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ORDINARY DUALITY VIS-A-VIS GEOMETRIC DUALITY

by

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ABSTRACT. The ordinary dual problem is characterized as an orthogonal projection of the corresponding geometric dual problem--a projection that can be obtained via a suboptimization. This characterization endows geometric duality with certain strong advantages over ordinary duality.

Keywords: Ordinary programming, geometric programming, duality theory, orthogonal projection, suboptimization.

CONTENTS

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1. Introduction. The ordinary dual problem described herein was first defined and studied by Falk [3]. It is an extension of the "Wolfe dual problem" [9] and is more suitable for ordinary programming.

The geometric dual problem described herein was first defined and studied by Peterson [4,5]. It is a generalization of the "extended geometric dual problem" first defined and studied by Duffin and Peterson in Chapter 7 of [2]. The latter problem is, in turn, an extension of the "posynominal programming dual problem" originally defined and studied by Duffin and Peterson [1] and further explored by Duffin, Peterson and Zener [2].

Although a characterization of the ordinary dual problem as an orthogonal projection of the corresponding geometric dual problem was first announced in [4,5], a complete proof and its implications are being given here for the first time.

This proof utilizes a (geometric programming) version of Fenchel's duality theorem given by Rockafellar [8]. In doing so, it also requires some of the convexity theory in [8]--especially the theory having to do with the "relative interior" (ri S) of an arbitrary convex set $S \subseteq E_N$ (N-dimensional Euclidean space).

2. The primal problem. A problem that is sufficiently general to encompass both the most general ordinary programming problem and the most general geometric programming problem is given in section 2.2 of [5].

3. The ordinary dual problem. The "ordinary Lagrangian" for problem A is

$$L_o(x,\kappa;\lambda) \triangleq G(x,\kappa) + \sum_I \lambda_i g_i(x^i) .$$

This Lagrangian gives rise to the following "ordinary dual problem."

PROBLEM B_o. Using the feasible solution set

$$T_o \triangleq \{\lambda \in E_{o(I)} | \lambda \geq 0 \ \underline{\text{and}} \inf_{\substack{(x,\kappa)\in C \\ x\in X}} L_o(x,\kappa;\lambda) \ \underline{\text{is finite}}\}$$

and the objective function value

$$H_o(\lambda) \triangleq \inf_{\substack{(x,\kappa)\in C \\ x\in X}} L_o(x,\kappa;\lambda),$$

calculate both the problem supremum

$$\psi_o \triangleq \sup_{\lambda\in T_o} H_o(\lambda)$$

and the optimal solution set

$$T_o^* \triangleq \{\lambda \in T_o | H_o(\lambda) = \psi_o\} .$$

Problem B_o is, of course, an extension of the "Wolfe dual problem" [9]. It was initially defined and studied by Falk [3] and is the appropriate dual problem for ordinary programming.

4. <u>The geometric dual problem.</u> In terms of the notation and definitions in section 3.3.2 of [5], the "geometric Lagrangian" for problem A is

$$L_g(x,\kappa;y,\lambda) \triangleq \langle x,y\rangle - H(y,\lambda) - \sum_J \kappa_j h_j(y^j) .$$

This Lagrangian gives rise to the following "geometric dual problem."

PROBLEM B. <u>Using the feasible solution set</u>

$$T \triangleq \{(y,\lambda) \in D \mid y \in Y, \ \ \underline{\text{and}} \ \ h_j(y^j) \leq 0, \ j \in J\}$$

<u>and the objective function value</u>

$$H(y,\lambda) \triangleq h_0(y^0) + \sum_I h_i^+(y^i,\lambda_i) \ ,$$

<u>calculate both the problem infimum</u>

$$\psi \triangleq \inf_{(y,\lambda)\in T} H(y,\lambda)$$

<u>and the optimal solution set</u>

$$T^* \triangleq \{(y,\lambda) \in T \mid H(y,\lambda) = \psi\} \ .$$

A derivation of problem B from the Lagrangian L_g is given in [6]. However, problem B was initially defined and studied in [4,5]. It is, of course, an extension of the posynomial programming dual problem originally defined and studied by Duffin and Peterson [1] and further explored by Duffin, Peterson and Zener [2].

5. The key reformulation. The key to the most fundamental relation between the corresponding dual problems B_o and B is to reformulate the ordinary Lagrangian minimization (used to define the objective function $H_o(\lambda)$ for problem B_o) as a special case of problem A. Since that minimization does not directly involve the constraints $g_i(x^i) \leq 0$, $i \in I$, it can actually be reformulated as a special case of the unconstrained version of problem A.

To obtain the unconstrained version of problem A, simply let both index sets I and J be empty and drop the (now unnecessary) subscript 0 from the symbol $g_0:C_0$. In addition, replace all remaining symbols with their script counterparts in order to avoid ambiguous notation when carrying out the desired reformulation.

The resulting unconstrained version of problem A can be given the following concise definition (in terms of the notation and definitions in section 2.1 of [5]).

PROBLEM $\mathcal{A}$. Using the feasible solution set

$$\mathcal{S} \triangleq \mathcal{X} \cap \mathcal{C},$$

calculate both the problem infimum

$$\varphi \triangleq \inf_{x \in \mathcal{S}} g(x)$$

and the optimal solution set

$$\mathcal{S}^* \triangleq \{x \in \mathcal{S} \mid g(x) = \varphi\} .$$

Similarly, the geometric dual of problem $\mathcal{A}$ can be given the following concise definition (in terms of the notation and definitions in section 3.1.4 of [5]).

PROBLEM $\mathcal{B}$. Using the feasible solution set

$$\mathcal{T} \triangleq \mathcal{Y} \cap \mathcal{D},$$

calculate both the problem infimum

$$\psi \triangleq \inf_{y \in \mathcal{T}} h(y)$$

and the optimal solution set

$$\mathcal{T}^* \triangleq \{y \in \mathcal{T} \mid h(y) = \psi\} .$$

To reformulate the ordinary Lagrangian minimization as a special case of problem $\mathcal{A}$, simply let the function domain

$$\mathcal{C} \triangleq C = \{(x^0, x^I, x^J, \kappa) \in E_n \mid x^k \in C_k,\ k \in \{0\} \cup I;\ (x^j, \kappa_j) \in C_j^+,\ j \in J\}$$

and let the function value

$$\mathcal{g}(x^0, x^I, x^J, \kappa; \lambda) \triangleq L_o(x, \kappa; \lambda) = g_0(x^0) + \sum_J g_j^+(x^j, \kappa_j) + \sum_I \lambda_i g_i(x^i).$$

Also, let the cone

$$\mathcal{X} \triangleq \{(x^0, x^I, x^J, \kappa) \in E_n \mid (x^0, x^I, x^J) \in X;\ \kappa \in E_{o(J)}\}.$$

The presence of λ as a parameter in the resulting problem $\mathcal{a}$ will be indicated notationally by replacing the symbol $\mathcal{a}$ with the symbol $\mathcal{a}(\lambda)$. Moreover, the presence of λ as a parameter in any other entity will be indicated notationally in the same way.

Now, <u>problem $\mathcal{a}(\lambda)$ consists of using the feasible solution set</u>

$$\mathcal{S} \triangleq \{(x, \kappa) \in C \mid x \in X\}$$

<u>to calculate both the problem infimum</u>

$$H_o(\lambda) \triangleq \inf_{(x,\kappa) \in \mathcal{S}} L_o(x, \kappa; \lambda)$$

<u>and the optimal solution set</u>

$$\mathcal{S}^*(\lambda) \triangleq \{(x, \kappa) \in \mathcal{S} \mid L_o(x, \kappa; \lambda) = H_o(\lambda)\}.$$

To determine the nature of the resulting dual problem $\mathcal{B}(\lambda)$, we need to compute both the conjugate transform $\mathscr{h}(\cdot;\lambda) : \mathscr{D}(\lambda)$ of the given function $\mathscr{g}(\cdot;\lambda) : \mathscr{C}$ and the dual $\mathscr{Y}$ of the given cone $\mathscr{X}$.

6. <u>The resulting dual problem</u>. To compute $\mathscr{h}(\cdot;\lambda) : \mathscr{D}(\lambda)$, first note that

$$\begin{aligned}
\mathscr{h}(y^0,y^I,y^J,\beta;\lambda) &= \sup_{(x^0,x^I,x^J,\kappa)\in\mathscr{C}} \{\langle y^0,x^0\rangle + \sum_I \langle y^i,x^i\rangle + \sum_J \langle y^j,x^j\rangle \\
&\qquad + \sum_J \beta_j\kappa_j - g_0(x^0) - \sum_J g_j^+(x^j,\kappa_j) - \sum_I \lambda_i g_i(x^i)\} \\
&= \sup_{x^0\in C_0} [\langle y^0,x^0\rangle - g_0(x^0)] + \sum_I \sup_{x^i\in C_i} [\langle y^i,x^i\rangle - \lambda_i g_i(x^i)] \\
&\qquad + \sum_J \sup_{(x^j,\kappa_j)\in C_j^+} [\langle y^j,x^j\rangle + \beta_j\kappa_j - g_j^+(x^j,\kappa_j)] .
\end{aligned}$$

Consequently, $(y^0,y^I,y^J,\beta) \in \mathscr{D}(\lambda)$ if and only if each term on the right-hand side of the preceding equations is finite. Of course, the first term is finite if and only if $y^0 \in D_0$, in which case the first term is equal to $h_0(y^0)$. The finiteness of the remaining terms can be conveniently characterized with two lemmas.

The following lemma characterizes the finiteness of the terms involving the index set I.

<u>Lemma</u> A. <u>Given that</u> $\lambda_i \geq 0$, <u>the</u> $\sup_{x^i\in C_i} [\langle y^i,x^i\rangle - \lambda_i g_i(x^i)]$ <u>is finite if and only if</u> $(y^i,\lambda_i) \in D_i^+$, <u>in which case</u>

$$\sup_{x^i \in C_i} [\langle y^i, x^i \rangle - \lambda_i g_i(x^i)] = h_i^+(y^i, \lambda_i) .$$

Proof. Simply observe that

$$\sup_{x^i \in C_i} [\langle y^i, x^i \rangle - \lambda_i g_i(x^i)] = \begin{cases} \sup_{x^i \in C_i} \langle y^i, x^i \rangle & \text{if } \lambda_i = 0 \\ \lambda_i h_i(y^i/\lambda_i) & \text{if } \lambda_i > 0 \text{ and } y^i \in \lambda_i D_i \\ +\infty & \text{if } \lambda_i > 0 \text{ and } y^i \notin \lambda_i D_i, \end{cases}$$

and then use the defining formula for $h_i^+ : D_i^+$. q.e.d.

The next lemma characterizes the finiteness of the terms involving the index set J.

Lemma B. The $\sup_{(x^j, \kappa_j) \in C_j^+} [\langle y^j, x^j \rangle + \beta_j \kappa_j - g_j^+(x^j, \kappa_j)]$ is finite if and only if both $y^j \in D_j$ and $h_j(y^j) + \beta_j \leq 0$, in which case

$$\sup_{(x^j, \kappa_j) \in C_j^+} [\langle y^j, x^j \rangle + \beta_j \kappa_j - g_j^+(x^j, \kappa_j)] = 0 .$$

Proof. First, observe that

$$\sup_{(x^j, \kappa_j) \in C_j^+} [\langle y^j, x^j \rangle + \beta_j \kappa_j - g_j^+(x^j, \kappa_j)]$$

$$= \sup_{\kappa_j \geq 0} [\sup_{x^j} \{\langle y^j, x^j \rangle + \beta_j \kappa_j - g_j^+(x^j, \kappa_j) \mid (x^j, \kappa_j) \in C_j^+\}]$$

$$= \sup_{\kappa_j \geq 0} [\beta_j \kappa_j + \sup_{x^j} \{\langle y^j, x^j \rangle - g_j^+(x^j, \kappa_j) \mid (x^j, \kappa_j) \in C_j^+\}]$$

$$= \sup_{\kappa_j \geq 0} \left[\beta_j \kappa_j + \begin{pmatrix} \sup\limits_{x^j} \{\langle y^j, x^j\rangle - \sup\limits_{d^j \in D_j} \langle x^j, d^j\rangle \mid \sup\limits_{d^j \in D_j} \langle x^j, d^j\rangle < +\infty\} & \text{if } \kappa_j = 0 \\ \\ \sup\limits_{x^j} \{\langle y^j, x^j\rangle - \kappa_j g_j(x^j/\kappa_j) \mid x^j/\kappa_j \in C_j\} & \text{if } \kappa_j > 0 \end{pmatrix} \right]$$

$$= \sup_{\kappa_j \geq 0} \left[\beta_j \kappa_j + \begin{pmatrix} 0 & \text{if } \kappa_j = 0 \text{ and } y^j \in \bar{D}_j \\ +\infty & \text{if } \kappa_j = 0 \text{ and } y^j \notin \bar{D}_j \\ +\infty & \text{if } \kappa_j > 0 \text{ and } y^j \notin D_j \\ \kappa_j h_j(y^j) & \text{if } \kappa_j > 0 \text{ and } y^j \in D_j \end{pmatrix} \right] ,$$

where the final step makes use of the fact that the zero function with domain $\bar{D}_j$ (the topological closure of D_j) is the conjugate transform of the conjugate transform of the zero function with domain D_j. Now, note that the last expression is finite only if $y^j \in D_j$, in which case the last expression clearly

$$= \sup_{\kappa_j \geq 0} [\beta_j \kappa_j + \kappa_j h_j(y^j)] .$$

But this expression is obviously finite if and only if $h_j(y^j) + \beta_j \leq 0$, in which case this expression is clearly zero q.e.d.

We have now shown that the function domain

$$\mathcal{D}(\lambda) = \{(y^0,y^I,y^J,\beta) \in E_\eta | y^0 \in D_0;\ (y^i,\lambda_i) \in D_i^+,\ i \in I;$$
$$y^j \in D_j,\ \beta_j \in E_1,\ \text{and}\ h_j(y^j)+\beta_j \leq 0,\ j \in J\};$$

and we have also shown that the function value

$$\mathcal{K}(y^0,y^I,y^J,\beta;\lambda) = h_0(y^0) + \sum_I h_i^+(y^i,\lambda_i) \triangleq H(y,\lambda).$$

Moreover, elementary considerations show that the cone

$$\mathcal{Y} = \{(y^0,y^I,y^J,\beta) \in E_\eta | (y^0,y^I,y^J) \in Y,\ \beta = 0\}.$$

Therefore, <u>problem</u> $\mathcal{B}(\lambda)$ <u>consists of using the feasible solution set</u>

$$\mathcal{J}(\lambda) \triangleq \{y \in E_n | (y,\lambda) \in T\}$$

<u>to calculate both the problem infimum</u>

$$\psi(\lambda) \triangleq \inf_{y \in \mathcal{J}(\lambda)} H(y,\lambda)$$

<u>and the optimal solution set</u>

$$\mathcal{J}^*(\lambda) \triangleq \{y \in \mathcal{J}(\lambda) | H(y,\lambda) = \psi(\lambda)\}.$$

Hence, the duality theory relating problems $\mathcal{A}$ and $\mathcal{B}$ can be used to deduce important relations between the corresponding dual problems B_o and B.

7. The fundamental relation. In view of the general duality theory in section 3.1.4 of [5], if the preceding dual problems $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ are consistent and have no duality gap, then

$$0 = H_o(\lambda) + \psi(\lambda) ;$$

in which event the negative $-H_o(\lambda)$ of the ordinary dual objective function value $H_o(\lambda)$ is simply the (sub)infimum $\psi(\lambda)$ of the geometric dual objective function value $H(y,\lambda)$ over y.

Thus, the set of all λ for which the preceding dual problems $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ are consistent and have no duality gap is of great interest. It is, of course, a subset of both the ordinary dual feasible solution set T_o and an orthogonal projection

$$\Lambda \stackrel{\Delta}{=} \{\lambda \in E_{o(I)} \mid \mathcal{J}(\lambda) \text{ is not empty}\}$$

of the geometric dual feasible solution set T.

8. The main consequences. Primal problems A that exhibit minimally useful relations of the preceding type (between their corresponding dual problems B_o and B) can be characterized in the following way.

DEFINITION. Problem A is projectible from its geometric dual problem B to its ordinary dual problem B_o if

$$T_o = \Lambda$$

and

$$0 \equiv H_o(\lambda) + \psi(\lambda) \quad \text{for each} \quad \lambda \in T_o \ .$$

The preceding terminology is appropriate because the two defining relations simply assert that (the epigraph of) the negative $-H_o:T_o$ of the ordinary dual objective function $H_o:T_o$ is just an orthogonal projection of (the epigraph of) the geometric dual objective function $H:T$.

The following proposition provides, in the context of closed convex programming, a rather weak condition (involving relative interiors) that is sufficiently strong to guarantee the projectibility of problem A.

PROPOSITION I. *If*

(i) *both the functions* $g_k:C_k$, $k \in \{0\} \cup I \cup J$, *and the cone* X *are convex and closed,*

(ii) *there exists a vector* (x^o, κ^o) *such that*

(a) $$x^o \in (\text{ri } X),$$

(b) $$x^{ok} \in (\text{ri } C_k) \qquad k \in \{0\} \cup I,$$

(c) $$(x^{oj}, \kappa_j^o) \in (\text{ri } C_j^+) \qquad j \in J \ ,$$

then

(I) *problem* A *is projectible from its geometric dual problem* B *to its ordinary dual problem* B_o,

(II) *the (sub)optimal solution set* $\mathcal{J}^*(\lambda)$ *is not empty for each* $\lambda \in \Lambda$.

Proof. First, note from the theory of relative interiors that problem $\mathcal{A}(\lambda)$ has a feasible solution $(x^o, \kappa^o) \in \mathcal{S}$ for each λ. Then, note that conclusions (I) and (II) are implied by (in fact, are equivalent to) the statement

problem $\mathcal{A}(\lambda)$ has a finite infimum $H_o(\lambda)$ if and only if problem $\mathcal{B}(\lambda)$ has a feasible solution $y \in \mathcal{J}(\lambda)$, in which event $0 = H_o(\lambda) + \psi(\lambda)$ and $\mathcal{J}^*(\lambda) \neq \emptyset$.

Now, observe that the preceding statement is, in turn, implied by Corollary 3A on page 23 of [5] together with the (unstated) dual of (Fenchel's) Theorem 5 on page 26 of [5] (which is itself proved as Theorem 31.4 on page 335 of [8]). Consequently, we need only show that the hypotheses of that corollary and theorem are implied by the hypotheses of this proposition.

Toward that end, we first note that elementary (though tedious) considerations show that $\mathcal{g}(\cdot;\lambda):\mathcal{C}$ inherits the convexity and closedness of the $g_k:C_k$, $k \in \{0\} \cup I \cup J$, and that $\mathcal{X}$ inherits the convexity and closedness of X.

Finally, to show that $(x^o, \kappa^o) \in (\text{ri}\,\mathcal{X}) \cap (\text{ri}\,\mathcal{C})$, we first use the formulas for $\mathcal{X}$ and $\mathcal{C}$ to derive comparable formulas for $(\text{ri}\,\mathcal{X})$ and $(\text{ri}\,\mathcal{C})$--two derivations that make crucial use of the following basic facts:

(A) $(\text{ri } U) = U$ when U is a vector space,

(B) $(\text{ri } V) = \bigtimes_1^{\eta} (\text{ri } V_k)$ when $V = \bigtimes_1^{\eta} V_k$ and the sets V_k are convex.

Fact (A) is established on page 44 of [8], and fact (B) can be obtained inductively from the formula at the top of page 49 of [8].

Now, the formula for $\mathcal{X}$ along with facts (A) and (B) implies that

$$(\mathrm{ri}\ \mathcal{X}) = \{(x^0,x^I,x^J,\kappa) \in E_{\eta} \mid (x^0,x^I,x^J) \in (\mathrm{ri}\ X);\ \kappa \in E_{o(J)}\}\ .$$

Moreover, the formula for $\mathcal{C}$ along with facts (A) and (B) implies that

$$(\mathrm{ri}\ \mathcal{C}) = \{(x^0,x^I,x^J,\kappa) \in E_{\eta} \mid x^k \in (\mathrm{ri}\ C_k),\ k \in \{0\} \cup I;\ (x^j,\kappa_j) \in (\mathrm{ri}\ C_j^+),\ j \in J\}\ .$$

In particular then, the hypothesized vector $(x^\circ,\kappa^\circ) \in (\mathrm{ri}\,\mathcal{X}) \cap (\mathrm{ri}\ \mathcal{C})$.

q.e.d.

The following proposition brings to light the most significant implications of the projectibility of problem A.

PROPOSITION II. <u>If problem</u> A <u>is projectible from its geometric dual problem</u> B <u>to its ordinary dual problem</u> B_o, <u>then the ordinary dual supremum</u> ψ_o <u>is finite if and only if the geometric dual infimum</u> ψ <u>is finite; in which case</u>

$$0 = \psi_o + \psi$$

<u>and</u>

$$T_o^* \supseteq \{\lambda \in E_{o(I)} \mid (y,\lambda) \in T^* \text{ for some } y \in E_n\}\ ,$$

<u>with equality holding if and only if the (sub)optimal solution set</u> $\mathcal{J}^*(\lambda)$ <u>is not empty for each</u> $\lambda \in T_o^*$.

Proof. The defining equation for the set Λ and the defining relations for problems B and $\mathcal{B}(\lambda)$ readily imply that

$$\psi = \inf_{\lambda\in\Lambda} \psi(\lambda) \tag{1}$$

and that

$$T^* = \{(y,\lambda) \in D | \lambda \in \Lambda,\ \psi(\lambda) = \psi, \text{ and } y \in \mathcal{J}^*(\lambda)\} . \tag{2}$$

Now, in view of the defining relations for problem B_o, the projectibility of problem A obviously implies that

$$0 = \psi_o + \inf_{\lambda\in\Lambda} \psi(\lambda) \tag{3}$$

and that

$$T_o^* = \{\lambda \in \Lambda | \psi(\lambda) = \psi\}, \tag{4}$$

with equation (1) also having been used in the derivation of equation (4).

Clearly, the initial conclusions of the proposition are implied by equations (1) and (3), while the final conclusions are implied by equations (2) and (4). q.e.d.

Taken together, Propositions I and II have the following important corollary.

COROLLARY 1. *If the hypotheses of Proposition* I *are satisfied, then* ψ_o *is finite if and only if* ψ *is finite; in which case*

$$0 = \psi_o + \psi$$

and

$$T_o^* = \{\lambda \in E_{o(I)} \mid (y,\lambda) \in T^* \ \text{for some} \ y\}.$$

The scope of Corollary 1 can, of course, be inferred from an examination of the hypotheses of Proposition I. In particular, since it is widely known that neither ordinary duality nor geometric duality is of much significance in nonconvex programming, the convexity of both $g_k{:}C_k$, $k \in \{0\} \cup I \cup J$ and X is not an unreasonably strong assumption. Furthermore, since there seems to be no significant convex programming problems A involving either a nonclosed function $g_k{:}C_k$ or a nonclosed cone X, the closedness of both $g_k{:}C_k$, $k \in \{0\} \cup I \cup J$ and X is also not an unreasonably strong assumption. (Actually, the replacement of either a nonclosed $g_k{:}C_k$ or a nonclosed X by its "closure" has a known, usually minor, effect on the problem infimum φ and optimal solution set S^*.) Consequently, the true scope of Corollary 1 actually hinges on how frequently hypothesis (ii) of Proposition I is satisfied in the context of (closed) convex programming--a question that will now be examined.

For many important problems A, the cone X is in fact a vector space; in which case fact (A) asserts that $(\text{ri } X) = X$. Hence, to treat such problems, it is convenient to replace condition (a) in hypothesis (ii) with the condition (a') $\quad X$ <u>is a vector space, and</u> $x^o \in X$,

which clearly does not disturb the validity of Proposition I and Corollary 1.

For many important problems A, the set C_k is the whole vector space E_{n_k}, $k \in \{0\} \cup I$; in which case fact (A) asserts that $(\text{ri } C_k) = E_{n_k}$, $k \in \{0\} \cup I$. Hence, to treat such problems, it is convenient to replace condition (b) in hypothesis (ii) with the condition

(b') $\quad C_k = E_{n_k} \qquad k \in \{0\} \cup I$,

which clearly does not disturb the validity of Proposition I and Corollary 1.

For many important problems A, the index set J is empty; in which case the condition $(x^{\circ j}, \overset{\circ}{\kappa}_j) \in (\text{ri } C_j^+)$, $j \in J$ is vacuously satisfied. Hence, to treat such problems, it is convenient to replace condition (c) in hypothesis (ii) with the condition

(c') $\quad J$ <u>is empty</u>,

which clearly does not disturb the validity of Proposition I and Corollary 1.

For many important problems A, conditions (a'), (b') and (c') are all satisfied; in which case the vector $x^{\circ} \triangleq 0$ obviously satisfies conditions (a), (b) and (c) in hypothesis (ii). Hence, to treat such problems, it is convenient to replace hypothesis (ii) with the hypothesis (ii') X *is a vector space*; $C_k = E_{n_k}$, $k \in \{0\} \cup I$; *and* J *is empty*, which does not, of course, disturb the validity of Proposition I and Corollary 1.

Some very important problems A (discussed in [5] and the references cited therein) that obviously satisfy hypothesis (ii') are: posynomial programming problems, quadratic programming problems (with either linear or quadratic constraints), linear regression problems (with constraints that bound norms), and optimal location problems. Although the most general ordinary programming problem (example 8 on page 12 of [5]) does not generally satisfy hypothesis (ii'), it is not difficult to see that it does satisfy the original hypothesis (ii).

9. Some important implications. We have just observed that the conclusion to Corollary 1 is valid for many (if not all) convex programming problems of interest. For all such problems A, the corresponding ordinary dual problem B_o can be obtained by orthogonally projecting the corresponding geometric dual problem B via a suboptimization--a property that endows geometric duality with the following strong advantages over ordinary duality.

For many important problems A (including all posynomial programming problems, all quadratic programming problems, all linear regression problems, and all optimal location problems), the corresponding geometric dual problem B can be expressed in terms of formulas that are as elementary as the formulas expressing the primal problem A. The fact that the corresponding ordinary dual problem B_o almost never has such an elementary representation is just a reflection of the fact that any suboptimization of a function represented in terms of elementary formulas rarely produces a function that can be represented in terms of elementary formulas.

Of course, the geometric dual problem B has an independent vector variable y that is not present in the corresponding ordinary dual problem B_o. However, there is no a priori reason why minimizing $H(y,\lambda)$ over y should be any more difficult than minimizing $L_o(x,\kappa;\lambda)$ over (x,κ). More importantly, there is also no a priori reason why minimizing $H(y,\lambda)$ over y and λ jointly should be any more difficult than first minimizing $L_o(x,\kappa;\lambda)$ over (x,κ) and then maximizing the result over λ; in fact, the latter maximinimization looks much more formidable than the former joint minimization.

Finally, the geometric dual problem B sensitizes more parameters in its primal problem A than does the corresponding ordinary dual problem B_o. As indicated in sections 3.1.5 and 3.3.5 of [5], this fact makes geometric duality more powerful than ordinary duality for parametric programming and post-optimality analysis.

In concluding, it is worth mentioning that the preceding duality between suboptimization and parameter deletion is generalized and more thoroughly studied in [7].

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