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SOME RELATIONSHIPS BETWEEN HIERARCHICAL SYSTEMS THEORY AND CERTAIN OPTIMIZATION PROBLEMS

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SUMMARY

In this paper are examined some relationships between multilevel hierarchical systems theory and four different types of optimization problems: a decomposable mathematical program, a sequential optimization problem, a continuous-time optimal control problem, and a welfare maximization problem involving profit distribution to consumers. In each instance an identification of the problem is made with a multilevel hierarchical system, a coordination principle is identified, and theorems originating in the context of the individual problems are restated in the terminology developed by Mesarović for multilevel hierarchical systems.

INTRODUCTION

The work of M. D. Mesarović and his associates (see [6] for a catalogue of works) represents a major new framework for the structuring of many important technical and economic decision problems. This work, being so original in contents and terminology, appears to us to require at this time further integration into some already established optimization theories. This paper represents an attempt at such further integration, by examining whether, or under what conditions, certain optimization problems (a decomposable mathematical programming problem, a sequential decision problem, an optimal control problem, and a welfare maximization problem from economic theory) can be coordinated by coordination principles developed by Mesarović and his associates.

The scope of the paper is limited in the following two respects. In the first place, we do not attempt to pass judgement about the general usefulness of hierarchical systems theory; we are here not concerned with conceptual advantages or disadvantages of the systems approach as opposed to other approaches. In the second place, we are here focusing essentially on static characterizations, i.e., the coordinability or noncoordinability of a given optimization problem by some coordination principle. In other words, we do not discuss dynamic coordination strategies or iterative computational procedures. In this, we follow Mesarović et al [6], where the discussion is also mainly directed towards the static concepts of applicability and coordinability.

DECOMPOSABLE MATHEMATICAL PROGRAMMING

Consider the following mathematical programming problem, denoted by (P):

$$\text{Maximize} \quad \sum_{j=1}^{n} f_j(m_j) \qquad \text{(P)}$$

subject to

$$\sum_{j=1}^{n} g_j(m_j) \leq a,$$

$$m_j \in M_j \quad (j = 1 \ldots n),$$

where the functions f_j are real-valued and the g_j are vector-valued. It is assumed that (P) has an optimal solution and the sets M_j are bounded. The application of the interaction balance principle to (P) will now be discussed. For that purpose, (P) must first be decomposed by way of interaction decoupling. There are many ways in which this can be achieved, but the following one is a natural one (and is also suggested in Mesarović et al [6], pp. 246-247): Namely, let each infimal decision problem $(P_j(\beta))$ $(j = 1 \dots n)$ be

$$\text{Maximize} \quad f_j(m_j) + \beta u_j \qquad (P_j(\beta))$$

subject to

$$g_j(m_j) + u_j \leq a,$$

$$(m_j, u_j) \in X_j = \{(m_j, u_j) \mid m_j \in M_j\},$$

where u_j is the interface input vector by which the infimal decision problem $(P_j(\beta))$ is coupled with the other infimal decision problems. The j^{th} component K_j of the interaction function

$$K(m) = (K_1(m), K_2(m) \dots K_n(m)) \text{ is } K_j(m) = \sum_{\substack{i=1 \\ i \neq j}}^{n} g_i(m_i),$$

where $m = (m_1, m_2 \dots m_n)$. The vector β is the coordination input. Everything that is said in what follows presupposes this particular decomposition of (P). For given β, let the set of optimal solutions to $(P_j(\beta))$ be denoted X_j^β. Let $u = (u_1, u_2 \dots u_n)$, and let x^β denote a pair (m, u) such that each pair $(m_j, u_j) \in X_j^\beta$. Finally, let $\hat{M}$ denote the set of optimal solutions to the original problem (P). The interaction balance principle can now be expressed by the following proposition:

$$(\forall\beta)\ (\forall x^{\beta})\ \{[(m,\ u) = x^{\beta}\ \text{ and }\ u = K(m)] \rightarrow m \in \hat{M}\}.$$

The interaction balance principle is applicable to the problem (P) if this proposition is true. If, in addition, the set of β's for which the proposition hypothesis holds is not empty, then the problem (P) is coordinable by the interaction balance principle.

Now suppose there exists some $\tilde{\beta}$ with some associated $x^{\tilde{\beta}} \in X_1^{\tilde{\beta}} \times X_2^{\tilde{\beta}} \times \ldots \times X_n^{\tilde{\beta}}$ with the property that $\tilde{u} = K(\tilde{m})$, where $x^{\tilde{\beta}} = (\tilde{m}, \tilde{u})$. That is, the coordinating condition $[(m, u) = x^{\beta}$ and $u = K(m)]$ is satisfied for $\tilde{\beta}$ and $(\tilde{m}, \tilde{u})$. The following proposition is then easy to demonstrate:

<u>Proposition 1:</u> If $[(\tilde{m}, \tilde{u}) = x^{\tilde{\beta}}$ and $\tilde{u} = K(\tilde{m})]$ holds, then $\tilde{m}$ and $\tilde{\beta}$ satisfy the following saddle point optimality conditions for (P):

$$\tilde{m} = (\tilde{m}_1, \tilde{m}_2 \ldots \tilde{m}_n) \text{ maximizes} \tag{1}$$

$$\sum_{j=1}^{n} f_j(m_j) - \tilde{\beta}\left(\sum_{j=1}^{n} g_j(m_j) - a\right)$$

over $M_1 \times M_2 \times \ldots \times M_n$,

$$\tilde{\beta} \geq 0, \tag{2}$$

$$\sum_{j=1}^{n} g_j(\tilde{m}_j) \leq a, \tag{3}$$

$$\tilde{\beta}\left(\sum_{j=1}^{n} g_j(\tilde{m}_j) - a\right) = 0. \tag{4}$$

$\tilde{m}$ is then an optimal solution to (P), and it follows that the interaction balance principle is applicable to (P). It is also easy to show the following proposition.

Proposition 2: Suppose $\tilde{m}$ and $\tilde{\beta}$ satisfy the optimality conditions (1) - (4).

Let

$$\tilde{u}_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} g_i(\tilde{m}_i)$$

and

$$\tilde{u} = (\tilde{u}_1, \tilde{u}_2 \ldots \tilde{u}_n),$$

then

$$[(\tilde{m}, \tilde{u}) = x^{\tilde{\beta}} \text{ and } \tilde{u} = K(\tilde{m})].$$

The conclusion, then, is that (P) is coordinable by the interaction balance principle under precisely the same circumstances as there exists a saddle point in m and β. For a discussion of those circumstances under the usual concavity - convexity assumptions, see Geoffrion [4].

It may be pointed out that the decomposition of (P) into the infimal decision problems $(P_j(\beta))$ exhibited above is a somewhat unusual one in a mathematical programming context, since in mathematical programming decomposition algorithms, there are usually no variables corresponding to the u_j.

SEQUENTIAL OPTIMIZATION

The purpose of this section is to illustrate how the interaction prediction principle developed by Mesarovič et al can be used in a particular N-stage sequential optimization problem. The initial state s_1 is given. The output at stage n, s_{n+1}, is a function only of the output of the previous stage and the control m_n applied at stage n:

$$s_{n+1} = P_n(s_n, m_n) \tag{5}$$

where $m_n \in M_n$ and s_{N+1} is the final output. Each set M_n is assumed finite.

For a given initial state s_1 and the process defined by (5) we want to minimize $g_o(s_1,m)$ where $m \in M_1 x \ldots x M_N$. Given (5) we define the following sets recursively

$$S_1 = \{s_1\} \tag{6}$$

$S_{n+1} = \{s \mid s = P_n(s_n, m_n)$ for some $s_n \in S_n$ and $m_n \in M_n\}$. Each set S_n can be interpreted as the state space at stage n.

To develop appropriate infimal decision problems, we consider the following set of optimization problems:

For each

$$s \in S_n, \ n=1 \ldots N, \tag{7}$$

Minimize

$$g_n(s, m_n, K_n(P_n(s, m_n), m))$$

over

$$m \in M_1 x \ M_2 x \ \ldots \ x M_N$$

where

$$K_N(s', m) = 0 \quad \text{for all} \quad s' \in S_{N+1},$$

$$K_n(s', m) = g_{n+1}(s', m_{n+1}, K_{n+1}) \quad \text{for all } s' \in S_{n+1}.$$

All functions are real-valued. The g_n's are the infimal objective functions and are assumed given though no relationship to g_o is specified. The K_n's specify the interfaces between different infimal units. Note that in (7) we have an entire set of optimization problems for each given n.

Suppose that the supremal unit uses the interaction prediction mode, i.e., that the supremal unit specifies the value of K_n for each $s \epsilon S_{n+1}$ by a real-valued function α_n. We obtain the following infimal decision problem:

For each

$$s \epsilon S_n \tag{8}$$

Minimize

$$g_n(s, m_n, \alpha_n(P_n(s, m_n)))$$

over

$$m_n \epsilon M_n.$$

Solving (8) yields a function $Q_n: S_n \rightarrow M_n$. These functions and the initial state s_1 induce a sequence $\{s_1^\alpha, m_1^\alpha, s_2^\alpha \ldots m_N^\alpha, s_{N+1}^\alpha\}$, where $s_{n+1}^\alpha = P_n(s_n^\alpha, m_n^\alpha)$ and $s_1^\alpha = s_1$.

Let $\hat{M}$ be the set of overall optimal controls (minimizing g_o). Let $m^\alpha = (m_1^\alpha, m_2^\alpha \ldots m_N^\alpha)$ with $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_N)$ being any coordination input. Following Mesarović et al we now say that the interaction prediction principle is applicable iff the following proposition is true:

$$(\forall \alpha)\ (\forall m^\alpha)\ \{[K_n(s, m^\alpha) = \alpha_n(s) \quad \text{for all} \quad s \epsilon S_{n+1}$$

$$\text{and for} \quad n = 1 \ldots N] \rightarrow m^\alpha \epsilon \hat{M}\}.$$

Assuming for the moment that the interaction prediction principle is applicable, then the question whether the system is coordinable by it depends on the existence of a coordination input satisfying the coordinating condition of the principle.

<u>Proposition 3:</u> Assuming the interaction prediction principle is applicable, then the system is coordinable by it.

This proposition can be shown constructively: the supremal unit first "predicts" $\alpha_N = 0$ for all $s_{N+1} \varepsilon S_{N+1}$ and requires the N^{th} infimal problem to be solved, yielding Q_N and g_N, then it "predicts" the $(N-1)^{st}$ interface, requires the $(N-1)^{st}$ infimal problem to be solved, etc. We obtain the backwards algorithm of dynamic programming as developed by Bellman [1], which could be expected from the definitions in (7).

The critical question is whether the interaction prediction principle is applicable. This is of course dependent upon the relation between the overall objective function g_o and the infimal objective functions g_n as recognized by Mesarovič et al [6] (cf. Proposition 5.1 and Propositions 5.26-5.30). Observe that if the problem at hand can be formulated as a dynamic programming problem, then the g_n's are easily defined and g_1 takes the place of g_o, which immediately implies that the interaction prediction principle is applicable.

The result here is fairly general in the sense that the interaction prediction principle appears applicable to a larger set of problems (with the underlying process described in (5)) than the set of problems that can be formulated in dynamic programming terms as suggested by the following example.

Example:

Consider the following two-stage process:

$P_1(s_1, m_1) = m_1^2$ and

$P_2(s_2, m_2) = s_2 + m_2^2$, with

$m_i \in M_i = \{-\pi/2, 0, \pi/2\}$ for $i = 1, 2$.

The overall objective function is given by

$$g_o(m) = \sin(m_1 + m_2) + \frac{3}{2} m_1 m_2 + m_1^2 + m_2^2.$$

A dynamic programming recursion cannot be formulated in any obvious way, since knowledge of the state s_2 is not sufficient to identify an associated optimal value of m_2 in the second stage: if $s_2 = (\pi^2)/4$ then an optimal associated m_2 is $\pi/2$ if $m_1 = -\pi/2$ and $-\pi/2$ if $m_1 = \pi/2$. Increasing the state space circumvents this difficulty; this is however artificial since it leads eventually to a complete enumeration of all possible decision sequences.

Consider the following infimal objective functions:

$$g_2(s, m_2, 0) = m_2^2,$$

$g_1(s, m_1, \alpha_1(P_1(s, m_1))) = \sin(m_1 + \alpha) + 3/2\ \alpha m_1 + m_1^2$, i.e. $\alpha_1(\cdot) \equiv \alpha$. Minimization of g_2 and g_1 over M_2 and M_1 gives an overall optimal pair $m_1 = m_2 = 0$ for g_o provided that α was predicted to be zero.

In general it is however not clear how infimal objective functions should be constructed; the only guideline that is available is that the g_n's should have a common minimizer m which optimizes g_o.

This is a decomposition in the spirit of Mesarović et al, though as it involves an infinite number of infimal decision problems it is not a direct application of their framework. The functional $K_t(m)$ is defined for each t as the value at time t of the function y which satisfies $\dot{y} = f(y, m,t)$ on $[0,t]$ with $y(0) = \bar{y}$. (We assume for simplicity that $K_t(m)$ is thus uniquely defined). Let $K(m)$ be the cartesian product of the $K_t(m)$, $t \in [0,T]$.

For a given β, let the set of optimal solutions to $(OC_t(\beta))$ be denoted X_t^β. Let X^β be the cartesian product of the X_t^β, and $\hat{M}$ be the set of solutions to (OC). The interaction balance principle is then applicable iff

$(\forall\beta \in \Gamma)(\forall x^\beta \in X^\beta)\{[(m,u) = x^\beta \text{ and } u = K(m)] \rightarrow m \in \hat{M}\}$. It is shown in Peterson [8], [9] that the conditions $(m, u) \in X^\beta$ and $u_t = K_t(m)$ imply that m is a solution to (OC). Therefore the interaction balance principle is applicable to this decomposition.

In general, however, solving the problems $(OC_t(\beta))$ for each t does not yield functions (m, u) which satisfy $u = K(m)$, and hence this decomposition is not always useful. Conditions sufficient to ensure that there exists a β (with $\beta(T) = 0$) such that $(m, u) \in X^\beta$ and $u = K(m)$ are given by the hypothesis of an appropriate version of Pontryagin's maximum principle (e.g. [10], Theorem 3, page 50), and the added assumptions that a solution to (OC) exists, and that $g_{t\beta}$ is pseudoconcave. Under such conditions, and for the decomposition used here, the problem (OC) is coordinable through the interaction balance principle.

OPTIMAL CONTROL

Consider the optimal control problem (OC):

$$\text{Maximize} \quad g(m) \equiv \int_0^T L(y, m, t)dt \tag{OC}$$

with respect to

$$m \text{ on } [0, T]$$

subject to

$$\dot{y} = f(y, m, t), \; y(0) = \bar{y}, \; m_t \in M_t.$$

Here L and f are known functions, T and $\bar{y}$ are known constants, M_t is a known set of permissible m_t values which may vary with time t, and m and y may be vector-valued. m_t and y_t are used to denote, respectively, $m(t)$ and $y(t)$, and $\dot{y}$ to denote the time derivative of y. A possible decomposition of this problem results, for each t, in the infimal decision problem $(OC_t(\beta))$:

$$\text{Maximize} \quad g_{t\beta}(m_t, u_t) \tag{$OC_t(\beta)$}$$

$$= L(u_t, m_t, t) + \beta_t f(u_t, m_t, t) + \dot{\beta}_t u_t$$

with respect to

$$(u_t, m_t) \in X_t = \{(u_t, m_t) \mid m_t \in M_t\},$$

where

$$\beta \in \Gamma \equiv \{\text{differentiable functions on } [0, T] \text{ with } \beta(T) = 0\}.$$

WELFARE MAXIMIZATION

This section discusses the application of the prediction principle to obtain Pareto-optimal states of a 'private ownership economy' (cf. [2], [3], [7], [11]).

Consider an economy with n consumers, k firms and q commodities. With each consumer i, $1 \leq i \leq n$, is associated a consumption set M_i from which a consumption vector m_i is chosen; an initial resource endowment r_i and a real-valued utility function $g_i: M_i \to E^1$. With the j^{th} firm, $1 \leq j \leq k$, there is associated a set M_{n+j} from which that firm is to select a production vector m_{n+j}. Each consumer i shares a proportion $\lambda_{i,n+j}$ of firm j's profits, hence $\sum_{i=1}^{n} \lambda_{i,n+j} = 1$.

The following assumptions are made for each consumer i:

(i) $M_i = E^q_+$ (the nonnegative orthant of E^q),

(ii) each g_i is continuous, strictly increasing and concave.

It is assumed for each firm j that:

(iii) each M_{n+j} is a convex, compact subset of E^q containing $0 \in E^q$.

It is further assumed that

(iv) each of the q components of the vector $\sum_{i=1}^{n} r_i$ is positive,

(v) $M_1 + \sum_{j=1}^{k} M_{n+j}$ contains an open neighborhood of $0 \in E^q$.

Consider the following welfare problem, denoted (WP):

$$\text{Pareto Maximize } [g_1(m_1) \ldots g_n(m_n)] \qquad \text{(WP)}$$

with respect to $m_1, \ldots, m_{n+k}$

subject to

$$\sum_{i=1}^{n} r_i - \sum_{i=1}^{n} m_i + \sum_{j=1}^{k} m_{n+j} = 0$$

$$m_i \in M_i,\ m_{n+j} \in M_{n+j},\ i = 1 \ldots n;\ j = 1 \ldots k.$$

Let Γ denote the set

$$\{(\beta, \alpha_{n+1} \dots \alpha_{n+k}) \mid \beta, \alpha_{n+j} \in E^q; \ \beta \geq 0 \ \text{ and } \ \beta \neq 0\}.$$

An element γ of Γ is the coordination input for the following decomposition of (WP):

For $1 \leq i \leq n$:

$$\text{Maximize } \ g_i(m_i) \qquad (C_i(\gamma))$$

with respect to m_i

subject to

$$\beta \cdot m_i = \beta \cdot r_i + \sum_{j=1}^{k} \lambda_{i,n+j} \, \beta \cdot \alpha_{n+j}$$

$$m_i \in M_i.$$

For $1 \leq j \leq k$:

$$\text{Maximize } \ \beta \cdot m_{n+j} \qquad (F_j(\gamma))$$

with respect to m_{n+j}

subject to

$$m_{n+j} \in M_{n+j} .$$

The problem $C_i(\gamma)$ is referred to as the i^{th} consumer's choice problem. The constraining equation represents a budget constraint. The problem $F_j(\gamma)$ is interpreted as the j^{th} firm's choice problem. The vector β represents a set of prices at which consumers and firms can buy and sell commodities.

Define

$$\tilde{q}(\gamma) \equiv (0, \alpha_{n+1} \dots \alpha_{n+k}), \quad \text{for} \quad \gamma \in \Gamma$$

and

$$g(\gamma, m) \equiv \left(\sum_{i=1}^{n} r_i - \sum_{i=1}^{n} m_i + \sum_{j=1}^{k} m_{n+j}, m_{n+1} \dots m_{n+k} \right)$$

for

$$(\gamma, m) \in \Gamma \times M_1 \times \dots \times M_{n+k}, \quad \text{where } m = (m_1 \dots m_{n+k}).$$

For each $\gamma \in \Gamma$, denote by m_i^γ the solution to $C_i(\gamma)$ (to $F_j(\gamma)$, if $n < i = n + j \leq n + k$), and define $m^\gamma \equiv (m_1^\gamma \dots m_{n+k}^\gamma)$. Let $\hat{M}$ denote the set of solutions to (WP). The prediction principle (of which the interaction prediction principle is only a special case, see Mesarovič et al [6], p. 99) is

$$(\forall \gamma \in \Gamma)(\forall m^\gamma)\{[q(\gamma, m^\gamma) = \tilde{q}(\gamma)] \rightarrow m^\gamma \in \hat{M}\}.$$

In accordance with general equilibrium theory the following definition is made:
(m^γ, β) is said to be a competititve equilibrium if $q(\gamma, m^\gamma) = \tilde{q}(\gamma)$. Then,

<u>Proposition 4:</u> If (m^γ, β) is a competititve equilibrium, then $m^\gamma \in \hat{M}$.

Hence the prediction principle is applicable to this decomposition of (WP).

<u>Proposition 5:</u> If $\hat{m} \in \hat{M}$, then there exists a $\gamma \in \Gamma$ such that (m^γ, β) is a competitive equilibrium with $m^\gamma = \hat{m}$.

This is the celebrated theorem on the existence of a competititve equilibrium developed by Arrow, Debreu and others. It can be proved either with an argument based on a fixed-point theorem ([11],pp. 61-102) or by a mathematical programming argument based on the existence of a saddle point (using ideas presented in [5] and in [7], pp. 18-21).

<u>Proposition 6:</u> $\hat{M}$ is non-empty.

The proof of this is straightforward. The last two propositions imply that (WP) is coordinable through the prediction principle.

It is interesting to note that the general equilibrium problem as posed in the earlier literature [2] omitted distribution of profits to consumers, in which case the $\lambda_{i,n+j}$ terms in the problem $C_i(\gamma)$ vanish. Under these circumstances, coordination of (WP) can be achieved through the interaction balance principle.

CONCLUSION

Our investigation has revealed that there are interesting relationships between the hierarchical systems theory and various optimization theories. In fact, it appears that a wide range of optimization problems can be coordinated using coordination principles from hierarchical systems theory. However, a particular optimization problem cannot always be coordinated using a given coordination principle. Consider, for example, the application of the interaction prediction principle to the decomposable mathematical programming problem (P) discussed earlier in the paper. For that purpose, (P) must again first be decomposed, and the simplest way may be the following: Let each infimal decision problem $(P_j(\alpha_j))$ be

$$\text{Maximize} \quad f_j(m_j) \qquad (P_j(\alpha_j))$$

subject to

$$g_j(m_j) + \alpha_j \leq a,$$

$$m_j \in M_j,$$

where α_j is the "predicted", or fixed, interface input vector for the j^{th} infimal decision problem. Let $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_n)$. Let m_j^{α} be an optimal solution to $(P_j(\alpha_j))$, and let $m^{\alpha} = (m_1^{\alpha}, m_2^{\alpha} \ldots m_n^{\alpha})$. The interaction prediction principle is now expressed by the proposition

$$(\forall\alpha)(\forall m^{\alpha})\{[m = m^{\alpha} \text{ and } K(m) = \alpha] \rightarrow m \in \hat{M}\}$$

where $K(m)$ and $\hat{M}$ are defined as earlier. It is easy to see that this proposition is not necessarily true for the problem (P), and hence the interaction prediction principle is not, in general, applicable.

It is worth pointing out that we have deliberately glossed over two difficulties associated with the application of the hierarchical systems theorem coordination principles to different optimization problems. First, finding that specific decomposition of a problem which allows coordination by some coordination principle may not be a trivial task. Secondly, even if one has found the "right" decomposition, there then reamins the task of computing through some algorithm those values of the coordination inputs for which the coordinating condition of the particular principle holds. As already pointed out in the Introduction, we have not given any attention to algorithmic schemes for computing coordination input values here. Whether the hierarchical systems theorem will turn out to be a success in practical appllcations may well depend on how easily these two difficulties can be resolved.

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REFERENCES

[1] R. E. Bellman, Dynamic Programming, Princeton University Press, Princeton, 1957.

[2] G. Debreu, "Valuation equilibrium and Pareto optimum", in K. Arrow and T. Scitovsky, eds., Readings in Welfare Economics, Richard D. Irwin, 1969, pp. 39-45.

[3] G. Debreu, Theory of Value, John Wiley, 1959.

[4] A. M. Geoffrion, "Duality in Non-linear Programming: A Simplified Applications-Oriented Development", SIAM Review, Vol. 13, No. 1, Jan. 1971, pp. 1-36.

[5] O. L. Mangarsarian, Nonlinear Programming, Mc Graw-Hill, 1969, pp. 80-82.

[6] M. D. Mesarović, D. Macko and Y. Takahara, Theory of Hierarchical, Multi-level Systems, Academic Press, New York and London, 1970.

[7] T. Negishi, General Equilibrium Theory and International Trade, North-Holland, 1972, pp. 11-27.

[8] D.W. Peterson, "A Sufficient Maximum Principle", I.E.E.E. Transactions on Automatic Control, Vol. 16, No. 1, Feb. 1971, pp. 85-86.

[9] D. W. Peterson, "On Sensitivity in Optimal Control Problems", Forthcoming in the Journal of Optimization Theory and Applications.

[10] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Wiley, New York, 1962.

[11] J. Quirk and R. Saposnik, Introduction to General Equilibrium Theory and Welfare Economics, Mc Graw-Hill, 1968.