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The Incentive for Creation of Complete Securities Markets*

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Proposed Running Head:

Complete Security Markets

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A result that appears to be well known is that in the absence of moral hazard, transaction costs, information costs, and institutional constraints a necessary condition for a securities market to be in equilibrium is that it be complete. Specifically if a securities market is incomplete and frictionless, then individuals within the economy can make a riskless profit by appropriately introducing new securities up until the point where the market is made complete. References to this result appear, for example, in the introduction of papers by Laffont [5], Green and Sheshinski [2], Forsythe [1], and Grossman [3]. Yet, to my knowledge, no author has published an explicit proof of this basic result. Therefore my purpose here is to provide a simple proof that emphasizes the coordinating effect complete markets have on the implicit prices with which individuals value uncertain future income.

The Model

Consider a model of exchange economy. Its notation and structure is identical to Hart's model [4] except that I have shortened it from three to two periods and assume von Neumann-Morgenstern utility functions for all individuals. The basic elements are as follows. $\mathcal{I} = \{1,\ldots,I\}$ are the individuals, $\mathcal{H} = \{1,\ldots,H\}$ are the commodities, $\mathcal{S} = \{1,\ldots,S\}$ are the states of nature, and $\mathcal{F} = \{1,\ldots,F\}$ are the securities. Individuals learn with certainty at the beginning of period two the identity of the state of nature. The two periods are referred to as now and then. The endowment that individual i receives now is $\omega_1^i = [\omega_{11}^i,\ldots,\omega_{1H}^i]' \gg 0$ and the endowment he receives then if state s occurs is $\omega_2^i(s) = [\omega_{21}^i(s),\ldots,\omega_{2H}^i(s)]' \gg 0.$[1] Commodities can not be carried over from now to then. If state s occurs the return then of security f is the commodity vector $a^f(s) = [a_1^f(s),\ldots,a_H^f(s)]' > 0$. Individual i purchases now the vector of securities $z^i = (z_1^i,\ldots,z_F^i)'$. His consumption now is the vector $x_1^i = (x_{11}^i, \ldots, x_{1H}^i)' \gg 0$ and his consumption then if s occurs is $x_2^i(s) = [x_{21}^i(s), \ldots, x_{2H}^i(s)]' \gg 0$.

This is a rational expectations model in which all individuals are price takers and share identical, accurate expectations concerning then's prices. Let $p_1 = (p_{11}, \ldots, p_{1H})$ and $\pi = (\pi^1, \ldots, \pi^R)$ be, respectively, now's commodity prices and now's securities prices and let $p_2(s) = [p_{21}(s), \ldots, p_{2H}(s)]$ be the expected commodity prices then if state s occurs. For convenience let $\omega_2^i = [\omega_2^i(1), \ldots, \omega_2^i(S)]$, $a^f = [a^f(1), \ldots, a^f(S)]$, $x_2^i = [x_2^i(1), \ldots, x_2^i(S)]$, and $p_2 = [p_2(1), \ldots, p_2(S)]$. Each individual picks (x_1^i, z^i, x_2^i) so as to maximize his expected utility

$$\sum_{s=1}^{S} d^i(s)\, U^i[x_1^i, x_2^i(s)] \tag{1}$$

subject to the S + 1 budget constraints:

$$p_1 x_1^i + \pi z^i \leq p_1 \omega_1^i; \tag{2}$$

$$p_2(s) x_2^i(s) \leq p_2(s)\, \omega_2^i(s) + p_2(s) \left[\sum_{f=1}^{F} z_f^i\, a^f(s) \right]$$

$$s = 1, \ldots, S. \tag{3}$$

U^i is i's differentiable, strictly monotonic von Neumann-Morgenstern utility function and $d^i(s)$ is his subjective probability that state s will occur. I assume that the maximum does not occur at a corner. Given a fixed set of securities, a <u>Radner equilibrium</u> is a set of prices (p_1, π, p_2) and a set consumption and trading plans $(x_1^1, \ldots, x_1^I, z^1, \ldots, z^I, x_2^1, \ldots, x_2^I)$ such that (a), for all $i \in \mathcal{I}$, (x_1^i, z^i, x_2^i) maximizes his expected utility subject to his budget constraints and (b) all markets are cleared: $\Sigma_i x_1^i \leq \Sigma_i \omega_1^i$, $\Sigma_i z_i \leq 0$, and $\Sigma_i x_2^i \leq \Sigma_i \omega_2^i$. I defer for the moment defining equilibrium when the set of securities traded is not fixed.

Let $\mathcal{L}$ be the Lagrange expression formed from (1) through (3). Given a set of prices (p_1, π, p_2), the first order conditions for individual i to be at a maximum are:

$$\frac{\partial \mathcal{L}}{\partial x^i_{1h}} = \Sigma_s d^i(s) \frac{\partial U^i[x^i_1, x^i_2(s)]}{\partial x^i_{1h}} - \lambda^i p_{1h} = 0, \; h = 1,\ldots,H; \quad (4)$$

$$\frac{\partial \mathcal{L}}{\partial x^i_{2h}(s)} = d^i(s) \frac{\partial U^i(x^i_1, x^i_2(s)]}{\partial x^i_{2h}(s)} - \delta^i_s \, p_{2h}(s) = 0, \; h=1,\ldots,H \text{ and } s=1,\ldots,S; \quad (5)$$

$$\frac{\partial \mathcal{L}}{\partial z^i_f} = -\lambda^i \pi^f + \Sigma_s \delta^i_s \, p_2(s) a^f(s) = 0, \; f = 1,\ldots,F. \quad (6)$$

The $S+1$ LaGrange multipliers λ^i and δ^i_s have the usual interpretations: λ^i is i's marginal utility of a unit of monetary income now and δ^i_s is i's marginal utility for i of a unit of monetary income then if s occurs. Define $\psi^i_s = \delta^i_s / \lambda^i$ to be i's <u>implicit price</u> now for income then if state s occurs. It is how much income now i is willing to pay for one unit of income then if s occurs.

For each individual i, equations (6) may be rewritten as F equations with the S implicit prices being unknowns:

$$\begin{aligned} q^1(1)\,\psi^i_1 + q^1(2)\,\psi^i_2 + \ldots + q^1(S)\,\psi^i_S &= \pi^1 \\ &\cdots \\ q^F(1)\,\psi^i_1 + q^F(2)\,\psi^i_2 + \ldots + q^F(S)\,\psi^i_S &= \pi^F \end{aligned} \quad (7)$$

where $q^f(s) = p_2(s)\, a^f(s)$ is the monetary return of security f in state s. Let $q^f = [q^f(1),\ldots,q^f(S)]$ and let $\psi^i = (\psi^i_1,\ldots,\psi^i_S)'$. Then (7) is

$$Q\,\psi^i = \pi' \quad (8)$$

where

$$Q = \begin{bmatrix} q^1 \\ \cdots \\ q^F \end{bmatrix} . \qquad (9)$$

The security market is <u>complete</u> if rank $Q = S$. Because Q is determined both by the physical returns $a^f(s)$ of the securities and by the prices p_2, completeness of the market is defined jointly by the physical returns and the equilibrium prices. A necessary, but not sufficient, condition for completeness is for the number of securities to be at least as great as the number of states of nature. It is not sufficient because, for a given array of physical returns $a = (a^1,\dots,a^F)$, the matrix Q may have full row rank for some admissible price vectors p^2 and less than full row rank for other admissible price vectors p^2.

Equation (9) is a restriction on each individual's implicit prices. Since implicit prices are generally nonnegative the set of vectors y that satisfy (9) in an admissible manner is $\Omega^{S-F} = \{ y \mid y \in R^S_+ \text{ and } Qy = \pi' \}$ where R^S_+ is the S-dimensional nonnegative orthant. This is a convex subset of R^S_+ that consists of a unique point only if rank $Q = S$, i.e. only if the market is complete. Therefore if the market is incomplete, then two individuals i and j who are in equlibrium may have different implicit prices $\psi^i \neq \psi^j$.

Analysis

A <u>full equilibrium</u> is a set of securities $\mathcal{F}$, a set of prices (p_1, π, p_2), and a set of plans $(x_1^1,\dots,x_1^I,z^1,\dots,z^I,x_2^1,\dots,x_2^I)$ such that (a) the prices (p_1, π, p_2) and plans $(x_1^1,\dots,x_2^I)$ are a Radner equilibrium and (b) no individual can profitably introduce a new security onto the market. Thus within a full equilibrium each individual i has exhausted his opportunities for maximization with respect to both his plans (x_1^i, z_1^i, x_2^i) and the set of possible securities. In a Radner equilibrium each individual takes the set of securities $\mathcal{F}$ as given and maximizes only with respect to his plans (x_1^i, z^i, x_2^i). The result that I prove is: in the absence of moral hazard, transaction costs, and information costs a necessary condition for the two period exchange economy to be in full equilibrium is that the securities market be complete. The only exception to this occurs when the market is incomplete and, at some Radner equilibrium, all individuals by chance have identical implicit prices. This is an unlikely occurence if individuals have heterogeneous utility functions, subjective probabilities, and endowments.

The proof is this. Assume, contrary to the result, that the economy is in full equilibrium and has an incomplete market. Because the market is incomplete Ω^{S-F} contains a multiplicity of points and unless individuals have identical utility functions, identical subjective probabilities, and identical endowment streams, it is likely that a pair of individuals $i,j \in \mathcal{I}$ exist who have unequal equilibrium implicit prices: $\psi^i \neq \psi^j$ where $\psi^i, \psi^j \in \Omega^{S-F}$.

Since the vectors ψ^i and ψ^j are distinct points, they are also disjoint convex sets in R_+^S. Therefore a hyperplane exists that separates them: a vector $q^{F+1} = [\, q^{F+1}(1),\ldots,q^{F+1}(S)\,] > 0$ and scalars π_+^{F+1} and π_-^{F+1} exist such that either

$$0 < q^{F+1}\psi^i < \pi_-^{F+1} < \pi_+^{F+1} < q^{F+1}\psi^j \qquad (10)$$

or

$$0 < q^{F+1}\psi^j < \pi_-^{F+1} < \pi_+^{F+1} < q^{F+1}\psi^i \qquad (11)$$

All the components of q^{F+1} may be chosen to be nonnegative because ψ^i and ψ^j are single points within R_+^S. Assume without loss of generality that (10) is satisfied. Let a third individual $k \in \mathcal{J}$ offer to sell at price π_+^{F+1} and buy at price π_-^{F+1} a new security -- labeled F+1 -- that has monetary returns q^{F+L}.[2] Given the offer price of π_-^{F+1} individual i wants to sell individual k some quantity of F+1 because the marginal utility he attaches to a unit of F+1 is

$$\Sigma_s \delta_s^i q^{F+1}(s) - \lambda^i \pi_-^{F+1} < 0. \tag{12}$$

That the left hand side represents i's marginal utility for F+1 follows from (6). The direction of the inequality follows from (10) becuase $\lambda_i > 0$ and $\psi_s^i = \delta_s^i / \lambda^i$. Similarly individual j wants to buy from individual k some quantity of F+1 at price π_+^{F+1}. If k astutely selects the prices π_-^{F+1} and π_+^{F+1}, then the quantities that i wants to sell and j wants to buy will be equal and k can make a riskless profit of $y(\pi_+^{F+1} - \pi_-^{F+1}) > 0$ where y is the positive quantity traded. Therefore, the market is not in full equilibrium because k has an incentive to introduce a new security. This contradicts the original assumption that the market is in equilibrium and proves the result.

Notice that this proof, and therefore this result, is based on a local argument: individuals i, j, and k can make themselves better off by trading small amounts of security F+1 among themselves. Nevertheless this does not mean that each of them will be better off once the economy reaches a new Radner equilibrium with F + 1 securities instead of F securities. Hart [4] has shown with an example that the introduction of a new security into an incomplete securities market may result in every individual being worse off in the new Radner equilibrium than they were in the initial Radner equilibrium. The reason is that the prices in the new Radner equilibrium will generally differ from the prices in the initial Radner equilibrium and these new prices may be unfavorable to all individuals.

Footnotes

[1] The transpose of a vector is indicated by a prime; thus $\omega_1^i = \{\omega_{11}^i, \ldots, \omega_{1H}^i\}'$ is a column vector. The notation $\omega_1^i \gg 0$ means that all components of ω_i^i are positive and the notation $a^f > 0$ means that all components of a_f are nonnegative and that at least one component is positive.

[2] Given a vector of monetary returns q^{F+1} and a vector of equilibrium prices p_2, it is always possible to construct a matrix of physical returns $a^{F+1} = [a^{F+1}(1),\ldots,a^{F+1}(s)] > 0$ such that, for all s, $p_2(s)\ a^{F+1}(s) = q^{F+1}(s)$.

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[4] O. Hart, "On the Optimality of Equilibrium when the Market Structure is Incomplete," J. of Econ. Theory 11 (December 1975), 418-443.

[5] J. J. Laffont, "Optimism and Experts against Adverse Selection in a Competitive Economy," J. of Econ. Theory 10 (June 1975), 284-308.