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APPLICATIONS OF RECENT RESULTS ON THE ASYMPTOTIC STABILITY OF OPTIMAL CONTROL TO THE PROBLEM OF COMPARING LONG RUN EQUILIBRIA

by

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Section 1: Introduction

The purpose of this article is to discuss some recent results on the global asymptotic stability of optimal control and show how they may be used with profit in the problem of comparing long run equilibria.

To be more specific in Section 1 we shall study optimal control problems of the form

$$(1) \quad \text{maximize} \int_o^\infty e^{-\rho t} \, \pi\,(x(t),\, \dot{x}(t),\, \alpha)\, dt$$

$$\text{s.t. } x(o) = x_o \text{ given}$$

where $\pi\,(x(t),\, \dot{x}(t),\, \alpha)$ is instantaneous payoff at time t which is assumed to be a function of the state of the system at time t, $x(t)$, the rate of change of the state $\frac{dx}{dt} \equiv \dot{x}(t)$ and a vector of parameters, α. Here $x(t) \in R^n$, $\dot{x}(t) \in R^n$, $\alpha \in R^m$ and $\rho \geq o$. Also the maximum in (1) is taken over the set of absolutely[1] continuous functions $x(\cdot)$ such that $x(o) = x_o$.

If $\pi(.,\, .,\, \alpha)$ is strictly concave in $(x, \dot{x})$ then there is at most one optimum path $x(t, x_o, \alpha)$ for each x_o, α. If optimum exists for each (x_o, α) then there is a function

$$h : R^n x R^m \to R^n$$

called the "optimal policy function" such that

$$(2) \quad \dot{x}(t, x_o, \alpha) = h[x(t, x_o, \alpha), \alpha]$$

Furthermore, h does not depend upon x_o or on t due to the time stationarity of (1).

What we shall discuss in this paper is a set of sufficient conditions on π and ρ such that there is a unique steady state $\bar{x}(\alpha)$ such that for all x_o

(3) $x(t, x_o, \alpha) \to \bar{x}(\alpha)$, $t \to \infty$.

Property (3) is called global asymptotic stability and is abbreviated: G.A.S.. The discussion of sufficient conditions on π for (3) to hold will be terse because this is covered in detail by my survey paper [5]. It will be helpful to the reader if he has a copy of [5] while reading this essay.

After brief discussion of sufficient conditions on π and ρ for G.A.S. we shall take up the problem of comparing long run equilibria. This problem is known by most readers as "comparative statics" or "comparative dynamics" and is the main concern of this article. Furthermore, something called "Samuelson's Correspondence Principle" is supposed to play a role in the exercise.

Before we indicate how sufficient conditions for G.A.S in optimal control problems are related to Samuelson's Correspondence Principle and since it plays a central role in this essay, let us remind readers how it is described in the Foundations and of some criticism that it has received.

Samuelson's Correspondence Principle (CP)

Samuelson, in the Foundations [24], considered the system of equations

(4) $0 = E(p, \alpha)$

where

$$E : R^n x R^m \to R^n$$

is a system of excess demand functions for n+1 goods as a function of the price vector $(p_1, \ldots, p_n)$ and the parameter vector $(\alpha_1, \ldots, \alpha_m)$. Note that there are only n independent equations by Walras Law. The n+1 good is numeraire. Equations (4) describe competitive equilibrium where excess demand equals zero (we are assuming no free goods). Now (4) is an equilibrium system. It tells us what the equilibrium price is but it does not tell us how the economic system gets to equilibrium.

Samuelson proposed the adjustment mechanism

(5) $\dot{p}_i = g_i (E_i (p, \alpha)), i = 1, 2, \ldots, n$

$p_i(o) = p_i o$ given, $i = 1, 2, \ldots, n$

where

$g_i(o) = o$, $g_i (E_i) > o$, for $E_i > o$, and $g_i (E_i) < o$, for $E_i < o$.

Mechanism (5) corresponds to the intuitive idea that price increases when excess demand is positive and vice versa.

After introducing (5) Samuelson studied its asymptotic stability and enunciated his <u>Correspondence Principle</u>: the hypothesis of asymptotic stability of (5) together with a priori information on $\partial E/\partial\alpha$ gives rise to useful restrictions on

$$\partial\overline{p}/\partial\alpha$$

where $\overline{p}(\alpha)$ is the equilibrium price (assumed locally unique) as a function of α. Samuelson applied his principle to other problems as well as general equilibrium but we shall concentrate on the equilibrium problem here for illustrative purposes.

I have deliberately stated the Correspondence Principle as a <u>methodological principle</u> rather than as a precise theorem in order to capture Samuelson's basic idea. Research of Quirk-Saposnik [22], et al has shown that it is not possible <u>in general</u> to use the hypothesis of asymptotic stability of (5) together with sign information on $\partial E/\partial\alpha$ to get comparative statics information on $\partial E/\partial\alpha$.

For example, look at the case where α is one dimensional. Without loss of generality assume $g_i (E_i) = E_i$. Differentiate (4) totally with α and solve for $\partial\overline{p}/\partial\alpha$,

(6) $$\partial\overline{p}/\partial\alpha = - \left(\frac{\partial E}{\partial p}\right)^{-1} \frac{\partial E}{\partial\alpha} .$$

The hypothesis that all eigenvalues of $\partial E/\partial p$ have negative real parts contains n restrictions and a priori sign restrictions on $\frac{\partial E}{\partial \alpha}$ contain n more restrictions. But there are n^2 elements in $\partial E/\partial p$. Since Sonnenschein's theorem [27] shows that the axioms of Arrow-Debreu-McKenzie general equilibrium theory are general enough to allow <u>any continuous function</u> from $R^n \to R^n$ to be an excess demand function for some n+1 goods general equilibrium system therefore it comes as no surprise that examples of higher dimensional systems satisfying all of the restrictions listed above could be created that give arbitrary sign to $\partial \overline{p}/\partial \alpha$. Yet, in such examples, $\partial E/\partial p$ was a stable matrix and $\partial E/\partial \alpha$ had apriori sign restrictions.

These findings led to a lot of research on the correspondence principle (Quirk-Saposnik [22]) that concluded that a priori sign information on $\partial E/\partial \alpha$ plus stability assumed on $\partial E/\partial p$ gave sign restrictions on $\partial \overline{p}/\partial \alpha$ from (6) if the system (6) was one dimensional or if $\partial E/\partial p$ "had a lot of zeroes in it." See Quirk-Saposnik [22].

This was discouraging enough. But while the Quirk-Saposnik type of research was going on the very mechanism (5) came under attack as a description of adjustment. The papers by Gordon and Hines [15] and Phelps and Winter in [21] come to mind. Gordon and Hines argued that speculative activity would destroy any such mechanism (5). They also asked "Whose maximizing behavior does such a mechanism describe?" They argued that it is mechanical and not based on self interested behavior. Phelps and Winter in their article in [21] developed the beginnings of an alternative disequilibrium dynamics.

After the "Phelps Volume" [21] was published there were several attempts in the literature to rationalize mechanisms of type (5) but no consensus seems in sight. Hence, the epitaph of the correspondence principle was written by Arrow and Hahn [1, p. 321] in their chapter on comparing equilibria: "It "isn't.""

Shortly after the Arrow Hahn book was published the Sonnenschein-Mantel-Debreu theorem [27], [19], and [14] appeared. This result showed that any continuous function E(p) could be an excess demand function for some economy populated by people with perfectly well behaved utility functions.

In view of the SMD result it seems that the CP is sure to fail to be of much use if one insists on the generality of abstract general equilibrium theory. Nevertheless, the CP in some form lurks in the "underworld" of economists of a more practical bent.

I will offer here a version of the CP that I think is somewhat immune to the criticisms listed above. Return now to the optimal control problem (1) together with the dynamics of the solution (2). Many intertemporal equilibrium systems studied in the literature: optimal economic growth, (re Journal of Economic Theory Symposium Volume Feb., 1976) adjustment cost models in the neoclassical theory of investment (re Lucas [16], Treadway [28], and Mortensen [20]), perfect foresight models (re Lucas and Prescott [17], Brock [4]), to name a few, can be fit into (1). By the device of describing general intertemporal equilibrium as the solution to a problem of maximizing a discounted sum of consumer surplus an economically interesting class of equilibrium models in modern capital theory is covered by (1).

Let

(7) $0 = h(\bar{x}(\alpha), \alpha)$

play the role of (4) and let (2) play the role of (5) in the CP.

Revised Correspondence Principle: For problems of type (1) whose solution generates the "equilibrium disequilibrium" adjustment process (2) the hypotheses of L.A.S. of the solution $\bar{x}(\alpha)$ to (7) with respect to (2) together with a priori economically natural structural assumptions on $\pi(x, \dot{x}, \alpha)$ leads to useful comparative statics information on $\partial\bar{x}/\partial\alpha$.

The revised CP is the main subject of this essay.

Turn now to a closely related idea. Arrow and Hahn [1], in their chapter on comparing equilibria gave little credence to the hope that the hypothesis of L.A.S. alone of equilibria of (5) would yield useful restrictions on $\partial\overline{p}/\partial\alpha$. They did show, however, how <u>sufficient conditions</u> for L.A.S. or G.A.S. of (5) such as "all goods are gross substitutes" yield useful restrictions on $\partial\overline{p}/\partial\alpha$. This is a "Correspondence Principle of sorts" in that stability hypotheses are closely linked to the problem of getting useful restrictions on $\partial\overline{p}/\partial\alpha$.

We shall investigate the same idea for (1) and (2) in this article.

<u>Problem</u>: <u>What restrictions do the various sufficient conditions for L.A.S. or G.A.S. of a steady state $\overline{x}$ of (2) together with the natural structure of $\pi(x, \dot{x}, \alpha)$ impose on $\partial\overline{x}/\partial\alpha$</u>?

In Section 3, we obtain the following result for

(8) $\pi = p\, f(x, \dot{x}) - \alpha_1'x - \alpha_2'\dot{x}.$

Here a'b denotes the scalar product of the vectors a and b.

Let

$$H^o(q, x) = \sup_{\dot{x}} [p\, f(x, \dot{x}) - \alpha_1'x - \alpha_2'\dot{x} + q'\dot{x}],$$

and let $(\overline{x}, \overline{q})$ be a steady state of (2) where $\overline{q}$ is the costate variable at steady state $\overline{x}$. Consider the three hypotheses:

(i) $H^o_{qx} = 0,$ at $(\overline{q}, \overline{x})$;

(ii) $Q \equiv \begin{pmatrix} H^o_{qq}, & \rho/2\, I_n \\ \rho/2\, I_n, & -H^o_{xx} \end{pmatrix}$ is positive definite at $(\overline{q}, \overline{x})$;

(iii) $R \equiv H^{o^{-1}}_{qq} H^o_{qx}$ is negative quasi definite at $(\overline{q}, \overline{x})$

where H^o_{qx}, H^o_{qq}, H^o_{xx} is the usual notation for second order partial derivatives and I_n denotes the nxn identity matrix. Any of these three hypotheses are sufficient for the G.A.S of $\overline{x}(\alpha)$ (re [ ]).

Theorem: For π given by (8) any of the hypotheses (i), (ii), (iii) imply that

$$(9) \quad \frac{\partial \bar{x}}{\partial \alpha_1}, \frac{\partial \bar{x}}{\partial \alpha_2}, \frac{\partial \bar{x}}{\partial(\alpha_1+\rho\alpha_2)}$$

are all negative quasi definite.

Remark: If (8) describes instantaneous profit for a competitive one product firm then

$$\bar{x}(\alpha_1, \alpha_2)$$

is the long run or steady state demand function for x. The quantity $\beta \equiv \alpha_1 + \rho\alpha_2$ is rent and

$$\frac{\partial \bar{x}}{\partial \beta}$$

negative quasi definite says that the "demand for $\bar{x}$ is downward sloping."

This is a nontrivial result because long run demands, $\bar{x}(\alpha)$ generated by optimization problems like (1) may not slope downward due to dynamic interactions (cf Mortensen [20, p. 664]).

The paper is organized as follows: Section One contains the introduction. Section Two develops abstract comparative statics results that will be used later. Section Three develops the result that was described above.

In Section Four we show that the three G.A.S. hypotheses described above imply

$$B(\rho) \equiv \frac{1}{q}\frac{d\bar{x}}{d\rho} \leqq 0.$$

The latter quantity

$$\frac{1}{q}\frac{d\bar{x}}{d\rho}$$

is a measure of capital deepening introduced by Burmeister and Turnovsky [12]. The Burmeister-Turnovsky article relates the negativity of $B(\rho)$ to the "Cambridge" controversy in capital theory.

Section Five provides a modified proof of a theorem of Mortensen [20]. His theorem relates the hypothesis of L.A.S. of the optimal steady state $\bar{x}$ to qualitative informaion on

$$\frac{\partial \bar{x}}{\partial \alpha}$$

Section Six closes with a summary.

Notations Equations will be numbered consectively in each section. The section number of an equation will be given only when necessary. Partial derivatives will be denoted by subscripts as in (i) - (iii) above. Upper bars will be dropped from equilibrium quantities whenever it is possible to do so without causing confusion. We say that an nxn matrix A is negative quasi definite if $x'Ax < 0$ for all $x \neq 0$. We resume the term "negative definite" for the case when A is symmetric and $x' Ax < 0$ for all $x \neq 0$. The symbol A' denotes the transpose of matrix A. Also $\dot{x}$ is short for dx/dt.

Before we begin we would like to point out that a paper by Burmeister and Long [29] attacks a somewhat similar question. They are interested in the implications of the L.A.S. hypothesis for the comparison of steady statics under changes in ρ. We focus on the formulation of "equilibrium disequilibrium" mechanisms $\dot{x} = h(x, \alpha)$ and the extension of Samuelson's Correspondence Principle to such mechanisms. To our knowledge Burmeister-Long [29] and Mortenson [20] are the first to explicitly formulate questions of the type: What are the implications of the L.A.S. hypothesis on the optimal steady states of a control problem for the problem of comparing steady states? Our paper continues to develop this line of thinking.

Let us get into the substance of the paper.

Section 2: Abstract Results on Comparing Optimal Steady State

This section will develop abstract results which will be given content in later sections.

Consider the system:

(1) $\dot{q} = \rho q - H^o_x (q, x, \alpha)$

(2) $\dot{x} = H^o_q (q, x, \alpha), \; x(o) = x_o.$

Equations (1), (2) are necessary for optimality in a large class of problems. The transversality condition,

(3) $\lim_{t\to\infty} q(t)' \, x(t) \, e^{-\rho t} = 0,$

has been shown to be necessary as well as sufficient for a general class of problems by Benveniste and Scheinkman [2].

A steady state $\bar{x}(\alpha)$ with its associated costate $\bar{q}(\alpha)$ must solve:

(4) $0 = \rho\bar{q}(\alpha) - H^o_x (\bar{q}(\alpha), \bar{x}(\alpha), \alpha)$

(5) $0 = H^o_q (\bar{q}(\alpha), \bar{x}(\alpha), \alpha).$

The transverality condition (3) is automatically satisfied at a steady state $\bar{x}(\alpha)$, $\bar{q}(\alpha)$. Hence questions (4) and (5) characterize optimal steady states.

Totally differentiate both sides of (4), (5) with respect to α (drop upper bars to ease typing) to obtain

(6) $0 = \rho q_\alpha - H^o_{xq} q_\alpha - H^o_{xx} x_\alpha - H^o_{x\alpha}$

(7) $0 = H^o_{qq} \, q_\alpha + H^o_{qx} x_\alpha + H^o_{q\alpha}$

Premultiply (6), (7) by the matrix (x'_α, q'_α) and obtain

(8) $$(x'_\alpha, q'_\alpha) \begin{pmatrix} H^o_{x\alpha} - \rho q_\alpha \\ - H^o_{q\alpha} \end{pmatrix} = - x'_\alpha \, H^o_{xx} x_\alpha + q'_\alpha H^o_{qq} q_\alpha - x'_\alpha H^o_{xq} q_\alpha + q'_\alpha H^o_{qx} x_\alpha.$$

The R.H.S. of (8) is nonnegative quasi definite because H^o is convex in q and concave in x, and the crossproduct terms cancel. This can be seen immediately by premultiplying and post multiplying R.H.S. (8) by a vector $a \in R^m$. Recall that $\alpha \in R^m$.

Add $\rho x'_\alpha$ to both sides of (8) and get for $a \in R^m$

$$(9) \quad a'(x'_\alpha,\ q'_\alpha) \begin{pmatrix} H^o_{x\alpha} \\ - H^o_{q\alpha} \end{pmatrix} a = a'(q_\alpha,\ x_\alpha)'\ Q(\alpha)\ (q_\alpha,\ x_\alpha)a$$

where

$$(10) \quad Q(\alpha) \equiv \begin{bmatrix} H^o_{qq} & \rho/2\ I_n \\ \rho/2\ I_n & - H^o_{xx} \end{bmatrix}.$$

The matrix $Q(\alpha)$ plays a central role in the stability hypotheses of Magill [10], Rockafellar [23], and Brock-Scheinkman [8]. Magill, for example, shows that if $Q(\alpha)$ is positive definite at the steady state $(x(\alpha_o)\ q(\alpha_a))$ then $x(\alpha_o)$ is locally asymptotially stable. Brock and Scheinkman [8] differentiate the function $V = \dot{q}'\ \dot{x}$ with t along solutions of (1) and (2) and show that $\frac{dV}{dt} = (\dot{q},\ \dot{x})'\ Q(\alpha)\ (\dot{q},\ \dot{x})$ so that the positive definiteness of $Q(\alpha)$ implies that V acts like a Lyaponov function. Hence the positive definiteness of Q implies G.A.S..

In the spirit of the correspondence principle we have an abstract "comparative statics" result:

<u>Theorem 1</u> If $Q(\alpha)$ is positive definite at $x(\alpha_o)$, $q(\alpha_o)$ then for all $a \in R^m$

$$(11) \quad a'(x'_\alpha,\ q'_\alpha) \begin{pmatrix} H^o_{x\alpha} \\ - H^o_{q\alpha} \end{pmatrix} a \geqq 0$$

at $x(\alpha_o)$, $q(\alpha_o)$

<u>Proof</u>: Obvious from (9).

Theorem 1 will be a useful tool when we turn to a problem where α enters the Hamiltonian with a specific structure.

Turn now to another abstract result. Solve (7) for q_α in terms of x_α:

$$(12) \quad q_\alpha = \left[- H_{qq}^{o^{-1}} H_{qx}^o\right] x_\alpha - H_{qq}^{o^{-1}} H_{q\alpha}^o$$

Insert (12) into (6) to get

$$(13) \quad 0 = - (\rho - H_{xq}^o) H_{qq}^{o^{-1}} H_{qx}^o x_\alpha - (\rho - H_{xq}^o) H_{qq}^{o^{-1}} H_{q\alpha}^o - H_{xx}^o x_\alpha - H_{x\alpha}^o.$$

Equation (13) will play a centrol role in the comparative statics analysis of the Lucas-Treadway-Mortensen model of optimal accumulation of capital by a profit maximizing firm under adjustment cost which will be carried out in the sequel.

For the case where the Hamiltonian is separable in q and x i.e. $H_{xq}^o = H_{qx}^o = 0$ equations (12) and (13) give us

$$(14) \quad q_\alpha = - H_{qq}^{o^{-1}} H_{q\alpha}^o$$

$$(15) \quad 0 = - \rho H_{qq}^{o^{-1}} H_{q\alpha}^o - H_{xx}^o x_\alpha - H_{x\alpha}^o.$$

Section 3: Applications to Adjustment Cost Models

Consider the model

$$(1)\quad \int_{o}^{\infty} e^{-\rho t}\, \pi(x, \dot{x}, \alpha)\, dt \qquad \text{s.t. } x(o) = x_o$$

where

$$(2)\quad \pi(x, \dot{x}, \alpha) = f(x, \dot{x}) - \alpha_1' x - \alpha_2' \dot{x}.$$

This model was studied by Lucas [16], Mortensen [20], and Treadway [28] among others. We shall calculate the quantities H^o_{qq}, H^o_{qx}, H^o_{xx}, etc. for this model. Here, as elsewhere, subscript notation will be used for partial derivatives. The function π will be assumed to be twice continuously differentiable, concave in $(x, \dot{x})$, and all optimum paths will be assumed to be interior to all natural boundaries throughout this article. We use here the convention that if we write A^{-1}, for example, we take it for granted that it is assumed that A^{-1} exists. Now put $\alpha \equiv (\alpha_1, \alpha_2)$ and

$$(3)\quad H^o(q, x, \alpha) \equiv \underset{u \in R^n}{\text{maximum}} \left\{ f(x, w) - \alpha_1' x - \alpha_2' u + q' u \right\}$$

Let

$$(4)\quad u^o(q, x, \alpha_2)$$

denote the optimum choice of u in (3). Note that u^o does not depend upon α_1.

Since

$$(5)\quad f_u(x, u^o) = \alpha_2 - q$$

defines u^o we obtain immediately

$$(6)\quad f_{ux} + f_{uu}\, u^o_x = 0,\ u^o_x = - f^{-1}_{uu}\, f_{ux}$$

$$(7)\quad f_{uu}\, u^o_q = - I_n,\ u^o_q = - f^{-1}_{uu}$$

$$(8)\quad f_{uu}\, u^o_{\alpha_2} = I_n,\ u^o_{\alpha_2} = f^{-1}_{uu}$$

$$(9)\quad u^o_{\alpha_1} = 0$$

From the above equations the formulae below follow quickly.

(10) $H^o_x = f_x(x, u^o) - \alpha_1,\ H^o_{xx} = f_{xx} + f_{xu}\, u^o_x = f_{xx} - f_{xu}\, f^{-1}_{uu}\, f_{ux}$

(11) $H^o_{x\alpha_1} = f_{xu}\, u^o_{\alpha_1} - I_n = -I_n,\ H^o_{x\alpha_2} = f_{xu}\, u^o_{\alpha_2} = f_{xu} f^{-1}_{uu}$

(12) $H^o_{xq} = f_{xu}\, u^o_q = -f_{xu}\, f^{-1}_{uu}$

(13) $H^o_q = u^o,\ H^o_{qq} = u^o_q = -f^{-1}_{uu}$

(14) $H^o_{qx} = u^o_x = -f^{-1}_{uu}\, f_{ux}$

(15) $H^o_{q\alpha_1} = u^o_{\alpha_1} = 0,\ H^o_{q\alpha_2} = u^o_{\alpha_2} = f^{-1}_{uu}$

Let us examine some of the abstract results obtained in the previous section. We record (2.13) here for convenience and analyze it first.

(16) $0 = -(\rho - H^o_{xq})\, H^{o\,-1}_{qq}\, H^o_{qx}\, x_\alpha - (\rho - H^o_{xq})\, H^{o\,-1}_{qq}\, H^o_{q\alpha} - H^o_{xx}\, x_\alpha - H^o_{x\alpha}.$

Examine (16) for $\alpha = \alpha_1$. By (11) and (15) we get

(17) $0 = -(\rho - H^o_{xq})\, H^{o\,-1}_{qq}\, H^o_{qx}\, x_{\alpha_1} - H^o_{xx}\, x_{\alpha_1} + I_n.$

Premultiply both sides of (17) by x'_{α_1} and manipulate to get

(18) $0 = -\rho\, x'_{\alpha_1}\, H^{o\,-1}_{qq}\, H^o_{qx}\, x_{\alpha_1} + x'_{\alpha_1}\, H^o_{xq}\, H^{o\,-1}_{qq}\, H^o_{qx}\, x_{\alpha_1} - x'_{\alpha_1}\, H^o_{xx}\, x_{\alpha_1} + x'_{\alpha_1}.$

Since x_{α_1} is an nxn matrix equation (18) is an nxn matrix equation. Pre and post multiply (18) by the nxl vector a to get

(19) $0 = -\rho\, (x_{\alpha_1} a)'\, \underset{(\ )}{H^{o\,-1}_{qq}}\, H^o_{qx}\, (x_{\alpha_1} a) + (H^o_{qx} x_{\alpha_1} a)'\, \underset{(+)}{H^{o\,-1}_{qq}}\, (H^o_{qx} x_{\alpha_1} a)$

$- (x_{\alpha_1} a)'\, \underset{(+)}{H^o_{xx}}\, (x_{\alpha_1} a) + a'\, \underset{(?)}{x'_{\alpha_1}}\, a.$

Notice that we used $H^{o'}_{qx} = H^{o}_{xq}$ here. The signs in parentheses are signs of each term in equation (19).

We now arrive at

<u>Theorem 1</u> If $\rho \geqq 0$ and $H^{o^{-1}}_{qq} H^{o'}_{qx}$ is negative quasi semi definite at the steady state $x(\alpha)$ then x_{α_1} is negative quasi semi definite.

Proof: We must show that for all vectors a

(20) $a' x_{\alpha_1} a \leqq 0.$

But

$$(x_{\alpha_1} a)' H^{o^{-1}}_{qq} H^{o}_{qx} (x_{\alpha_1} a) \leqq 0$$

by hypothesis. The other two terms of (19) are nonnegative by convexity of H^o in q and concavity of H^o in x. Hence

$$a' x'_{\alpha_1} a \leqq 0.$$

But

$$a' x'_{\alpha_1} a = a' x_{\alpha_1} a$$

This ends the proof.

Theorem 1 is a typical example of a comparative statics result that may be derived from a G.A.S. hypothesis. For the assumption that

$$R \equiv H^{o^{-1}}_{qq} H^{o}_{qx}$$

is negative quasi definite is just the sufficient condition for G.A.S. reported in Brock-Scheinkman [9] and Magill [18]. We digress in order to sketch how R negative quasi definite implies L.A.S..

Brock and Scheinkman [9] show that the hypothesis that $H^{o^{-1}}_{qq} H^{o}_{qx} \equiv R$ is negative quasi definite implies the local asymptotic stability of the optimal solution (2.2) of (3.1) which we record here for convenience

$$\dot{x} = h(x, \alpha), \; x(o) = x_o$$

for $x_o = \bar{x}$. They do this by putting $V = \dot{x}' G\dot{x}$, $G \equiv H^{o\,-1}_{qq}$ and calculating

$$\dot{V} = \ddot{x}' G\dot{x} + \dot{x}' G\ddot{x} + \dot{x}' \dot{G} \dot{x}$$

$$= 2\, \dot{q}'\dot{x} + \dot{x}'[R+R']\dot{x} + \dot{x}' \dot{G} \dot{x}$$

$$< 2\, \dot{q}'\dot{x} + \dot{x}' \dot{G}\dot{x} .$$

Since $\dot{G} = 0$ at steady states and $\dot{q}'\dot{x} \leqq 0$ because

$$q(x(t)) = W_x(x(t)), \qquad \dot{q} = W_{xx}\dot{x},\ \dot{q}'\dot{x} = \dot{x}' W_{xx}\dot{x} \leqq 0$$

therefore

$$\dot{V} < 0$$

and V is a Lyaponov function.

Recall that $W(x_o) \equiv \text{maximum} \int_o^\infty e^{-\rho t} \pi(x, \dot{x}, \alpha)\, dt \quad x(o) = x_o$ is concave in x_o provided that π is concave in $(x, \dot{x})$ which we assume. Hence, $\dot{x}' W_{xx} \dot{x} \leqq 0$ if it exists. See Brock [5] and, especially Brock and Scheinkman [19] for details.

Thus we see that R plays a central role as a sufficient condition for L.A.S.. It turns out to be very powerful in analyzing a large class of adjustment cost models of Lucas-Mortensen-Treadway type. See [9] for the details. Turn back now to more discussion of x_α.

Notice that the nxn matrix x_{α_1} is not necessarily symettric but our result gives sufficient conditions for it to be negative <u>quasi</u> semi definite, i.e.

(21) $a' x_{\alpha_1} a \leqq 0$

for all $a \varepsilon R^n$. Since α_1 is the vector of wage rates for x (21) says that the long run factor demand curve $x(\alpha)$ is "downward sloping" in a generalized sense. Turn now to similar results on x_{α_2}.

Replace α by α_a in (16) above and use (11), (15) to obtain

(22) $$0 = -(\rho I_n - H^o_{xq}) H^{o\,-1}_{qq} H^o_{qx} x_{\alpha_2} - (\rho I_n - H^o_{xq}) H^{o\,-1}_{qq} f^{-1}_{uu} - H^o_{xx} x_{\alpha_2} - f_{xu} f^{-1}_{uu} .$$

Now by (13)

$$H^{o\,-1}_{qq} = - f_{uu}$$

so that

$$(23) \quad (\rho - H^o_{xq}) H^{o\,-1}_{qq} f^{-1}_{uu} = - (\rho - H^o_{xq}).$$

Also

$$(24) \quad - f_{xu} f^{-1}_{uu} = H^o_{xq}$$

by (14). Hence by (23) and (24)

$$(25) \quad - (\rho - H^o_{xq}) H^o_{qq} f^{-1}_{uu} - f_{xu} f^{-1}_{uu} = \rho I_n - H^o_{xq} + H^o_{xq} = \rho I_n .$$

Thus (22) simplifies down to

$$(26) \quad 0 = - \rho H^{o\,-1}_{qq} H^o_{qx} x_{\alpha_2} + H^o_{xq} H^{o\,-1}_{qq} H^o_{qx} x_{\alpha_2} + \rho I_n - H^o_{xx} x_{\alpha_2}.$$

Now follow the argument from equation (17) leading up to Theorem 1 to obtain

<u>Theorem 2</u>: If $\rho > 0$ and $H^{o\,-1}_{qq} H^o_{qx}$ is negative quasi semi-definite then x_{α_2} is negative quasi semi definite.

It is worthwhile and instructive to obtain Theorem 2 in a different manner. Look at the steady state equations:

$$(27) \quad 0 = \rho q - H^o_x = \rho q + \alpha_1 - f_x$$

$$(28) \quad 0 = H^o_q = u^o$$

$$(29) \quad 0 = \alpha_2 - q - f_u.$$

From (27), (28) and (29) we obtain that steady state x must satisfy

$$(30) \quad 0 = \rho\alpha_2 + \alpha_1 - \rho f_u (x, o) - f_x (x, o).$$

Put

$$\beta = \rho\alpha_2 + \alpha_1$$

and differentiate (30) totally w.r.t. β to get

(31) $0 = I_n - \rho f_{ux} x_\beta - f_{xx} x_\beta$

(32) $x_\beta = (f_{xx} + \rho f_{ux})^{-1}$.

Now obviously

(33) $x_{\alpha_1} = x_\beta, x_{\alpha_2} = \rho x_\beta$.

Hence if x_{α_1} is negative quasi semi definite and $\rho > 0$ then so also is x_{α_2} which gives another proof of theorem 2.

We close this discussion of the implications of the negative quasi definiteness of $H^{o\,-1}_{qq} H^o_{qx}$ to the problem of comparing long run equilibria in the Lucas-Treadway-Mortensen (LTM) model by noticing how equations (17), (18), (19) utilize the special structure of H^o as a function of α to "force" the discovery $H^{o\,-1}_{qq} H^o_{qx}$ as the central quantity to determine the sign of

$$a' x_{\alpha_1} a .$$

It is fascinating to note that the negative quasi definiteness of $H^{o\,-1}_{qq} H^o_{qx}$ is a very expeditious G.A.S. hypothesis for the LTM model as well as playing a central role in determining the sign of $x_{\alpha_1}, x_{\alpha_2}$. See [9] for the details. Turn now to the comparative statics implications of the separability of the Hamiltonian in (q, x).

Separability of H^o occurs when $H^o_{xq} = 0 = H^o_{qx}$ for all (q, x), α. Record (2.14), (2.15) for convenience.

(34) $q_\alpha = - H^{o\,-1}_{qq} H^o_{q\alpha}$

(35) $0 = - \rho H^{o\,-1}_{qq} H^o_{q\alpha} - H^o_{xx} x_\alpha - H^o_{x\alpha}$.

Replace α by α_1 in (35), use (11) and (15) to obtain

(36) $0 = - H^o_{xx} x_{\alpha_1} + I_n$.

Hence,

<u>Theorem 3</u>: If $H^o_{xq} = 0$ for all x, q, α then if $\rho > 0$

x_{α_1}, x_β, x_{α_2}

are all negative quasi semi definite.

Proof: The matrix x_{α_1} is negative quasi semi definite from (36). Equation (33) gives the rest of the theorem. This ends the proof.

The separability of the Hamiltonian is Scheinkman's [26] G.A.S. hypothesis. Again we see the intimate connection between a G.A.S. hypothesis and strong comparative statics results.

Let us use the abstract results (2.8) and (2.9) to get some more comparative statics for the L.M.T. model. We record (2.8) and (2.9) for convenience.

$$(37)\quad (x_\alpha', q_\alpha') \begin{pmatrix} H^o_{x\alpha} - \rho q_\alpha \\ - H^o_{q\alpha} \end{pmatrix} = - x_\alpha' H^o_{xx} x_\alpha + q_\alpha' H^o_{qq} q_\alpha - x_\alpha' H^o_{xq} q_\alpha + q_\alpha' H^o_{qx} x_\alpha$$

$$(38)\quad x_\alpha' H^o_{x\alpha} - q_\alpha' H^o_{q\alpha} = \rho x_\alpha' q_\alpha - x_\alpha' H^o_{xx} x_\alpha + q_\alpha' H^o_{qq} q_\alpha - x_\alpha' H^o_{xq} q_\alpha + q_\alpha' H^o_{qx} x_\alpha .$$

Notice that (38) is the same as (2.9) and (38) trivially follows from (37).

Replace α by α_1 in (37), use $H^o_{x\alpha_1} = - I_n$, $H^o_{q\alpha_1} = 0$ from (11), (15) for the L.M.T. model to obtain

$$(39)\quad - x_{\alpha_1}' - \rho x_{\alpha_1}' q_{\alpha_1} = - x_{\alpha_1}' H^o_{xx} x_{\alpha_1} + q_{\alpha_1}' H^o_{qq} q_{\alpha_1} - x_{\alpha_1}' H^o_{xq} q_{\alpha_1} + q_{\alpha_1}' H^o_{qx} x_{\alpha_1} .$$

Pre and post multiply both sides of (39) by $a \varepsilon R^n$ to obtain

$$(40)\quad - a' x_{\alpha_1}' a - \rho a' x_{\alpha_1}' q_{\alpha_1} a = - a' x_{\alpha_1}' H^o_{xx} x_{\alpha_1} a + a' q_{\alpha_1}' H^o_{qq} q_{\alpha_1} a .$$

Notice that the cross product terms cancel to give R.H.S. (40).

From (40)

$$(41)\quad - a' x_{\alpha_1}' a = (q_{\alpha_1} a, x_{\alpha_1} a)' Q(\alpha) (q_{\alpha_1} a, x_{\alpha_1} a)$$

where

$$Q(\alpha) \equiv \begin{bmatrix} H^o_{qq} & \rho/2\ I_n \\ \rho/2\ I_n, & -\ H^o_{xx} \end{bmatrix}.$$

We can now prove

Theorem 4: (i) The matrix

$$x_{\alpha_1} + \rho\ x'_{\alpha_1}\ q_{\alpha_1}$$

is negative quasi semi definite. (ii) If $Q(\alpha)$ is positive semi definite then

$$x_{\alpha_1},\ x_\beta,\ x_{\alpha_2}$$

are all negative quasi semi definite.

Proof: Part (i) follows immediately from (40) because H^o is convex in q and concave in x. To obtain part (ii) first note from (41) that $Q(\alpha)$ positive semi definite implies directly that x'_{α_1} and, hence, x_{α_1} is negative quasi semi definite. That x_β and x_{α_a} are negative quasi semi definite follows directly from equation (33). This ends the proof.

Once again we see that a G.A.S. hypothesis, viz $Q(\alpha)$ positive semi definite, yields strong comparative statics results.

Remark When $\pi = pf(x, \dot{x}) - \alpha_1'x - \alpha_2'\dot{x}$ where p is product price we may obtain comparative statics results on p by noticing that the choice of optimum path is homogeneous of degree 0 in (p, α_1, α_2). Put $\bar{\alpha}_1 = p^{-1}\alpha_1$, $\bar{\alpha}_2 = p^{-1}\alpha_2$. Then use the results obtained above to obtain qualitative results for $\partial x/\partial p$.

We turn now to the impact on steady state x when the discount ρ is increased.

Section 4: Relationships Between G.A.S. Hypotheses and Generalized Capital Deepening.

Consider the steady state equations

$$(1)\quad 0 = \rho q - H_x^o(q, x)$$

$$0 = H_q^o(q, x)$$

Burmeister-Turnovsky [ ] introduce the measure of capital deepening

$$B(\rho) \equiv q' x_\rho$$

where x_ρ is the derivative of the steady state with respect to ρ. An economy is called "regular" at ρ_o if

$$B(\rho_o) \leqq 0.$$

The motivation for introducing $B(\rho)$ is discussed in detail in [12]. It is a measure of sorts for the impact on the steady state capital stock constellation when the interest rate ρ changes in a growth model.

It turns out that the following Theorem may be proved.

Theorem: All of the following G.A.S. hypotheses at $q(\rho_o)$, $x(\rho_o)$ are sufficient for $B(\rho) \leqq 0$ at $\rho = \rho_o$: (i) Q is positive semi definite, (ii) $H_{qq}^{o\,-1} H_{qx}^o$ is negative quasi definite, or (iii) $H_{xq}^o = 0$.

Proof: We demonstrate (i) first. Differentiate (1) totally w.r.t. ρ to obtain

$$(2)\quad 0 = \rho q_\rho + q - H_{xq}^o q_\rho - H_{xx}^o x_\rho$$

$$(3)\quad 0 = H_{qq}^o q_\rho + H_{qx}^o x_\rho.$$

Premultiply (2) by x_ρ' to get

$$(4)\quad 0 = \rho x_\rho' q_\rho + x_\rho' q - x_\rho' H_{xq}^o q_\rho - x_\rho' H_{xx}^o x_\rho.$$

Premultiply (3) by q_ρ' to get

$$(5)\quad 0 = q_\rho' H_{qq}^o q_\rho + q_\rho' H_{qx}^o x_\rho.$$

(6) $0 = \rho x_\rho' q_\rho + x_\rho' q - x_\rho' H_{xx}^o x_\rho + q_\rho' H_{qq}^o q_\rho = (q_\rho, x_\rho)' Q(q_\rho, x_\rho) + x_\rho' q_\rho.$

It follows immediately from (6) that (i) implies

$$x_\rho' q_\rho = q_\rho' x_\rho \leqq 0.$$

In order to derive the second result solve (3) for q_ρ in terms of x_ρ and insert the solution into (2) to get

(7) $q_\rho = - H_{qq}^{o\,-1} H_{qx}^o x_\rho$

(8) $0 = - \rho H_{qq}^{o\,-1} H_{qx}^o x + q + H_{xq}^o H_{qq}^{o\,-1} H_{qx}^o x_\rho - H_{xx}^o x_\rho$

Premultiply both sides of (8) by x_ρ' to obtain

(9) $0 = - \rho x_\rho' H_{qq}^{o\,-1} H_{qx}^o x_\rho + x_\rho' q + x_\rho' H_{xq}^o H_{qq}^{o\,-1} H_{qx}^o x_\rho - x_\rho' H_{xx}^o x_\rho.$

Now

$$x_\rho' H_{xq}^o H_{qq}^{o\,-1} H_{qx}^o x_\rho = (H_{qx}^o x_\rho)' H_{qq}^{o\,-1} H_{qx}^o x_\rho \geqq 0$$

since H^o is convex in q. Also

$$- x_\rho' H_{xx}^o x_\rho \geqq 0$$

because H^o is concave in x. Hence from (9) either (ii) or (iii) is sufficient for

$$q' x_\rho \leqq 0.$$

This ends the proof.

Remark: Parts (i) and (ii) of this theorem are due to E. Burmeister [7] and M. Magill [18] respectively.

A fascinating discussion of the implications of the L.A.S. hypothesis in the problem of comparing equilibria when ρ changes and the relation of their problem to the Cambridge Controversy in Capital Theory as well as the "Hahn Problem" is contained in Burmeister and Long [29].

This concludes the presentation of results that we have obtained on the implications of sufficient conditions on $\pi(x, \dot{x}, \alpha)$ and ρ for the G.A.S. of optimal paths for qualitative results on steady states. Notice that all of the results are of the following character: A G.A.S. hypothesis and a hypothesis about how α enters H^o is placed upon the Hamiltonian $H^o(q, x, \alpha)$ of the system to obtain results on comparing steady states. This suggests that a general theory of comparing steady states is waiting to be discovered. This is so because "Hamiltonian like" constructs are very general. For example, such a construct can be invented for dynamic games where some arguments enter in a passive way and are determined by equilibrium forces in much the same way as competitive prices are determined and other arguments enter in an active way and are determined in much the same way as Cournot oligopoly models determine output levels. See Brock [3] for a development in general "Hamiltonian" terms of this class of dynamic industrial organization games which are due to Edward Prescott and his students at Carnegie.

Furthermore, the analysis of Cass and Shell [13] neatly develop, with a emphasis, the Hamiltonian formalism of modern growth theory for both descriptive and optimal growth models. A "Hamiltonian like" formalism underlies virtually any dynamic model that has a recursive structure. Hence results such as those obtained in this article which depend upon hypotheses placed upon the Hamiltonian alone should extend to more general models.

Turn now to the development of results on comparing equilibria that depend only on the L.A.S. hypothesis of the steady state x.

Section 5: Example of Use of The Natural Structure of A Dynamic Model and the L.A.S. Hypothesis to Get Strong Comparative Statics Results

This section presents a simplified proof of a theorem of Mortensen. Return to the adjustment cost model of Lucas-Treadway and Mortensen

(1) maximize $\int_o^{\infty} (Pf(x, \dot{x}) - \alpha_1' x - \alpha_2' u) e^{-\rho t}$.

s.t. $x(o) = x_o$, $\dot{x} = u$

Assume P, α_1, α_2 are time independent. Write the necessary conditions for optimality in Euler equation form,

(2) $\frac{d}{dt} (f_u - \alpha_2) = f_x - \alpha_1 + \rho (f_u - \alpha_2)$.

Let $\overline{x}$ be a steady state and let $x_o = \overline{x} + \Delta x_o$. Under very general conditions (See Magill [18]) equation (2) may be approximated thusly for small Δx_o.

(3) $\frac{d}{dt} (f_{uu} \Delta u + f_{ux} \Delta x) = f_{xu} \Delta u + f_{xx} \Delta x + \rho f_{ux} \Delta x + \rho f_{uu} \Delta u$

(4) $\Delta\dot{x} = \Delta u$, $\Delta x(o) = \Delta x_o$.

Here (3) was obtained from (2) by expansion in a Taylor series at $(\overline{x}, o)$ and using the equations of steady state to cancel off the first order terms. All second derivatives are evaluated at the steady state $(x, u) = (\overline{x}, o)$.

Remark: It is an open question whether such a quadratic approximation is valid for (2) when $\overline{x}$ is not L.A.S.

Now if f is strictly concave in $(x, \dot{x})$ we saw before that there is a policy function $h(x, P, \alpha)$ such that optimum trajectories satisfy

(5) $\dot{x} = h(x, P, \alpha)$, $x(o) = x_o$

Linearize h around the steady state $\overline{x}$ to get

(6) $\Delta\dot{x} = h_x(\overline{x}; P, \alpha) \Delta x$, $\Delta x(o) = \Delta x_o$.

Equations (3), (4), and (6) describe the same trajectory therefore the matrix $M \equiv h_x(\overline{x};\ p,\ \alpha)$ must solve a quadratic matrix equation that is generated by (3). I.e.

(7) $BM^2 + (C-C' - \rho B)M = A + \rho C$

where

(8) $B = f_{uu},\ C = f_{ux},\ C' = f_{xu},\ A = f_{xx}.$

Equation (7) is obtained by plugging

(9) $\Delta\dot{x} = M\ \Delta x,\ \Delta\ddot{x} = M\Delta\dot{x} = M^2\Delta x$

into (3) and equating coefficients.

Equation (7) is difficult to solve for the optimal adjustment matrix M except in the one dimensional case. But, nonetheless, we can say a good deal about M in terms of A, B, C, ρ.

For instance, we know that for $f(x,\ \dot{x})$ strictly concave there is just one steady state $\overline{x}$ and it is G.A.S. for the case $\rho = 0$ where the maximum is interpreted in the sense of the overtaking ordering (cf. Brock and Haurie [6] and references). Scheinkman's [25] result tells us that, except for hairline cases, if $\rho > 0$, and ρ is small enough then G.A.S. will hold. Hence, we know, except for hairline cases that M is a stable matrix when ρ is small.

We may also employ the G.A.S. hypotheses listed above to find conditions on A, B, C, ρ that guarantee that M is a stable matrix. See the Brock "Survey" paper [5] where this exercise is carried out in detail.

In this section we are interested in the following: What restrictions does the stability of M imply on the comparison of steady states? This question is more in the spirit of the original Samuelson Correspondence Principle which asserted that stability of M would generate comparative statics results. This brings us to Mortensen's Theorem.

<u>Theorem</u> (Mortensen [20]). Assume B is negative definite. If M is a stable matrix then $\overline{x}_{\beta}$, $\frac{\partial h}{\partial \beta}(\overline{x};\, p,\, \alpha_1,\, \alpha_2)$, $\beta \equiv \alpha_1 + \rho\, \alpha_2$ are both symmetric and negative definite iff C is symmetric. Moreover, the characteristic roots of M are all real if C is symmetric.

Proof: We will give a proof here because some parts of our proof are different then Mortensen. First, by direct use of the steady state equations

(10) $\overline{x}_{\beta} = (A + \rho C)^{-1}$, $\frac{\partial h}{\partial \beta}(\overline{x};\, p,\, \alpha_1,\, \alpha_2) = -\, M(A + \rho C)^{-1}$

Second, the following lemma is needed

<u>Lemma</u>: At any steady state, if the quadratic approximation (3) is valid then

(11a) $M'B - BM = C - C'$

(11b) $A + \rho C = (M' - \rho I_n)\, BM.$

Proof of the Lemma: Equation (11b) follows directly from (11a) and (7). So we need establish (11a) only. Write the necessary conditions of optimality in Hamiltonian form:

(12) $\dot{q} = \rho q - H_x^o\, (q,\, x)$

(13) $\dot{x} = H_q^o\, (q,\, x)$, $x(o) = x_o$

(14) $\lim_{t \to \infty} e^{-\rho t}\, q(t)'x(t) = 0$

In order to see that (14) is necessary for optimality and is the correct transversality condition in general see Benveniste and Scheinkman [2].

Following Magill [18] look at the necessary conditions for optimality of the linear quadratic approximation written in Hamiltonian form.

(15) $\Delta\dot{q} = \rho\Delta q - H_{xq}^o\, \Delta q - H_{xx}^o\, \Delta x$

(16) $\Delta\dot{x} = H_{qx}^o\, \Delta x + H_{qq}^o\, \Delta q$

(17) $\lim_{t \to \infty} e^{-\rho t}\, \Delta q'(t)\, \Delta x(t) = 0$

Now

(18) $\Delta q = W \Delta x$

for some matrix W. The intuition behind this is compelling. For if $R(x_o) \equiv \text{maximum} \int_0^{\infty} e^{-\rho t} \pi(x, u, \alpha)\, dt$, s.t. $x(o) = x_o$, $\dot{x} = u$ and if R is twice continuously differentiable at $x_o = \overline{x}$ then (18) holds where $W = \frac{\partial^2 R}{\partial x^2} (\overline{x})$. This is so because by definition of q

$$q = \frac{\partial R}{\partial x} ;$$

Hence

$$\Delta q = \frac{\partial^2 R}{\partial x^2} \Delta x = W \Delta x$$

for the linear approximation.

Turn back to (16),

(18) $\Delta \dot{x} = H^o_{qx} \Delta x + H^o_{qq} \Delta q = H^o_{qx} \Delta x + H^o_{qq} W \Delta x = M \Delta x.$

Hence

(19) $M = H^o_{qx} + H^o_{qq} W.$

Now calculate

$M'B - BM$

from (19).

We obtain

(20) $M'B - BM = (H^o_{xq} + W H^o_{qq}) B - B (H^o_{qx} + H^o_{qq} W)$

But

$$B \equiv f_{uu},\ H^o_{qq} = - f^{-1}_{uu},\ H^o_{xq} = - f_{xu} f^{-1}_{uu},\ H^o_{qx} = - f^{-1}_{uu} f_{ux}$$

from (3.13), (3.12), and (3.14).

Thus

(21) $M'B - BM = H^{o}_{xq} f_{uu} - f_{uu} H^{o}_{qx} = f_{xu} - f_{ux}$

$= C - C'$.

This ends the proof of the lemma.

<u>Remark</u>: Notice that the proof of the lemma did not assume $\overline{x}$ was L.A.S. or G.A.S.. In fact, the result (21) holds for all problems where $f(x, \dot{x})$ is quadratic and concave in $(x, \dot{x})$. Mortensen does not need to assume that $\overline{x}$ is L.A.S. to get (21).

Let us continue with the proof of the theorem.

Examine the following equalities which follow directly from (11b)

(22a) $A + \rho C = M'BM - \rho BM$

(22b) $-(A + \rho C)M^{-1} = \rho B - M'B$.

Notice that by (11a) both $A + \rho C$ and $(A + \rho C)M^{-1}$ are symmetric provided that $C = C'$. Hence by (10) it follows that $\overline{x}_\beta$ and $\frac{\partial h}{\partial \beta}$ are both symmetric since the increase of a symmetric matrix is symmetric.

We shall show now that if $C = C'$ and $\overline{x}$ is L.A.S. then $\overline{x}_\beta$ and $\frac{\partial h}{\partial \beta}$ are negative definite. Since B is negative definite and the determinant of M is non zero (M has all eigenvalues with real parts negative) therefore the L.H.S. of (22 a, b) will be negative definite provided that $N \equiv -BM$ is negative definite. Now $-B$ being positive definite possesses a square root. Therefore, there is a non-singular matrix T such that

(23) $-B = T'T$

Therefore

(24) $(T^{-1})' NT^{-1} = (T^{-1})' (-BM)T^{-1} = (T^{-1})' (T'TM)T^{-1}$

Now BM is symmetric because $C = C'$. Hence its eigenvalues are all real. Thus the eigenvalues of M must be all real since (24) implies the eigenvalues of M are the same as those of $N \equiv -BM$. Since M is a stable matrix all the roots must be negative. Hence N is a negative definite symmetric matrix. Look at the L.H.S. of (22a, 22b). The R.H.S. is negative definite because N, B are. Therefore the L.H.S. is negative definite also. Since the inverse of a negative definite matrix is negative definite, therefore $\bar{x}_\beta$, $\frac{\partial h}{\partial \beta}$, must be negative definite. This ends the proof of Mortensen's theorem.

Remark: The only part of Mortensen's theorem that needs the L.A.S. of $\bar{x}$ is the negative definiteness of $\bar{x}_\beta$ and $\frac{\partial h}{\partial \beta}$. The symmetry of $\bar{x}_\beta$ and $\frac{\partial h}{\partial \beta}$ as well as the characteristic roots of M being real require only the symmetry of C alone.

Mortensen's theorem is an excellent example of how the indigenous structure of the adjustment cost model interplays with the L.A.S. hypothesis to produce strong qualitative results.

Corollary (i) $H^{o^{-1}}_{qq} H^{o}_{qx} = f_{ux} \equiv C'$

(ii) If C is negative quasi definite then $\bar{x}$ is L.A.S. and both $\bar{x}_\beta$, $\frac{\partial h}{\partial \beta}(\bar{x}, P, \alpha_1, \alpha_2)$ are negative quasi definite.

Proof: The first formula follows directly from

(25) $H^{o}_{qq} = - f^{-1}_{uu}, \; H^{o}_{qx} = - f^{-1}_{uu} f_{ux},$

and $C \equiv f_{xu}$. Here equations (25) are recorded from (3.13), (3.14) for convenience.

Now C is negative quasi definite iff C' is. Hence the negative quasi definiteness of C is simply the sufficient condition for G.A.S. reported in Brock-Scheinkman [18] and Magill [18]. Turn now to the task of showing that $\bar{x}_\beta$ and $\frac{\partial h}{\partial \beta}$ are negative quasi definite. Look at 22a, b. Since for any $a \varepsilon R^n$

$$a'(M'B - BM)a = a'(C - C')a = 0,$$

and $\bar{x}_\beta$, $\frac{\partial h}{\partial \beta}$ are negative quasi definite iff $A + \rho C$, $-(A + \rho C)M^{-1}$ are negative quasi definite therefore by (22a) and (22b) we need only show that

$$BM$$

is positive quasi definite in order to finish the proof. Recall that

(26) $\bar{x}_\beta = (A + \rho C)^{-1}, \; \frac{\partial h}{\partial \beta} = - \left[(A + \rho C)M^{-1}\right]^{-1}.$

In order to see that BM is positive quasi definite calculate thus

(27) $$BM = f_{uu} (H^{o}_{qq} W + H^{o}_{qx}) = - W - H^{o\,-1}_{qq} H^{o}_{qx} .$$

The first equality follows from (25) and the second follow because

$$f_{uu} = - H^{o\,-1}_{qq} .$$

Now W is negative semidefinite by concavity of the state valuation function of the associated linear quadratic approximation of the original problem around the steady state $\overline{x}$ (Magill [18]). Hence the R.H.S. of (27) is positive quasi definite. This ends the proof.

Remark: Some additional structure is needed on the problem above and beyond the stability of M in order to get the negative quasi semi definiteness of $\overline{x}_{\beta}$. This is so because Mortensen [20, p. 663] provides a two dimensional example when M is stable, C is not symmetric, and

$$\frac{\partial \overline{x}_2}{\partial \beta_2} > 0$$

Section 6: Summary

This article has shown how a Samuelson type of "Correspondence Principle" can be developed for a class of optimal control problems that arise in modern capital theory. More particularly we have shown how three sufficient conditions for the global asymptotic stability of optimal steady state that have been obtained recently by researchers imply strong comparative statics results on the optimal steady state when they are combined with specific structural hypotheses on the Hamiltonian of the optimal control problem.

Furthermore, we have used a result of Mortensen to point out how the L.A.S. hypothesis alone on the optimal steady state implies strong comparative statics results when combined with certain structural assumptions on the Hamiltonian.

The purpose of this article has been to show that there is hope that a version of Samuelson's Correspondence Principle can be developed for the recursive equilibrium systems typical of the modern "equilibrium-disequilibrium dynamics" that is immune to the criticism leveled against the Correspondence Principle in the case of Arrow-Debreu-McKenzie general equilibrium theory.

More work needs to be done exploring the implications of different G.A.S. hypotheses for comparative statics in the context of the optimal control model studied in this paper as well as more general recursive equilibrium systems before we can have much confidence that the ideas presented in this paper will be of much use for dynamic economics. In particular <u>some</u> structure will have to be replaced upon $\pi(x, \dot{x}, \alpha)$ above and beyond concavity in $(x, \dot{x})$. This is so because examples were created in [5] where given arbitrary dynamics $\dot{x} = F(x)$, a $\pi(x, \dot{x}, \alpha)$ could be found that generated $\dot{x} = F(x)$ as the optimum dynamics. The examples did not make much sense from an economic point of view but they showed that concavity of π in $(x, \dot{x})$ is not enough to get useful restrictions on the optimal dynamics.

Footnotes

1. A function $x(\cdot)$ is absolutely continuous if $\frac{dx}{dt}$ exists almost everywhere.

2. The reader is reminded, however, that Sonnenschein's theorem did not exist at the time that Quirk et al were working.

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