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Discussion Paper 262

December, 1976

"Best Replay Strategies in the Champsaur-Dreze-Henry Procedure"

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Mathematical Studies
in Economics &
Management Sciences

DISCUSSION PAPER NO. 262

BEST REPLAY STRATEGIES IN THE CHAMPSAUR - DREZE - HENRY PROCEDURE

by

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ABSTRACT

This paper studies the properties of the discrete-time procedure of Champsaur, Drèze and Henry when the behavioral assumption on the consumers is changed. The properties of the procedure when consumers are assumed to behave competitively are similar to those studied by Champsaur, Drèze and Henry who assume that consumers report their preferences correctly.

BEST REPLAY STRATEGIES IN THE CHAMPSAUR-DREZE-HENRY PROCEDURE *

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1. Introduction

Champsaur, Drèze and Henry ([1], Section 4) have introduced a discrete-time procedure (the CDH procedure) for the allocation of resources in an economy with several public goods and one private good. Under this procedure, a Planning Board (P.B.) asks consumers about their preferences and producers about their costs; it then changes the allocation of resources on the basis of this information, according to certain specified rules. Under classical assumptions and provided that the consumers and producers report correct information on preferences and costs, the procedure will lead the economy from any given initial allocation to a Pareto optimum.

In this paper we are concerned with the performance of a slight variant of the CDH procedure in which consumers are no longer assumed to report truthfully but rather to behave in a non-cooperative, self-interested way: taking all relevant information as given, each consumer is assumed to send messages to the P.B. so that given the announcements of all agents the resulting change in allocation under the rules of the procedure is the one most favored by this consumer.

The formal modelling is that of a non-cooperative game with the announced preferences of the consumers as their strategy choices and with the utility to each consumer of the new allocation as pay-off.

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* I would like to thank Claude Henry, John Roberts and Henry Tulkens for their many helpful comments on earlier drafts of this paper. This research was supported by the Fonds National de la Recherche Scientifique of Belgium while I was at CORE and by the National Science Foundation under a grant to the Center for Mathematical Studies in Economics and Management Science at Northwestern University.

The principal results of this paper are the following:

(i) given a minor assumption on the honesty of consumers, it is shown that at a Nash equilibrium, a change in allocation occurs if and only if it is possible when consumers report truthfully.

(ii) Nash equilibria exist in all relevant cases.

From these results and those of Champsaur, Dreze and Henry it follows that even with misrepresentation of preferences the procedure still converges to a Pareto optimum.

2. The procedure and its properties.

The economy has n consumers $(i=1,\ldots,n)$, K public goods x_k $(k=1,\ldots,K)$ and one private good y.

Consumers' preferences can be represented by strictly quasi-concave utility functions U_i and the production function $f\colon R^K \to R$ is concave and continuously differentiable. A program $(x,y) = (x_1,\ldots,x_K,y_1,\ldots,y_n)$ is feasible if and only if

$$\Sigma_i\, y_i \leq f(x)$$

Suppose the allocation is $(x,y) = (x_1,\ldots,\, x_K,\, y_1,\ldots,\, y_n)$ and the P.B. is considering changing the level of one public good by an amount a. Under the CDH procedure, to decide which good to change and in what direction (increase or decrease) the P.B. asks the agents two types of questions. To each consumer i, for each good k it asks:

1. How much, in terms of the numeraire private good, is consumer i willing to pay to enjoy a level $x_k^+ = (x_k + a)$ of public good k?

2. How much compensation (again in terms of the private good) would i require to be as well off with $x_k^- = (x_k - a)$ of public good k?

Consumer i's truthful answers to these questions are π_k^{i+} and π_k^{i-}, respectively.

π_k^{i+} is defined as $\pi_k^{i+} = y_i - \bar{y}_i$ where $\bar{y}_i$ is such that

$$U_i(x_1,\ldots,x_k^+,\ldots,x_K,\ \bar{y}_i) = U_i(x_1,\ldots,x_k,\ldots,x_K,\ y_i)$$

and similarly $\pi_k^{i-} = \bar{\bar{y}}_i - y_i$ where $\bar{\bar{y}}_i$, if such a finite value does exist, is such that

$$U_i(x_1,\ldots,x_k^-,\ldots,\bar{\bar{y}}_i) = U_i(x_1,\ldots,x_k,\ldots,x_K,y_i)$$

To the producer of public goods, two similar questions are asked:

(i) what is the minimum cost to increase the level of production of public good k by an amount a?

(ii) What is the maximum amount saved by decreasing the level of production of public good k by an amount a?

The answers to these questions are respectively defined to be γ_k^+ and γ_k^-.

Then (see [1]) each good k falls in only one of the three following categories:

(1) $\Sigma_i \pi_k^{i+} - \gamma_k^+ > 0$

(2) $\gamma_k^- - \Sigma_i \pi_k^{i-} > 0$

(3) $\Sigma_i \pi_k^{i+} - \gamma_k^+ \leq 0$ and $\gamma_k^- - \Sigma_i \pi_k^{i-} \leq 0$

The P.B. chooses any good in (1) or (2) and adjusts the level of the public good accordingly. If k is such that $\Sigma_i \pi_k^{i+} - \gamma_k^+ > 0$ then the level of the good is increased to $x_k^+ = (x_k + a)$ and each consumer i is required to pay π_k^{i+} to cover the cost. Moreover he receives a share δ^i of the social surplus $(\Sigma_j \pi_k^{j+} - \gamma_k^+)$. Thus he is left with an amount of private good given by:

$$y_i^1 = y_i - \pi_k^{i+} + \delta^i(\Sigma_j \pi_k^{j+} - \gamma_k^+), \quad \text{where} \quad \delta^i > 0, \ \Sigma_i \delta^i = 1.$$

On the other hand, if k is such that $\gamma_k^- - \Sigma_i \pi_k^{i-} > 0$ then the new level of that good is $x_k^- = (x_k - a)$ and each consumer now has y_i^2 of private good:

$$y_i^2 = y_i + \pi_k^{i-} + \delta^i(\gamma_k^- - \Sigma_j \pi_k^{j-})$$

It is easily seen that along this procedure all utilities increase.

If there are no public goods in categories (1) and (2), the rules of the procedure are that at the next iteration the allocation is not changed but the step size is divided by two.

The procedure leads the economy from any feasible initial allocation $(x_1,\ldots,x_K,y_1,\ldots,y_n)$ to a Pareto optimum.

In our analysis costs are still assumed to be reported truthfully, so producers announce the true γ_k^+ and γ_k^-, but consumers announce benefits ψ_k^{i+} and compensations ψ_k^{i-} that are not necessarily equal to their true values π_k^{i+} and π_k^{i-}.

The procedure we consider is slightly different from that described by Champsaur, Dreze and Henry: instead of requiring strict positivity of the excess of benefits over costs $(\Sigma_i \pi_k^{i+} - \gamma_k^+ > 0)$ or of savings in cost over compensations $(\gamma_k^- - \Sigma_i \pi_k^{i-})$ in order to adjust the amount of good k only non-negativity will be required.

If $\Sigma_i \psi^{i+}_k - \gamma^+_k \geq 0$ then good k may be increased

if $\gamma^-_k - \Sigma_i \psi^{i-}_k \geq 0$ then good k may be decreased

This modification is needed because with the original rule ($\Sigma_i \pi^{i+}_k - \gamma^+_k > 0$ or $\gamma^-_k - \Sigma_i \pi^{i-}_k > 0$), lack of compactness of the strategy sets means it is not possible to define the best-replay strategy of a consumer (see section III-2). An alternative solution to the problem would have been to require that $\Sigma_i \pi^{i+}_k - \gamma^+_k \geq \varepsilon$ or $\gamma^-_k - \Sigma_i \pi^{i-}_k \geq \varepsilon$, with ε a fixed, positive value given a priori, would be needed to change the level of public good k. This alternative could slow down the convergence of the procedure and it would have caused problems when one is close to an optimum, unless maybe one specified that the rule would decrease ε when the step a is decreased. The solution adopted here is clearly more straightforward although it can lead to a problem of oscillation: if both $\Sigma_i \psi^{i+}_k - \gamma^+_k$ and $\gamma^-_k - \Sigma_i \psi^{i-}_k$ are equal to zero for some k, this good could be increased then decreased by the amount of the step. This situation, as we shall see below, could happen only if $\Sigma_i \pi^{i+}_k - \gamma^+_k$ and $\gamma^-_k - \Sigma_i \pi^{i-}_k$ are both equal to zero. To avoid oscillations one can specify that an increase of a good by the amount a cannot be followed by a decrease of that same good by the same step size.

III. Best replay strategies.

We shall no longer assume that consumers report benefits or compensations truthfully. Instead we shall suppose that consumer i, given ψ^{j+}_k and ψ^{j-}_k, $j=1,\ldots,n$, $j\neq i$, $k=1,\ldots,K$, and given the cost variables γ^+_k and γ^-_k for each k, will announce any ψ^{i+}_k's and

ψ^{i-}_{k}'s that maximize the utility to him of the resulting change in allocation.

III.1. Direction of misrepresentation

Lemma 1. Given the ψ^{j+}_{k}'s and ψ^{j-}_{k}'s of the other consumers and γ^{+}_{k} and γ^{-}_{k}, $k = 1,\ldots,k$, consumer i underreports a benefit and overreports a compensation.

This result is important because:

(1) it means that even with misrepresentation the P.B. will never change the allocation of resources in the wrong direction. The worst that can happen is that the allocation is not changed.

(2) it also means that the rules of the procedure remain consistent: no good can be simultaneously increased and decreased.

Proof

(a) $\psi^{i+}_{k} \leq \pi^{i+}_{k}$. Suppose, to the contrary, that $\psi^{i+}_{k} = \pi^{i+}_{k} + \varepsilon$, $\varepsilon > 0$.

Only one of the following situations can occur: 1. The overreporting does not change the decision, the good can be increased if consumer i reports only π^{i+}_{k}; 2. The overreporting does change the decision; 3. The overreporting is not enough to change the decision.

1. $\Sigma_{j\neq i}\, \psi^{j+}_{k} + \pi^{i+}_{k} - \gamma^{+}_{k} \geq 0$ then a fortiori, $\Sigma_{j\neq i}\psi^{j+}_{k} + \psi^{i+}_{k} - \gamma^{+}_{k} \geq 0.$

In this case if the consumer had not overreported the good k could nonetheless have been increased. The overreporting hurts the consumer because by announcing $\psi^{i+}_{k} = \pi^{i+}_{k} + \varepsilon$ he is left with strictly less private good, and no more public goods. Indeed:

$$y_i^1 = y_i - \Psi_k^{i+} + \delta^i(\Sigma_j \Psi_k^{j+} - \gamma_k^+)$$

$$= y_i - \pi_k^{i+} - \varepsilon + \delta^i(\Sigma_{j\neq i}\Psi_k^{j+} + \pi_k^{i+} + \varepsilon - \gamma_k^+)$$

$$= y_i - \pi_k^{i+} + \delta^i(\Sigma_{j\neq i}\Psi_k^{j+} + \pi_k^{i+} - \gamma_k^+) - \varepsilon(1 - \delta^i)$$

$$< y_i - \pi_k^{i+} + \delta^i(\Sigma_{j\neq i}\Psi_k^{j+} + \pi_k^{i+} - \gamma_k^+) \quad \text{since} \quad \varepsilon(1-\delta^i) > 0$$

2. $\Sigma_{j\neq i}\Psi_k^{j+} + \pi_k^{i+} - \gamma_k^+ < 0$ but $\Sigma_{j\neq i}\Psi_k^{j+} + \Psi_k^{i+} - \gamma_k^+ \geq 0$

Here the good is increased because the consumer overreports . This consumer at the new allocation is left with:

$$y_i^1 = y_i - \Psi_k^{i+} + \delta^i(\Sigma_{j\neq i} \Psi_k^{j+} + \Psi_k^{i+} - \gamma_k^+)$$

$$= y_i - \pi_k^{i+} - \varepsilon + \delta^i(\Sigma_{j\neq i}\Psi_k^{j+} + \pi_k^{i+} + \varepsilon - \gamma_k^+)$$

$$= y_i - \pi_k^{i+} + \delta^i(\Sigma_{j\neq i}\Psi_k^{j+} + \pi_k^{i+} - \gamma_k^+) - \varepsilon(1 - \delta^i)$$

$$< y_i - \pi_k^{i+}$$

by definition of π_k^{i+} this amount $y_i - \pi_k^{i+}$ is the quantity of private good that consumer i wants to have in association with the new x_k^+ to be as well off at the new allocation as he was at the initial situation. If in this case the consumer overreports he will have x_k^+ of public good k but strictly less private good than $y_i - \pi_k^{i+}$. Thus he loses utility by overreporting.

3. $\Sigma_{j\neq i}\Psi_k^{j+} + \pi_k^{i+} - \gamma_k^+ < 0$ and $\Sigma_{j\neq i}\Psi_k^{j+} + \Psi_k^{i+} - \gamma_k^+ < 0$

In this case, the allocation will not change ; the consumer therefore stays at his initial utility level but does not gain from overreporting.

(b) $\Psi^{i-}_{k} \geq \pi^{i-}_{k}$ again assume a contrario that $\Psi^{i-}_{k} = \pi^{i-}_{k} - \varepsilon$; $\varepsilon > 0$.

If the allocation is changed consumer i will have an amount y_i^2 of private good where y_i^2 is given by

$$y_i^2 = y_i + \Psi^{i+}_{k} + \delta^i(\gamma^-_k - \Sigma_j \Psi^{j-}_{k})$$

$$= y_i + \pi^{i+}_{k} + \delta^i(\gamma^-_k - \Sigma_{j\neq i}\Psi^{j-}_{k} - \pi^{i-}_{k}) - \varepsilon(1 - \delta^i)$$

The same reasoning as above shows that underreporting a compensation makes the consumer worse off.

III. 2. The one public good case.

With $K = 1$ we can drop the index k in our notation.

Lemma 2 If, when his announcement does not influence the change in allocation, the consumer reports truthfully, his best replay strategies are:

$$\Psi^{i+} = \max\ [0, \min\ (\pi^{i+}, \gamma^+ - \Sigma_{j\neq i}\Psi^{j+})]$$

$$\Psi^{i-} = \max\ [\pi^{i-}, \gamma^- - \Sigma_{j\neq i}\Psi^{j-}]$$

Proof

In choosing Ψ^{i+} consumer i can be in one of **three** situations:

(1) $\Sigma_{j\neq i} \Psi^{j+} - \gamma^+ \geq 0$

(2) $\Sigma_{j\neq i} \Psi^{j+} - \gamma^+ \leq 0$ and $\Sigma_{j\neq i}\Psi^{j+} + \pi^{i+} - \gamma^+ \geq 0$

(3) $\Sigma_{j\neq i} \Psi^{j+} - \gamma^+ \leq 0$ and $\Sigma_{j\neq i}\Psi^{j+} + \pi^{i+} - \gamma^+ \leq 0$

We shall consider each case separately:

(1) Here consumer i can have the increase in the public good even if he

volunteers to pay nothing. If he were to announce $\psi^{i+} > 0$ he would have the same amount of public good but $(1-\delta^i)\ \psi^{i+}$ (>0) less of private good. Clearly his best replay is:

$$\psi^{i+*} = 0 = \max[0, \min(\pi^{i+}, \gamma^{+} - \Sigma_{j\neq i}\psi^{j+})]$$

since in this case $\min(\pi^{i+}, \gamma^{+} - \Sigma_{j\neq i}\psi^{j+}) = \gamma^{+} - \Sigma_{j\neq i}\psi^{j+} < 0$

(2) If the consumer reports truthfully, the good can be increased. But this remains true if he announces $\psi^{i+} = \gamma^{+} - \Sigma_{j\neq i}\psi^{j+}$ that is if he volunteers to pay exactly enough to cover that part of the cost of increasing the public good, which is not covered by the other consumers' announcements. As above, if he announces a payment greater than $\gamma^{+} - \Sigma_{j\neq i}\psi^{j+}$ he will have no more public good but less private good.

Therefore

$$\psi^{i+*} = \gamma^{+} - \Sigma_{j\neq i}\psi^{j+} = \max[0, \min(\pi^{i+}, \gamma^{+} - \Sigma_{j\neq i}\psi^{j+})]$$

$$\text{since} \quad \gamma^{+} - \Sigma_{j\neq i}\psi^{j+} \leq \pi^{i+} \quad \text{because} \quad \Sigma_{j\neq i}\psi^{j+} + \pi^{i+} - \gamma^{+} \geq 0$$

(3) $\Sigma_{j\neq i}\psi^{j+} - \gamma^{+} < 0$ and $\Sigma_{j\neq i}\psi^{j+} + \pi^{i+} - \gamma^{+} < 0$

Lemma 1 states that the consumer would not overreport a benefit. So in this situation there is no announcement by consumer i that would make the increase in the public good possible. We have assumed that in this case the consumer reports truthfully.

$$\psi^{i+*} = \pi^{i+} = \max[0, \min(\pi^{i+}, \gamma^{+} - \Sigma_{j\neq i}\psi^{j+})]$$

A similar argument shows that when

$$\gamma^{-} - \Sigma_{j\neq i}\psi^{j-} - \pi^{i-} \geq 0$$

Consumer i should ask for a compensation equal to $\gamma^{-} - \Sigma_{j\neq i}\psi^{j-}$. And if

$$\gamma^{-} - \Sigma_{j\neq i}\psi^{j-} - \pi^{i-} < 0$$

his announcement does not change the decision, so he says the truth.

Therefore:

$$\psi^{i-*} = \max[\pi^{i-}, \gamma^{-} - \Sigma_{j\neq i}\psi^{j-}]$$

III. 3. The case with several public goods

As in the one public good case, the i-th consumer will choose his best replay strategy by looking at the sign of:

$$\Sigma_{j\neq i}\psi^{j+}_{k} + \pi^{i+}_{k} - \gamma^{+}_{k} \quad \text{and} \quad \gamma^{-}_{k} - \Sigma_{j\neq i}\psi^{j-}_{k} - \pi^{i-}_{k}$$

For goods such that $\Sigma_{j\neq i}\psi^{j+}_{k} + \pi^{i+}_{k} - \gamma^{+}_{k} < 0$ he will announce $\psi^{i+*}_{k} = \pi^{i+}_{k}$ because we assume again that he says the truth when his announcement cannot change the decision.

Similarly if $\gamma^{-}_{k} - \Sigma_{j}\psi^{j-}_{k} - \pi^{i-}_{k} < 0$ he will announce $\psi^{i-*}_{k} = \pi^{i-}_{k}$.

If there are goods such that $\Sigma_{j\neq i}\psi^{j+}_{k} - \gamma^{+}_{k} \geq 0$ he will volunteer to pay nothing $(\psi^{i+*} = 0)$ for the same reason as in the one public good case.

The difference between the one public good and the several public goods case appears when there are one or more public goods h for which

$$\Sigma_{j\neq i}\psi^{j+}_{h} - \gamma^{+}_{h} < 0 \quad \text{and} \quad \Sigma_{j\neq i}\psi^{j+}_{h} + \pi^{i+}_{h} - \gamma^{+}_{h} \geq 0$$

and/or one or more goods ℓ such that

$$\gamma_\ell^- - \Sigma_{j\neq i}\psi_\ell^{j-} - \pi_\ell^{i-} \geq 0$$

because in this case, where consumer i knows that he can make several changes possible, he can in fact choose the change in allocation he prefers.

To find out which change he prefers consumer i computes:

for each h $\Phi_h^{i+} = \gamma_h^+ - \Sigma_{j\neq i}\psi_h^{j+}$ and $u_i(h) = u_i(x_1,\ldots,x_h^+,\ldots,x_K,y_i - \Phi_h^{i+})$

for each ℓ $\Phi_\ell^{i-} = \gamma_\ell^- - \Sigma_{j\neq i}\psi_\ell^{j-}$ and $u_i(\ell) = u_i(x_1,\ldots,x_\ell^-,\ldots,x_K,y_i + \Phi_\ell^{i-})$

He prefers the increase in the level of public good h' if

$$u_i(h') \geq u_i(h) \quad \text{for all } h$$
$$\geq u_i(\ell) \quad \text{for all } \ell$$

On the other hand he prefers the decrease in the level of public good ℓ' if

$$u_i(\ell') \geq u_i(h) \quad \text{for all } h$$
$$\geq u_i(\ell) \quad \text{for all } \ell$$

<u>Theorem 1</u> If consumers report truthfully when their announcements do not change the decision, consumer i's best replay strategy is:

(1) if the change in allocation he prefers is the increase in public good h':

$$\psi_k^{i+} = \pi_k^{i+} \qquad \text{if} \quad \Sigma_{j\neq i}\psi_k^{j+} + \pi_k^{i+} - \gamma_k^+ < 0$$

$$\psi_{h'}^{i+} = \gamma_{h'}^+ - \Sigma_{j\neq i}\psi_{h'}^{j+}$$

$$\psi_h^{i+} = 0 \qquad h\neq h' \text{ and } \Sigma_{j\neq i}\psi_h^{j+} + \pi_h^{i+} - \gamma_h^+ \geq 0$$

$$\psi_k^{i-} = \pi_k^{i-} \qquad \text{if} \quad \gamma_k^- - \Sigma_{j\neq i}\psi_k^{j-} < 0$$

$$\psi_\ell^{i-} = \gamma_\ell^- + \varepsilon \qquad \text{if} \quad \gamma_\ell^- - \Sigma_{j\neq i}\psi_\ell^{j-} - \pi_\ell^{i-} \geq 0 \ ; \ \varepsilon > 0$$

(2) If the change in allocation he prefers is the decrease in the level of public good ℓ':

$$\psi_k^{i+} = \pi_k^{i+} \qquad \text{if} \quad \Sigma_{j\neq i}\psi_k^{j+} + \pi_k^{i+} - \gamma_k^+ < 0$$

$$\psi_h^{i+} = 0 \qquad \text{if} \quad \Sigma_{j\neq i}\psi_h^{j+} + \pi_h^{i+} - \gamma_h^+ \geq 0$$

$$\psi_k^{i-} = \pi_k^{i-} \qquad \text{if} \quad \gamma_k^- - \Sigma_{j\neq i}\psi_k^{j-} - \pi_k^{i-} < 0$$

$$\psi_{\ell'}^{i-} = \gamma_{\ell'}^- - \Sigma_{j\neq i}\psi_{\ell'}^{j-}$$

$$\psi_\ell^{i-} = \gamma_\ell^- + \varepsilon \quad \text{if} \quad \gamma_\ell^- - \Sigma_{j\neq i}\psi_\ell^{j-} - \pi_\ell^{i-} \geq 0 \quad \text{and} \quad \ell \neq \ell'; \ \varepsilon > 0 .$$

Proof

(1) For goods k consumer i's announcement does not alter the decision, so, by assumption, he says the truth.

For goods $h \neq h'$ consumer i wants to make the increases in these goods impossible. Two cases can occur:

(i) $\Sigma_{j\neq i}\psi_h^{j+} - \gamma_h^+ > 0$; since consumer i cannot announce a negative benefit, he cannot prevent this change from being possible. His best replay is nonetheless (see above) $\psi_h^{i+} = 0$.

(ii) $\Sigma_{j\neq i}\psi_h^{j+} - \gamma_h^+ < 0$; here consumer i has to announce a positive benefit to make this change possible. By announcing $\psi_h^{i+} = 0$ he makes the sum of benefits less than the cost.

For goods ℓ, by demanding a sufficiently high compensation he can always make $\gamma_\ell^- - \Sigma_j\psi_\ell^{j-}$ negative. By announcing $\psi_\ell^{i-} = \gamma_\ell^- + \varepsilon$ he makes $\gamma_\ell^- - \Sigma_j\psi_\ell^{j-}$ negative even in the exceptional case where $\Sigma_{j\neq i}\psi_\ell^{j+} = 0$.

(2) The proof is similar to the one of (1) and will be omitted.

To summarize consumer i's best choice of Ψ_k^{i+} and Ψ_k^{i-} is as follows:

1. If this consumer can change no decision he should announce any benefit less or equal to his true one and any compensation greater or equal to his true one. We will assume he chooses his true benefits and his true compensations.

2. If one single change in allocation can result from an appropriate choice of consumer i's announcements, if it is an increase in the level of a public good, he should announce a benefit just sufficient to cover the cost. If on the other hand this one change is a decrease, he should ask for a compensation that equals the excess of savings in costs over the demanded compensations by other consumers.

3. If there are more than one change feasible for consumer i he should consider each change individually and then compute how much private good he would have at every possible new allocation. Then he must decide which new allocation he prefers. For the public good chosen in this way consumer i proceeds as in 2. For the other goods he must announce benefits such that total benefits over all consumers are less than costs, if such a choice is possible. If not, he announces a zero benefit. The compensation he announces for the other goods that could be decreased are such that these changes become impossible.

IV. Nash equilibria

A Nash equilibrium is a set of announcements Ψ_k^{i+*} and Ψ_k^{i-*}, $i=1,\ldots,n$, $k=1,\ldots,K$, such that for each i given that the other consumers have announced ψ_k^{j+*} and ψ_k^{j-*}, $j \neq i$, consumer i's best replay strategy is to announce precisely these values ψ_k^{i+*} and ψ_k^{i-*}, $k=1,\ldots,K$.

IV.1. The one public good case

IV.1.a: If $\Sigma_i \pi^{i+} - \gamma^+ < 0$ and $\gamma^- - \Sigma_i \pi^{i-} < 0$ then

$$\forall\ i \quad \psi^{i+*} = \pi^{i+} \text{ and } \psi^{i-*} = \pi^{i-}$$ is a Nash equilibrium set of strategies. The proof is trivial.

IV.1.b Lemma 3

(i) If $\Sigma_i \pi^{i+} - \gamma^+ \geq 0$ then for all i:

$$\psi^{i+*} = \pi^{i+} - a_i ;\ a_i \geq 0 \quad \pi^{i+} - a_i \geq 0$$

$$\Sigma_i a_i = \Sigma_i \pi^{i+} - \gamma^+$$

$$\psi^{i-*} = \pi^{i-}$$

is a Nash equilibrium set of strategies.

(ii) If $\gamma^- - \Sigma_i \pi^{i-} \geq 0$, then for all i:

$$\psi^{i+*} = \pi^{i+}$$

$$\psi^{i-*} = \pi^{i-} + a_i \quad a_i \geq 0 \quad \Sigma_i a_i = \gamma^- - \Sigma_i \pi^{i-}$$

is a Nash equilibrium set of strategies.

Proof

(i) Consumer i, to choose his best replay strategy, will study the signs of $\Sigma_{j\neq i} \psi^{j+*} + \pi^{i+} - \gamma^+$ and $\gamma^- - \Sigma_{j\neq i} \psi^{j-*} - \pi^{i-}$

$$\Sigma_{j\neq i} \psi^{j+*} + \pi^{i+} - \gamma^+ = \Sigma_{j\neq i} (\pi^{j+} - a_j) + \pi^{i+} - \gamma^+$$

$$= \Sigma_j \pi^{j+} - \Sigma_{j\neq i} a_j - \gamma^+$$

$$\text{but } \Sigma_{j\neq i} a_j = \Sigma_j \pi^{j+} - \gamma^+ - a_i$$

$$= \Sigma_j \pi^{j+} - \Sigma_j \pi^{j+} + \gamma^+ + a_i - \gamma^+$$

$$= a_i$$

$$\geq 0$$

$$\gamma^- - \Sigma_{j \neq i} \psi^{j-*} - \pi^{i-} = \gamma^- - \Sigma_{j \neq i} \pi^{j-} - \pi^{i-}$$

$$< 0$$

Therefore, consumer i's best replay is:

$$\psi^{i+} = \gamma^+ - \Sigma_{j \neq i} \psi^{j+*} = \gamma^+ - \Sigma_{j \neq i} \pi^{j+} + \Sigma_{j \neq i} a_j$$

$$= \gamma^+ - \Sigma_{j \neq i} \pi^{j+} + \Sigma_{j \neq i} \pi^{j+} + \pi^{i+} - \gamma^+ - a_i$$

$$= \pi^{i+} - a_i$$

$$= \psi^{i+*}$$

$$\psi^{i-} = \pi^{i-} = \psi^{i-*}$$

(ii) The proof is similar to the one above.

Note The a_i as they have been defined above always exists. They are just a way of sharing the surplus.

IV. 2. The case with several public goods.

If for all k $\Sigma_j \pi_k^{j+} - \gamma_k^+ < 0$ and $\gamma^- - \Sigma_J \pi_k^{j-} < 0$ there is no difference between the one public good case and the several public goods case.

Similarly if only one change in allocation is possible then consumers will report truthfully for all other goods and the Nash equilibrium strategy for the one good that can be increased or decreased is the same as in IV.1.b.

If several changes in allocation are possible at an iteration, the characterization of the Nash equilibrium strategies becomes more complicated.

Theorem 2. Index by $h \in H \subset \{1,\ldots,K\}$ the public goods such that

$$\Sigma_j \pi^{j+}_h - \gamma^+_h \geq 0$$

by $\ell \in L \subset \{1,\ldots,K\}$ those for which

$$\gamma^-_k - \Sigma_j \pi^{j-}_k \geq 0$$

1. If there exists $a_{ih} \geq 0$; $\pi^{i+}_h - a_{ih} \geq 0$; $\Sigma_i a_{ih} = \Sigma_i \pi^{i+}_h - \gamma^+_h$ and

   (i) a set of consumers $J(h')$, $h' \neq h$, $h' \in H$ such that

   (a) $\Sigma_{j \in J(h')} \pi^{j+}_{h'} - \gamma^+_{h'} < 0$ and $\forall\, i \notin J(h')$

$$\Sigma_{j \in J(h')} \pi^{j+}_{h'} + \pi^{i+}_{h'} - \gamma^+_{h'} \geq 0$$

   (b) $\forall\, i \notin J(h')$

$$U_i(x_1,\ldots,x^+_h,\ldots,y_i - \pi^{i+}_h + a_{ih})$$

$$\geq U_i(x_1,\ldots,x^+_{h'},\ldots,y_i - (\gamma^-_h - \Sigma_{j \in J(h')} \pi^{j+}_{h'}))$$

   (ii) for all $\ell \in L$ there is a consumer i' for whom

$$U_{i'}(x_1,\ldots,x^+_h,\ldots,y_{i'} - \pi^{i'+}_h + a_{i'h})$$

$$\geq U_{i'}(x_1,\ldots,x^-_\ell,\ldots,y_{i'} + (\gamma^-_\ell - \Sigma_{j \neq i} \pi^{j-}_\ell))$$

Then there exists a Nash equilibrium set of strategies given by:

for $k \notin H$	for $i=1,\ldots,n$	$\psi^{i+*}_k = \pi^{i+}_k$
for $h' \in H$ $h' \neq h$	for $j \in J(h')$	$\psi^{j+*}_{h'} = \pi^{i+}_{h'}$
	for $i \notin J(h')$	$\psi^{i+*}_k = 0$
for h	for $i=1,\ldots,n$	$\psi^{i+*}_h = \pi^{i+}_h - a_{ih}$
for $k \notin h$	for $i=1,\ldots,n$	$\psi^{i-*}_k = \pi^{i-}_k$
for $\ell \in L$	for $i=1,\ldots,n$ $i \neq i'$	$\psi^{i-*}_\ell = \pi^{i-}_\ell$
	for i'	$\psi^{i'-*}_\ell = \gamma^-_\ell + \epsilon$

2. If there exists $a_{i\ell} \geq 0 \quad \Sigma_i a_{i\ell} = \gamma_\ell^- - \Sigma_j \pi_\ell^{j-}$

and

(i) there exists $J(h)$ for all $h \in H$ such that

(a) as above is satisfied

(b) for all $i \notin J(h)$

$$U_i(x_1,\ldots,x_\ell^-,\ldots,y_i + \pi_\ell^{i-} + a_{i\ell})$$
$$\geq U_i(x_1,\ldots,x_h^+,\ldots,y_i - (\gamma_\ell^+ - \Sigma_{j\in J(h)}\pi_h^{j+}))$$

(ii) for all $\ell' \neq \ell$ there is a consumer i' such that:

$$U_{i'}(x_1,\ldots,x_\ell^-,\ldots,y_{i'} + \pi_\ell^{i-} + a_{i'\ell})$$
$$\geq U_{i'}(x_1,\ldots,x_{\ell'}^-,\ldots,y_{i'} - (\gamma_{\ell'}^- - \Sigma_{j\neq i'}\pi_{\ell'}^{j-}))$$

Then there exists a Nash equilibrium characterized by:

for $k \notin H$	for $i=1,\ldots,n$	$\psi_k^{i+*} = \pi_k^{i+}$
for $h \in H$	for $j \in J(h)$	$\psi_h^{j+*} = \pi_h^{i+}$
	$i \notin J(h)$	$\psi_h^{i+*} = 0$
for $k \notin L$	for $i=1,\ldots,n$	$\psi_k^{i-*} = \pi_k^{i-}$
for ℓ	for $i=1,\ldots,n$	$\psi_\ell^{i-*} = \pi_\ell^{i-} - a_{i\ell}$
for $\ell' \in L \quad \ell' \neq \ell$	for $i=1,\ldots,n \quad i\neq i'$	$\psi_{\ell'}^{i-*} = \pi_{\ell'}^{i-}$
	i'	$\psi_{\ell'}^{i'-*} = \gamma_{\ell'}^- + \varepsilon.$

In words, this means that there is, in case (1) at least one good h such that, for any other possible increase of a public good ($h' \in H$, $h' \neq h$), there are enough consumers ($i \notin J(h')$) who prefer the change in h and can prevent the change in h', even when the others ($j \in J(h')$) report their benefits truthfully. For any possible decrease ($\ell \in L$), there is always one consumer at least who prefers the increase in public good h to the decrease in public good ℓ. The situation is not symmetrical because one consumer can always prevent a decrease by asking for a sufficiently large compensation while for an increase, a single consumer, because he is limited to saying a non negative benefit, may not be able to prevent this increase.

Proof

1. Take a consumer i:

for goods k $\quad \Sigma_{j\neq i}\Psi_k^{j+*} + \pi_k^{i+} - \gamma_k^{+} = \Sigma_j \pi_k^{j+} - \gamma_k^{+} \leq 0$

so consumer i announces $\Psi_k^{i+} = \pi_k^{i+} = \Psi_k^{i+*}$

for good h $\quad \Sigma_{j\neq i}\Psi_h^{j+*} + \pi_h^{i+} - \gamma_h^{+} = a_{ih} \geq 0$ (see p. 14 and 15)

and $\Phi_h^{i+} = \gamma_h^{+} - \Sigma_{j\neq i}\Psi_h^{j+*} = \pi_h^{i+} - a_{ih}$

for goods h'

if consumer $i \in J(h')$ $\quad \Sigma_{j\neq i}\Psi_{h'}^{j+*} + \pi_{h'}^{i+} - \gamma_{h'}^{+} = \Sigma_{\substack{j\in J(h')\\ j\neq i}}\pi_{h'}^{j+} + \pi_{h'}^{i+} - \gamma_{h'}^{+}$

< 0 so consumer i

announces $\Psi_{h'}^{i+} = \pi_{h'}^{i+} = \Psi_{h'}^{i+*}$ since $i \in J(h)$

if consumer $i \notin J(h')$ $\quad \Sigma_{j\neq i}\Psi_{h'}^{j+*} + \pi_{h'}^{i+} - \gamma_{h'}^{+} = \Sigma_{j \notin J(h')}\pi_{h'}^{j+} + \pi_{h'}^{i+} - \gamma_{h'}^{+} \geq 0$

and $\Phi_{h'}^{i+} = \gamma_{h'}^{-} - \Sigma_{j\in J(h')}\pi_{h'}^{j+}$

but by assumption, this consumer i, $i \notin J(h')$ prefers the increase in public good h to the increase in public good h'; therefore;

$$\Psi_{h'}^{i+} = 0 = \Psi_{h'}^{i+*} \quad \text{if } i \notin J(h')$$

for goods $k \notin L$ $\bar{\gamma}_k - \Sigma_{j\neq i}\Psi_k^{j-*} - \pi_k^{i-} = \bar{\gamma}_k - \Sigma_{j\neq i}\pi_k^{j-} - \pi_k^{i-} < 0$

$$\text{hence} \quad \Psi_k^{i-} = \pi_k^{i-} = \Psi_k^{i-*}$$

for goods $\ell \in L$

if $i\neq i'$ $\bar{\gamma}_\ell - \Sigma_{j\neq i}\Psi_\ell^{j-*} - \pi_\ell^{i-} = \bar{\gamma}_\ell - \Sigma_{\substack{j\neq i\\ j\neq i'}}\pi_\ell^{j-} - (\bar{\gamma}_\ell + \varepsilon) - \pi_\ell^{i-}$

$$= \bar{\gamma}_\ell - \Sigma_{j\neq i'}\pi_\ell^{j-} - \bar{\gamma}_\ell - \varepsilon$$

$$= - \Sigma_{j\neq i'}\pi_\ell^{j-} - \varepsilon < 0$$

and $\Psi_\ell^{i-} = \pi_\ell^{i-} = \Psi_\ell^{i-*}$ if $i\neq i'$

if $i=i'$ $\bar{\gamma}_\ell - \Sigma_{j\neq i}\Psi_\ell^{j-*} - \pi_\ell^{i-} = \bar{\gamma}_\ell - \Sigma_{j\neq i}\pi_\ell^{j-} - \pi_\ell^{i-} \geq 0$

but consumer $i = i'$ prefers the increase in good h to the decrease in good ℓ so

$$\Psi_\ell^{i-} = \bar{\gamma}_\ell + \varepsilon = \Psi_\ell^{i-*} \quad i = i'.$$

2. The proof is similar to the one above and will be omitted.

V. What decisions are taken

Theorem 3. At any Nash equilibrium the allocation of resources is changed if and only if it can be changed had the consumers reported truthfully.

Proof

The "only if" was proved in lemma 1. Now we shall prove that if a change is possible with correct revelation, then a change must be feasible at the Nash equilibrium.

Suppose that for some good k' either:

$$\Sigma_i \pi_{k'}^{i+} - \gamma_{k'}^{+} \geq 0 \qquad \text{or} \qquad \gamma_{k'}^{-} - \Sigma_i \pi_{k'}^{i-} \geq 0$$

so one change is feasible when consumers reveal their preferences correctly. Assume that at a Nash equilibrium no change is feasible:
for good k' we have therefore:

$$\Sigma_i \psi_{k'}^{i+*} - \gamma_{k'}^{+} < 0 \quad \text{and} \quad \gamma_{k'}^{-} - \Sigma_i \psi_{k'}^{i-*} < 0$$

Assume now that the increase is feasible: $\Sigma_i \pi_{k'}^{i+} - \gamma_{k'}^{+} > 0$; any consumer i such that

$$\Sigma_{j \neq i} \psi_{k'}^{j+*} + \pi_{k'}^{i+} - \gamma_{k'}^{+} < 0 \quad \text{announces} \quad \psi_{k'}^{i+*} = \pi_{k'}^{i+}$$

But there must be at least one consumer for whom this magnitude is positive and he should announce

$$\psi_{k'}^{i+*} = \gamma_{k'}^{+} - \Sigma_{j \neq i} \psi_{k'}^{j+*}$$

which contradicts the fact that no change was feasible at the Nash equilibrium. A similar reasoning applies to the case where it was a decrease in the level of public good that was feasible.

VI The paths of the procedures.

In the CDH procedure with correct revelation, if at a given iteration several changes were possible the P.B. could choose any one of these changes. On the contrary when consumers play their Nash equilibrium strategies, to a certain extent it is they, the consumers, who choose the change: all the changes that would have been feasible had the consumers reported truthfully do not appear possible to the P.B. when consumers play their Nash equilibrium strategies.

If the P.B. has a specified rule for choosing one change when several are possible, it may still be able to implement this rule by a priori limiting the number of changes considered.

Suppose for example that the P.B., whenever possible, wants to choose a decrease in the level of a public good. If it assumes that consumers play their Nash equilibrium strategies rather than report truthfully, the P.B. can ask consumers only about decreases: if one is feasible (with correct revelation) it will also appear feasible at the Nash equilibrium when consumers report their compensations only.

If on the other hand the P.B. has an order in which it wants to increase public goods, it can implement this rule when consumers play their Nash equilibrium strategies by considering one good at a time: the P.B. asks consumers about their benefits when the first good on its list is to be increased. By doing so, the problem reduces to the one public good case and if the good can be increased with correct announcements it can also be at the Nash equilibrium.

The rule that would specify that the P.B. must choose the public good for which the surplus $(\Sigma_i \pi_k^{i+} - \gamma_k^+$ or $\gamma_k^- - \Sigma_i \pi_k^{i-})$ is the greatest is an example of a rule that could not be implemented when consumers play their Nash equilibrium strategies: in this situation the surplus is always equal to zero.

VII Conclusions

This paper studies the consequences of assuming, in the Champsaur-Dreze and Henry procedure, that consumers consider the announcements of others as given and select their stated benefits and compensations to maximize the utility to them of the resulting change in allocation.

Whatever rational consumers announce, a decision to change the allocation of resources in the wrong direction -- for example increasing a good that should be decreased -- will never be taken.

Furthermore if consumers play their Nash equilibrium strategies they will not reveal their preferences correctly, in general, but the change in allocation will be the same as one that could have been chosen by the P.B. had the consumers reported their preferences correctly. So this procedure is successful in the Green-Laffont [3] sense.

This work is similar to that of Roberts [4] which studies the continuous procedure of Drèze and de la Vallée Poussin [2]. An interesting difference between the two types of results is that in the continuous case a consumer may overreport or underreport his marginal willingness to pay, depending upon the value of the other consumers' announcements, the cost elements and δ^i, the coefficient that determines the share of the surplus he receives. In the discrete case, it was proved that the bias was systematic. This is probably due to the fact that in the discrete procedure, consumers are allowed to differentiate between a decrease and an increase in the level of each public good. In the continuous case they announce one marginal rate of substitution: if the sum of these over all consumers is greater than the marginal cost the level of that public good is increased; if the sum is less than the marginal cost, the level of the public good is decreased.

Roberts' results are stronger than those obtained here in two ways: in the continuous case the Nash equilibrium is unique and stable.

In the discrete case, in general, the Nash equilibria are not stable: starting from any given set of announcements, if consumers play their best replay strategies against the announced benefits (compensations) at the previous round, the message will not converge to a Nash equilibrium. However it would be interesting to study the properties of a procedure where starting from any set of announcements, consumers play their best replay strategies, taking the last announcements of the others as given. The allocation is changed whenever either $\Sigma_i \psi_k^{i+} - \gamma_k^+$ or $\gamma_k^- - \Sigma_i \psi_k^{i-}$ is positive. A Nash equilibrium would be defined as an allocation and a set of benefits and compensations such that no consumer would want to change his announcements. It seems that such a procedure would be stable.

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