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Linear Functionals of Convex Sets with Applications to Economics, Game-Theory and Social Choice*

Ehud Kalai
and
Roger B. Myerson
Northwestern University

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1. Introduction

Let $\mathcal{C}$ denote the set of all compact convex subsets of the n-dimensional Euclidean space, R^n. $\mathcal{C}$ is a convex cone ($A,B \in \mathcal{C}$ imply $A+B = \{a+b : a \in A, b \in B\} \in \mathcal{C}$ and $rA = \{ra : a \in A\} \in \mathcal{C}$ for every $r \geq 0$). The main purpose of this paper is to characterize all the uniformly continuous (relative to the Hausdorff metric on $\mathcal{C}$) real valued linear functions ($F(A+B) = F(A) + F(B)$ and $F(rA) = rF(A)$ for $r \geq 0$) on $\mathcal{C}$. In section 2 we give examples to show that these functions are common in Economics and Game-Theory. Section 3 is devoted to the proof of an extended version of Theorem 1 below. Section 4 exhibits a method of recovering the measures associated with our characterization and presents an open problem.

For $C \in \mathcal{C}$ and $p \in R^n$ let $V_C(p)$ denote the support of C in the direction p, i.e. $V_C(p) = \sup_{c \in C} c \cdot p$. Let S denote the boundary of the unit ball in R^n.

Theorem 1: F is a uniformly continuous, real valued linear function on $\mathcal{C}$ if and only if there exist non-negative real numbers a and b and probability measures μ and η on S such that for every $C \in \mathcal{C}$

$$F(C) = aE_\mu V_C - bE_\eta V_C$$

where E_μ denotes the expected value relative to the measure μ.

2. Examples

References to show the frequency of occurrence of linear functionals in Economics are too numerous to list (e.g. Debreu [1959] and Gale [1960]). We give the following examples in order to show four specific cases where the function F of Section 1 is natural.

Example 1: Decomposition of Production

We think of each of the coordinates of a point in R^n as representing output level of a certain good. Thus a point $x \in R^n$ represents output x^1 of good 1, x^2 of good 2 and so on. A set $C \in \mathcal{C}$ represents the feasible output levels for a producer. Given such a set C, let F(C) be the profit (or cost) associated with the producer's choice in C. If a set A represents the feasible production levels in one location and B representes the feasible production levels in another location then $A + B$ represents the feasible production levels from the two locations. $F(A+B) = F(A) + F(B)$ means that as far as profit is concerned the problem is decomposable and the decision can be made jointly or separately. Similar considerations will justify the requirement that $F(kC) = kF(C)$ for every positive integer k, $F(rC) = rF(C)$ for every positive rational number r, and continuity argument will result in $F(tC) = tF(C)$ for every non-negative real number t. Of course the quantity F(C) could represent many other concepts for which decomposition is desired.

Notice that $E_\mu V_C$ is the expected worth of the feasible production set C when prices have the distribution μ. Thus, by Theorem 1, it follows that every uniformly continuous function which is decomposable can be viewed as the difference of the expected worth of the set C according to two different fixed probability distributions on potential prices.

Example 2: <u>The Utility of Participating in a Cooperative Game</u>

Linear, real valued functions arise in cooperative game theory when we consider the von-Neumann Morgenstern utility that an individual has for the options of participating in various cooperative games. For simplicity we illustrate this point on a class of very simple games of this type (see Aumann-Peleg [1960] for the general case and Nash [1950] for the special case that we consider here). In addition linear functions of convex sets arise naturally when we consider possible extensions of the Shapley Value (Shapley [1953]) to the family of cooperative games without sidepayments (Kalai-Myerson [1977]).

We consider two fixed individuals with fixed von-Neumann Morgenstern utility functions. A two-person game of these two individuals is represented by a pair $(a,C) \in R^2 \times \mathcal{C}$. $a = (a^1,a^2)$ represents the V-M utility levels resulting to the two of them when they do not cooperate. C represents the utility levels available to them when they do cooperate. Thus $c = (c^1,c^2) \in C$ if and only if there is a joint strategy yielding utility levels c^1 and c^2 to the two players respectively. We are interested in the V-M utility that player 1 may have for participating in the various games that he may encounter with 2. (See Roth [1976] for a treatment of the sidepayments case as a generalization of the Shapley Value.) Consider three games, (a,A) (b,B) and the game resulting from the following lottery. With probability α they will play (a,A) and with probability $1-\alpha$ (b,B). If we consider the ex-anté feasible V-M utility level associated with the third game (c,C) we obtain

$$(c,C) = (\alpha a + (1-\alpha)b, \alpha A + (1-\alpha)B).$$

Thus it follows from the V-M theory that player 1's utility for the games must satisfy

$$u(\alpha a + (1-\alpha)b, \alpha A + (1-\alpha)B) = \alpha u(a,A) + (1-\alpha)u(b,B).$$

By considering all the games with a fixed non-cooperation point, or by choosing a non-cooperation point that varies linearly with the choice of the feasible set we obtain that u must induce a linear, real valued function on $\mathcal{C}$.

Example 3: <u>Decentralization of Social Choice Decisions</u>

We let a set C denote the feasible production levels of n-public goods. F(C) represents society's utility from the choice made out of C. The linearity of F means that if C is decomposed into sets that add up to C then making the choice on every one of the component sets will not effect society's utility of the final outcome.

Example 4: Utilitarian Social Welfare Functions

We consider a society of n individuals each having a von-Neumann Morgenstern utility function $u_i (1 \leq i \leq n)$. A set $C \subset R^n$ represents (as in Example 2) feasible utility levels of the n individuals in a certain situation. Thus if society's options in a certain situation are given by a set A then $C = \{(u_1(a), u_2(a), \ldots, u_n(a)) : a \in A\}$. C must be convex if we assume that the society has the option to randomize over alternatives in A. We let F(C) denote the V-M utility of the society when it makes the choice from C. The argument given in Example 2 holds also in this case and implies that F must be linear on C.

Notice that if society's utility of a choice set C is given by the function $V_C(p)$ $(= \sup_{c \in C} c \cdot p)$ for some $p \in R^n$ then society has a utilitarian social welfare function with the interpersonal weights given by $p = (p_1, p_2, \ldots, p_n)$. (See Harsanyi [1955] for an aximatization of this Social Welfare Function.) If society has a utility function of the form $u(C) = E_\mu V_C$ where μ is a probability distribution on the boundary of the unit ball in R^n then it is basically a utilitarian society which has a probability distribution μ over the interpersonal weights. Thus Theorem 1 tells us that every uniformly continuous V-M utility function for the society can be viewed as the difference of two utilitarian functions with two different probability distributions over the interpersonal weights.

3. Main Results

Let $\mathcal{K}$ denote a convex cone of closed convex sets in R^n with a common recession cone $T \neq R^n$. Thus we assume that $\mathcal{K}$ satisfies the following conditions.

1. Every element of $\mathcal{K}$ is closed and convex.
2. $A, B \in \mathcal{K}$ imply that $A+B \in \mathcal{K}$ (+ denotes set addition)
3. $A \in \mathcal{K}$ and $t > 0$ imply that $tA \in K$.
4. There exists a closed convex cone in R^n, $T(\neq R^n)$, such that for every $A \in \mathcal{K}$ the recession cone of A is T.
   Recall that the recession cone is the set of directions to which A is unbounded, i.e.
   $T = \{v \in R^n : A + \{v\} \subset A\}$ (see Rockafellar [1970]).

Notice that when $T = \{0\}$ then $\mathcal{K} = \mathcal{C}$.

The cases where $T = -R^n_+$ are very common in Game-Theory and Economics (free disposal). We let W denote the polar cone of T $(= \{w \in R^n : w \cdot r \leq 0$ for every $r \in T\})$ and for every $A \in \mathcal{K}$ we let V_A denote the support function of A then the effective domain of V_A is W. V_A is continuous, convex, and homogeneous of degree 1 on W. And there is a 1-1 correspondence between convex sets with T as a recession cone and functions on W which satisfy these conditions. We let $\overline{W}$ denote the intersection of W with the boundary of the unit ball in R^n, $\overline{W} = \{v \in W : \|v\| = 1\}$ where $\| \;\|$ denotes the Euclidean norm.

For $A, B \in \mathcal{K}$ the Hausdorff distance between A and B is defined as usual to be the supremum of radei of Euclidean balls which are centered around a point of one of the two sets A, B, without intersecting the other set. We discuss continuity of functions on $\mathcal{K}$ relative to this Hausdorff metric.

A real valued function F on $\mathcal{K}$ is linear if for every $A, B \in \mathcal{K}$ and every $r > 0$, $F(A+B) = F(A) + F(B)$ and $F(rA) = rF(A)$.

<u>Theorem 2</u>: F is a real valued uniformly continuous linear function on $\mathcal{K}$ if and only if there is some signed regular Borel measure M on $\overline{W}$ such that for every $C \in \mathcal{K}$

$$F(C) = \int_{\overline{W}} V_C(p)\,dM(p).$$

Let X be a normed linear space over the real numbers, let Y denote a convex cone in X, and let f be a real valued linear function on Y. We define $\| f\|_Y = \sup \{\frac{|f(y)-f(z)|}{\|y-z\|} : y, z \in Y\}$ and we say that f is bounded on Y if $\|f\|_Y < \infty$.

<u>Lemma 1</u>: A real valued bounded linear function f defined on a convex cone Y can be extended to a bounded linear function $\overline{f}$ on the entire space X with $\|\overline{f}\| = \|f\|_Y$.

<u>Proof of Lemma 1</u>: We let $Z = Y - Y$ then Z is a subspace of X. We extend f to $\overline{\overline{f}}$ defined on Z by

$$\overline{\overline{f}}(y_2 - y_z) = f(y_2) - f(y_z).$$

By the linearity of f it follows that $\overline{\overline{f}}$ is well defined, linear, and $\|\overline{\overline{f}}\|_Z = \|f\|_Y$. The Hahn-Banach Theorem completes the proof of the lemma.

Proof of Theorem 2: Let $\overline{W}^*$ denote the set of all real valued functions defined on $\overline{W}$ which are the restrictions to $\overline{W}$ of some support function V_A with effective domain W. Every element in $\overline{W}^*$ can be associated uniquely with some V_A for some unique convex set $A \in \mathcal{K}$. We let $\overline{V}_A$ denote the restriction of V_A to $\overline{W}$. Thus we can defnine F^* on $\overline{W}$ by $F^*(\overline{V}_A) = F(A)$. So F^* is a well defined, linear, real valued function on $\overline{W}^*$. Also F^* is uniformly continuous (with the sup norm) because F is (see Artstein [1970] or Kalai [1975]). To show that F^* is bounded on $\overline{W}^*$ we assume to the contrary that there exists a sequence of functions $\overline{V}_{A(i)}$ and $\overline{V}_{B(i)}$ on $\overline{W}^*$ such that

$$R(A(i),B(i)) = \frac{|F^*(\overline{V}_{A(i)})-F^*(\overline{V}_{B(i)})|}{\| \overline{V}_{A(i)}-\overline{V}_{B(i)}\|} \to \infty \text{ as } i \to \infty.$$

By the uniform continuity of F^* there exists a $\delta > 0$ such that if $\| \overline{V}_{A(i)}-\overline{V}_{B(i)}\| \leq \delta$ then $\| F^*(\overline{V}_{A(i)})-F^*(\overline{V}_{B(i)})\| \leq 1$. If we consider the sequence $(A^1_{(i)},B^1_{(i)}) \leq \frac{1}{\delta}$ for all i but also $R(A^1_{(i)},B^1_{(i)}) \to \infty$ as $i \to \infty$, a contradiction. Thus F^* is a real valued, linear, bounded function on $\overline{W}^*$. Since $\overline{W}^*$ is a convex cone in the space $C_o(\overline{W})$ consisting of all continuous functions on $\overline{W}$ we can extend, by Lemma 1, F^* to be a linear bounded real valued function on $C_o(\overline{W})$. Now Riesz Representation Theorem completes the proof of the theorem.

Corollary: Theorem 1 follows immediately from Theorem 2 by the Hahn Decomposition Theorem for signed measures. Notice also that if M is determined uniquely for a given F then Hahn's theorem implies that a,b,μ and η of Theorem 1 are determined uniquely after the obvious normalization.

4. Uniqueness, Recovering the Measure, and an Open Question

We give an example in R^2 to show how to recover the measure M underlying a given uniformly continuous linear F defined on $\mathcal{C}$. We believe that this method can be applied to the general case. In particular this would show that the measure M described by Theorem 2 is unique.

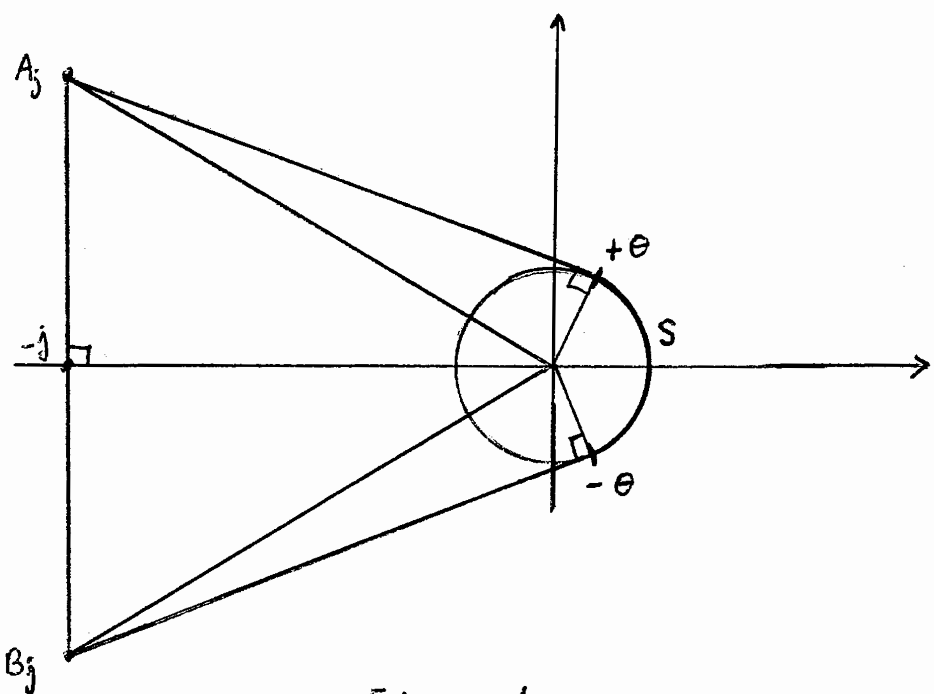

Figure 1

We let S be the interval described in Figure 1. S_j is the convex-hull of S and the points A_j and B_j. T_j is the convex-hull of A_j, B_j and (0,0). We claim that $M(S) = \lim_{j\to\infty} [F(S_j)-F(T_j)]$. To justify this claim,consider the support functions of S_j and T_j

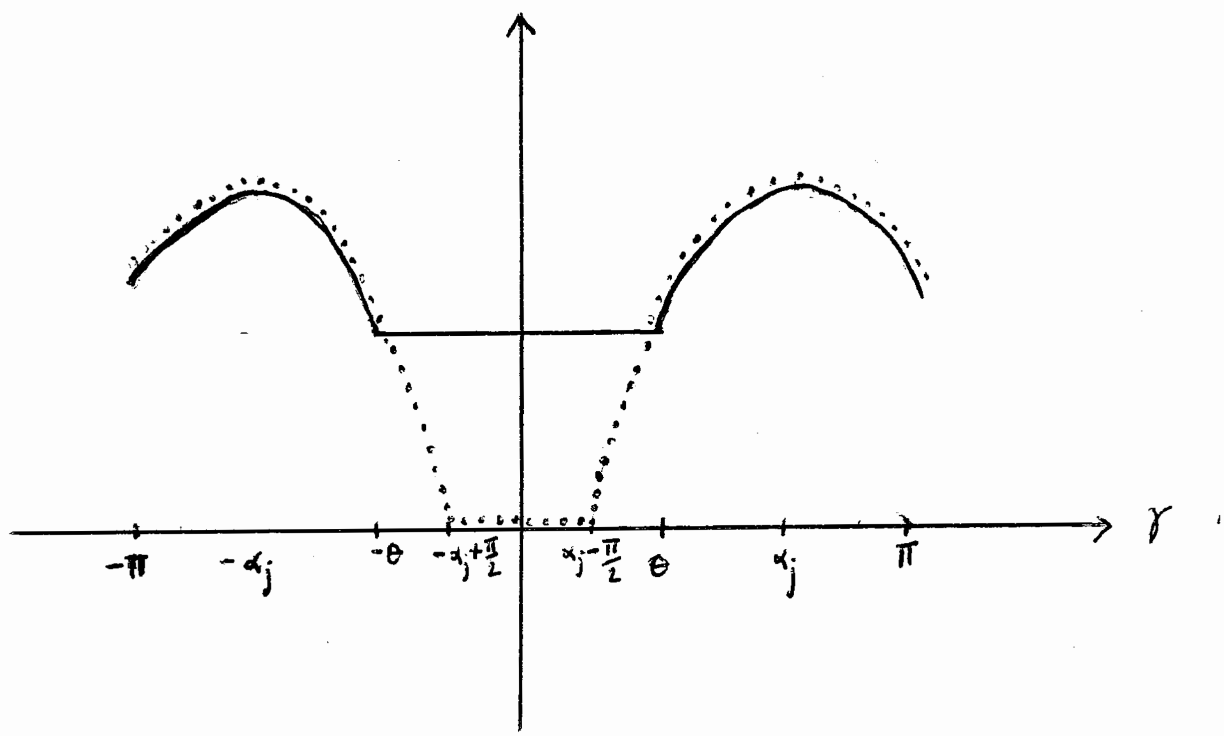

Figure 2.

$V_{T_j}(\gamma) = \max\ (0, |A_j|\cos(\gamma+\alpha_j),\ |A_j|\cos(\gamma-\alpha_j))$, the dotted line in Figure 2.

$V_{S_j}(\gamma) = \max\ (1, |A_j|\cos(\gamma+\alpha_j),\ |A_j|\cos(\gamma-\alpha_j))$, the solid line in Figure 2.

Since $-\alpha_j + \frac{\Pi}{2} \to -\theta$ and $\alpha_j - \frac{\Pi}{2} \to \theta$ as $j \to \infty$ it follows that

$$\int V_{S_j}(\gamma) - V_{T_j}(\gamma)\, dM(\gamma) \to \int_S 1\, dM(\gamma) = M(S) \text{ as } j \to \infty.$$

Let $F = (F_1, F_2, \ldots, F_n)$ consist of n uniformly continuous linear real valued functions on $\mathcal{X}$. Let $M_1, M_2, \ldots, M_n$ be the corresponding measures. A question that arises naturally from the examples given in Section 2 is whether $F(C) \in C$ for every $C \in \mathcal{K}$. This would enable us to consider the actual choice that the decision maker makes out of the alternative in C. Necessary and sufficient conditions on the M_i's that will induce this feasibility property would be of great interest. The condition that F(C) be on the boundary of C for every $C \in \mathcal{K}$ (or Pareto Optimality) is also desirable but unfortunately it is inconsistent with continuity (see Kalai-Myerson [1977]).

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