increments to public <u>investment</u> levels at time t and q(t) even though utility functions are time <u>dependent</u> and depend upon <u>stocks</u> of the public good as well as investment levels and still (7.10) may be achieved.

Budget balance requires, for each t

$$(7.21)\qquad \sum_{h=1}^{N} C_h(t) = q(\Sigma m_j(t)).$$

Equation (7.20) and (7.21) constitute the dynamic versions of the static formulae derived in Sections Two-Four above.

We must apologize for the sketchy treatment given above but, hopefully, the objective of showing that our design technique can be easily extended to the case of public capital goods was achieved by the above few paragraphs at a minimal cost in space. A complete treatment requires another paper.

The problem of incentive design for efficiency becomes especially interesting when posed in the context of Samuelson-Diamond-Cass type of overlapping generations models where the economy goes on forever. For the open endedness of these models opens up new problems of characterizing efficient paths of public and private capital accumulation as well as the problem of incentive design. The well-known work of Cass [3], Majumdar [11] and others treat the characteristics of efficient paths for private goods.

8. Suggestions for Further Research and Summary

It remains to be seen whether our technique is useful for the construction of Smith [15] type mechanisms when the price of the public good may depend upon the messages that each agent sends. It is reasonable to conjecture that the method of using the necessary conditions of Nash equilibrium together with the necessary conditions of Pareto Optimality in order to find the partial differential equation (2.5) that the C_h must satisfy in order to generate Pareto Optima that is espoused in this paper should be applicable to Smith's case as well.

Furthermore, it is of interest tó see if the family of mechanisms that Green and Laffont [5] call "Groves" mechanisms which were motivated by the well-known work of Groves in team theory can be usefully designed by use of our methods.

In this paper we derived two basic conditions that tax functions must satisfy in order to generate Pareto Optimal allocations of public goods. In Section Two we treated the case of all goods were either pure public goods or pure private goods and showed that the tax functions must satisfy for each public good k, using the notation of Section Two:

$$(8.1) \qquad \sum_h C_{hk} = q_k\left(\sum_{j=1}^{N} m_{jk}\right), \ k=1,2,\ldots,k$$

in order that Pareto optimality and budget balance obtain respectively. Groves and Ledyard's [9] famous quadratic tax functions satisfy (8.1), (8.2). Furthermore, (8.1), (8.2) were derived following a natural and elementary chain of reasoning.

In Section Three, in order to show the virtue of the method used to (8.1) and (8.2), we extend the analysis of Section Two to the case of designing taxes on an exogenously fixed coalition structure that guarantee Pareto Optima for a mixed cooperative-noncooperative equilibrium concept.

Section Four develops and characterizes a general class of quadratic tax functions that guarantee Pareto Optimal equilibrium allocations.

General consumption externalities are treated in Section Five. A general abstract context for the design of individually incentive compatible mechanisms is presented in Section Six. Here recent work of Smale [14] on the necessary conditions for Pareto Optimality under constraints is used to give a characterization of such mechanisms in terms of convex cones generated by gradients of tax functions. The characterization is shown to contain those developed in the previous sections.

Finally, Section Seven shows that our methods can be used to design mechanisms for the efficient accumulation of public capital and to investigate systematically and to simplify the structure of these mechanisms.

<u>Questions that Remain</u>

They are, to list a few:

1. Can <u>one</u> set of $\{D_h\}_{h=1}^{N}$ be designed so that for <u>all</u> coalition structures $S \equiv \{S_1, S_2, \ldots, S_I; S_{I+1}\}$, for each S-equilibrium in the sense of D.3.1., the Samuelson-Lindahl efficiency condition and the budget balance condition are satisfied? The answer to 1 is likely to be "no", in view of the work of Bennett and Cohn [1]. Then it is of interest to,
2. Classify the sets C of coalition structures such that there is one set of $\{D_h\}_{h=1}^{N}$, such that for each S-equilibrium in the sense of D.3.1. for each $S \in C$, the Samuelson-Lindahl efficiency condition and the budget balance condition holds.

3. Make up a notion of $< \{D_h\}_{h=1}^{N}, C >$ - core such that the core is non-empty and points in it are efficient allocations. Here C is the set of allowable coalitions. The $< \{D_h\}_{h=1}^{N}, C >$ - core is the set of all utility allocations that are not blocked by a coalition in C, where the tax functions are $C_h = \alpha_h q \cdot (\sum_{j=1}^{N} m_j) + D_h$, h = 1,2,...,N. This is necessary so that, for example, the Groves-Ledyard equilibrium could be called competitive, in the sense that the core converges to the set of G.L. "competitive equilibria" as the number of players tends to infinity in a precise way. It must be realized that it is not obvious how to formulate this question, since the class C is restricted.

4. Make up a notion of "partial equilibrium $< p, q, w_1, \ldots, w_N, C, \{D_h\}_{h=1}^{N} >$ core" where $p, q, w_1, \ldots, w_N, C, \{D_h\}_{h=1}^{N}$ are private goods price, public goods price, incomes of 1, ..., N resp., set of "allowable" coalitions, and tax functions. This would be a natural notion of demand for private and public goods, when coalitions C are allowed to form and tax functions are $\{D_h\}_{h=1}^{N}$. Is the core non-empty for each $p, q, w_1, \ldots, w_N$, and can $\{D_h\}_{h=1}^{N}$ be found that lead to efficient allocations in the core?

5. In the standard G. L. equilibrium with no public goods; i.e., an Arrow-Debreu equilibrium, a coalition can gain in a finite player competitive equilibrium, but this "gain" goes to zero as N tends to infinity. Does this type of theorem hold good for the Groves-Ledyard equilibrium as well? If so, might not we be worrying excessively about "coalition problems" in the Groves-Ledyard case?

6. If it is shown to be impossible to design one set of $\{D_h\}_{h=1}^{N}$ that "works" for a reasonably large class of coalitions, can we design a set of sets $\{D_h(m;\mathcal{S})\}_{h=1}^{N}$, one for each coalition structure $\mathcal{S} \in \mathcal{C}$, that work? Can we economize, in some useful way, on the number of different $\{D_h(m;\mathcal{S})\}_{h=1}^{N}$ required as $\mathcal{S}$ ranges over $\mathcal{C}$? Obviously, the smaller the number of $\{D_h(m;\mathcal{S})\}_{h=1}^{N}$ that we design, the smaller the information requirement of the system.

7. (Due to Steve Slutsky of Cornell University's Department of Economics.) In the large number of players case, will the gain to player h in calculating his desired level of public goods, given the other people's messages, and his tax function over the simple expedient of simply choosing his m_h to minimize his taxes, given the other people's messages, be enough to compensate him for the trouble of exploring and calculating his MRS for public goods? In the real world such a problem might be important. After all in the case of U.S. Government supplied public goods, such as national defense, it is hard to imagine any taxpayer doing anything but hiring an accountant to minimize his tax load. Is this kind of criticism really important before the G.L. system can be made operational?

FOOTNOTES

1/ If negative m_{hk} is allowed for each h,k, then (2.2) will hold with equality. Negative m_{hk} allows public bads and/or allows individual h to communicate desires to reduce the amount of k produced.

2/ Elaine Bennett and David Cohn [1] have established that coalitions can cheat Groves-Clarke type mechanisms. However, the same logic applies to the Groves-Ledyard mechanism (2.14) as well. Bennett and Cohn call a mechanism "group incentive compatible" if no group of agents can make themselves better off by misrepresenting their preferences. Bennett and Cohn use a result of Green and Laffont [5] to show that no revelation mechanism that is satisfactory (a weak requirement) and group incentive compatible exists.

3/ This definition may be reformulated by making use of the old trick that given Pareto Optimum allocation, a set of weights may be found so that the Pareto Optimum may be realized as the solution to the first order condition gotten by maximizing the weighted sum of utilities over the set of feasible allocations. We want our equilibrium concept to be:

$(\bar{x}_1, \ldots, \bar{x}_N, \bar{m}_1, \ldots, \bar{m}_N)$ is an equilibrium relative to $S_1, S_2, \ldots, S_I: S_{I+1}$ if $\{\bar{m}_s\}_{s\in S_i}$, $\{\bar{x}_s\}_{s\in S_i}$ is Pareto Optimum over the members of S_i given .

$\{\bar{m}_s\}_{s\notin S_i}$, $\{\bar{x}_s\}_{s\notin S_i}$ for each i, and $\{\bar{m}_j\}_{j\in S_{I+1}}$ is a non-cooperative equilibrium given $\{\bar{m}_j\}_{j\notin S_{I+1}}$, $\{\bar{x}_j\}_{j\notin S_{I+1}}$. But if $\{\bar{m}_s\}_{s\in S_i}$, $\{\bar{x}_s\}_{s\in S_i}$ is a Pareto Optimum given $\{\bar{m}_s\}_{s \in S_i}$, $\{\bar{x}_s\}_{s \in S_i}$, then there must exist a nontrivial set of nonnegative weights $\{\lambda_t\}_{t \in S_i}$ such that $\{\bar{x}_s\}_{s \in S_i}$, $\{\bar{m}_s\}_{s \in S_i}$ satisfies the first order necessary conditions to the problem

$$(*) \quad \text{maximize} \sum_{t \in S_i} \lambda_t U_t$$

$$\text{s.t.} \sum_{t \in S_i} p\, x_t + \sum_{t \in S_i} C_t \leq \sum_{t \in S_i} w_t$$

<u>given</u> $\{\bar{x}_s\}_{s \in S_i}$, $\{\bar{m}_s\}_{s \in S_i}$

Hence we may reformulate Definition 3.1 in terms of (*).

[4/] Even when there is no regulatory body to enforce the agreement the force of the state may be used anyway under certain conditions. For suppose a group wanted to form a coalition C of size N and wanted to prevent the formation of subcoalitions of C as well as entry and exit of individuals from C. Labor unions are of this character.

One way to enforce C might be to lean on the law of contract which is enforced by the state free to C. I.e., suppose that for C to be effective some number, say Y% of a particular group, A, of individuals must join C. The C-organizer goes around to each member of A with a contract that reads as follows:

> We would like to form C to bargain for our rights. In the past we've been done in by free riders who chisel on the organization. This contract is designed to police free riders and provide for a viable C.
>
> The undersigned agrees to put up X dollars in escrow to be deposited in the account of Coalition C at bank ________ which will be forfeited if the undersigned is found guilty of violating the bylaws of C. A copy of the bylaws of C is attached to this contract.
>
> This contract becomes legally binding if and only if Y% of the members of A sign. If Y% of the members of A have not signed by date Z, the X dollars will be returned immediately to the undersigned.

The amount X, percent Y, and date Z may be varied to suit the particular needs of C. X is chosen large enough to deter strikebreakers if C calls a walkout of the members of A. Furthermore, Y must be large enough so that a walkout is effective. Also, the date Z may be set further into the future the more time consuming the task of obtaining Y% of A as signatories. The indenture on the contract may include dues, provisions for meetings, organizers' salaries, mechanisms of trial for violators of the C bylaws, etc.

In theory, the harnessing of the power of the state (which is free to C) in enforcing the law of contract may be a useful way to voluntarily provide A-specific public goods but in practice legal hassles and court entanglements may prove to be fatal to this scheme.

If the scheme could be made to work it could be used to police the type of coalition structure studied above and thus the solution concept treated above may become more interesting than it is at present.

[5]/ Actually these are only required to hold for <u>demand</u> vectors $(\bar{m}_1,\ldots,\bar{m}_N)$ but in many applications it will be useful to construct $\{D_h\}_{h=1}^N$ such that (4.2), (4.3) hold for <u>all</u> $(m_1,\ldots,m_N)$.

[6]/ In detail

$$\frac{\partial}{\partial m_h}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} A^h_{ij}\, m_i m_j\right) = \frac{\partial}{\partial m_h}\left(\sum_{j=1}^{N} A^h_{hj}\, m_h m_j\right)$$

$$+ \frac{\partial}{\partial m_h}\left(\sum_{i\neq h}\sum_{j=1}^{N} A^h_{ij}\, m_i m_j\right) = 2 A^h_{hh}\, m_h + \sum_{j\neq h} A^h_{hj}\, m_j$$

$$+ \sum_{i\neq h} A^h_{ih}\, m_i = 2\sum_{j=1}^{N} A^h_{hj}\, m_j .$$

If $\sum_{h=1}^{N} \sum_{j=1}^{N} A_{hj}^{h} m_j = 0$ for all $(m_1, \ldots, m_N)$,

then $\sum_{h=1}^{N} \sum_{j=1}^{N} A_{hj}^{h} m_j = \sum_{j=1}^{N} (\sum_{h=1}^{N} A_{hj}^{h}) m_j = 0$

implies $\sum_{h=1}^{N} A_{hj}^{h} = 0$, $j = 1,2,\ldots,N$.

7/ The same proof shows that there are no twice continuously differentiable tax functions C_1, C_2 that solve (4.6)-(4.8) either. To see it replace each A_{ij}^{h} by $\frac{\partial C_h}{\partial m_{ij}}$ and follow the same steps.

8/ The procedure of finding Pareto Optima by maximizing a nonnegative weighted sum of utilities should be viewed as a "deus ex machina" type of procedure for remembering the correct (Smale [14, pp. 213-222]) first order necessary conditions which are: there exist $\{\lambda_h\}_{h=1}^{N}$, Λ all nonnegative, $(\lambda_1,\ldots,\lambda_N,\Lambda)\neq 0$ such that (5.4) is satisfied. The problem of deriving first order conditions for a Pareto Optimum is discussed in more detail below in Section 6.

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