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EFFECTIVE PRICE MECHANISMS [1]

by

Donald G. Saari [2]

and

Carl P. Simon [3]

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[2] Department of Mathematics, Northwestern University, Evanston, Illinois 60201; supported in part by NSF Grant MPS71-03407-A03

[3] Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109; supported in part by NSF Grant MPS75-08563.

ABSTRACT

It is known that the price mechanism whereby the rate of change of a price is proportional to the excess demand of the corresponding commodity need not converge to a competative equilibrium for a pure exchange economy with more than two commodities. On the other hand, there exist convergent price mechanisms, similar to the Newton iterative process, where the rate of change of the prices is determined by the excess demand and the marginal excess demands of all the commodities. This is a considerable informational requirement. It is shown that this requirement cannot be substantially reduced for any convergent price mechanisms, that is for price mechanisms expressed in terms of a difference or differential equation where the solutions converge to a competative equilibrum.

1. INTRODUCTION

We consider a pure exchange economy whose utility functions satisfy the classical axioms of convexity and monotonicity. By using the Kakutani fixed point theorem, one can show that such an economy has a competitive equilibrium (See Debreu (1959) or Arrow-Hahn (1972), for example.) A more difficult task is to determine the actual value of the price at such a competitive equilibrium. This leads to a deeper problem, which goes back to Walras, namely, to find some sort of adaptive mechanism on prices which leads, hopefully in a natural fashion, to these equilibrium points from any initial price system. In this paper, we will describe some necessary conditions which a price mechanism must satisfy in order that it be effective, i.e., from any initial price system in any given (classical) pure exchange economy it always yields a path which terminates at a competitive equilibrium of the system.

Let $\mathbb{R}_+ = [0,\infty)$, the non-negative real numbers. In an economy with c commodities, let P denote the normalized <u>price simplex</u>

$$P = \{\underline{p} = (p_1,\ldots,p_c) \in \mathbb{R}_+^c \mid p_1^2 + \ldots + p_c^2 = 1\}.$$

For any distribution of commodities, let $\zeta(\underline{p})$ denote the usual <u>aggregate excess demand function</u> at price $\underline{p}$. That is $\zeta(\underline{p})$ is the difference between the commodity bundle demanded at price $\underline{p}$ and the total supply. When $\zeta(\underline{p}^*) = \underline{0}$, supply equals demand and p^* is a <u>competitive equilibrium.</u> Since $\underline{p}\cdot\zeta(\underline{p}) = 0$ for all $\underline{p}$, by Walras' law, $\zeta(\underline{p})$ can be viewed as a tangent vector field on the price simplex P.

Since the zeroes of ζ are our desired equilibria, an obvious choice for a price mechanisms is the differential equation

$$\frac{d\underline{p}}{dt} = \zeta(\underline{p}) \quad .$$

This mechanism would be an effective one (roughly in the sense of the first paragraph above) if the solution curve of this differential equation through any given point near the boundary of P would always tend to a zero of ζ as $t \to +\infty$. That this is not the case was shown by Scarf (1960). He constructed an economy whose excess demand function has only one zero, and this zero is unstable in the sense that all orbits starting near it tend <u>away</u> from it. The construction of such counterexamples has been simplified by a theorem of H. Sonnenschein (1972) -- a result which has recently been generalized by others. (See section 4.) Sonnenschein's theorem is that any smooth vector field on P satisfying Walras' Law on a compact set in the interior of P can be realized as the excess demand function for some classical pure exchange economy. One can then use index theorems to construct examples like Scarf's.

Are there any effective price mechanisms? Recently Kellog, Li, and Yorke (1975), extending an ideas of M. Hirsch (1963), described an adaptive method which computes fixed points of a map from a convex bounded set B into itself by determining paths from arbitrary points on the boundary of B to fixed points in the interior of B. Since the problem of finding a zero of a map (or vector field) F is equivalent to the problem of finding a fixed point of the identity map minus F, one can use the Kellog-Li-Yorke result to find zeroes of ζ, i.e., price equilibria. So, there does exist a price mechanism with the property that, from any point $\underline{p}$ near the boundary of the price simplex P, the mechanism tends asymptotically toward a zero of ζ. Kellog et. al. do impose a technical condition on the eigenvalues of $D\zeta(p)$ (the Jacobian matrix of ζ at $\underline{p}$). Near the competitive equilibrium, this price mechanism behaves somewhat like a high-dimensional version of Newton's Method whose differential equation is

$$\frac{d\underline{p}}{dt} = - (Df(\underline{p}))^{-1} f(\underline{p}).$$

This means, of course, that this mechanism requires a considerable amount of information on the part of the price adjuster (or auctioneer) in that it requires knowledge not only of $\zeta(\underline{p})$ but also of every first partial derivative of ζ at every point.

A more recent effective price mechanism has been discussed by S. Smale (1975). Not only does he relax the assumptions on the eigenvalues of $D\zeta(p)$, but he also discusses the implications of his method in the search for price equilibria. Although the eigenvalue condition is relaxed, Smale still requires them to be non-zero at a zero of ζ. This is not severe since it is satisfied for an open dense set of vector fields. Also, his algorithm is somewhat closer to Newton's method than is that of Kellog, Li, and Yorke.

The purpose of this paper is to study how much information a price adjuster needs to have in order that his price mechanism be effective for all standard economies. The above "Generalized Newton Method" (G.N.) requires knowledge of $\zeta(\underline{p})$ and the gradients of all but one of its component functions.

In other words, if the price adjuster uses the mechanism $\dot{\underline{p}} = \zeta(\underline{p})$, he needs to know c quantities at each price; if he uses one of these quasi-Newton methods, he needs to know $(c - 1)^2 + (c - 1)$ quantities at each price, including how the $j^{\underline{th}}$ commodity affects the rate of charge of the demand for the $k^{\underline{th}}$ commodity for all j and k.

For practical problems, this is a staggering amount of information. Consequently, the natural question is whether there exist effective price mechanisms with a more modest demand on information content; say one which depends on $\zeta(p)$ and the gradients of only some of the component functions. We investigate this question in this paper, and our results show that the informational requirement cannot be relaxed by any significant amount. That is, should some price mechanism require a "low amount of information" (in Theorem 1 we define what is meant by this), then there can be found a classical exchange economy for which the mechanism is <u>not</u> effective.

Of course, <u>if</u> we have a priori knowledge concerning a given vector field $\zeta(p)$, we may be able to design a simpler mechanism. However, this is merely an exchange of type of information used, and, as Roy Radner pointed out to us, the expense of determining this second type of information may be very high.

We assume throughout this paper that the paths of prices follow the trajectories of a differential equation. It is not clear that this method adversely effects the information content. For example, other commonly used methods of tracing paths are the sandwich method and the method of complementary pivoting. (See Scarf (1973), Merrill (1972), and Saigal (1976)). H. Scarf pointed out that these methods require the same kinds and amount of information as do the generalized Newton methods.

Before we give a mathematical formulation of our goal, we will simplify the notation and reduce the dimension of the problem by eliminating the normalization constraint on the prices and the constraint on ζ given by

Walras' Law. The constraint on the prices implies that P is the $(c-1)$ dimensional manifold obtained by intersecting the unit sphere in $\mathbb{R}^c$ with the closed positive orthant of $\mathbb{R}^c$. Notice that this manifold can be represented by a single chart, for example its projection into the hyperplane $p_c = 0$.

Let $n = c-1$. We can associate each $\underline{p} \in P$ with a vector $\underline{q}$ in $\mathbb{R}^n$ in many ways. For example, we could use the diffeomorphism

$$(p_1,\ldots,p_n,p_c) = (q_1,\ldots,q_n,\sqrt{1-q_1^2-\ldots-q_n^2}\ ),$$

since the last coordinate of $\underline{p}$ is determined by the first n. A more natural way would be to consider the hyperplace Q in $\mathbb{R}^c$ which contains $\underline{0}$ and is normal to the vector $(1,1,\ldots,1)$ in $\mathbb{R}^c$. Then, project P onto a subset U of Q in a natural one-to-one fashion via a projection map g. The vectorfield $\zeta(\underline{p})$ on P can be identified with a vectorfield $f(q)$ on U where $q = g(\underline{p})$ and

$$f(\underline{q}) = Dg_p(\zeta(\underline{p})) \equiv \begin{pmatrix} \frac{\partial g_1}{\partial p_1}(p) & \cdots & \frac{\partial g_1}{\partial p_c}(p) \\ \vdots & & \vdots \\ \frac{\partial g_c}{\partial p_1}(p) & \cdots & \frac{\partial g_c}{\partial p_c}(p) \end{pmatrix} \begin{pmatrix} \zeta_1(\underline{p}) \\ \vdots \\ \zeta_c(\underline{p}) \end{pmatrix}$$

Similarly, any vector field on U "lifts" to a well-defined vector field on P which satisfies Walras' Law. The differential equation $\frac{d\underline{p}}{dt} = \zeta(\underline{p})$ on p is equivalent to the differential equation $\frac{d\underline{q}}{dt} = f(q)$ on U.

In the latter system, we have eliminated the constraints without loss of information. Thus, we shall work with $\underline{q}$ and $f(\underline{q})$ rather than $\underline{p}$ and $\zeta(\underline{p})$. Indeed, the Kellog-Li-Yorke and the Smale algorithms are applied to f and $\underline{q}$. We can now describe in more detail the purpose of this paper.

Assume that there are $c = (n+1)$ commodities and that $M(\underline{x};y_{11},\ldots,y_{1n},y_{21},\ldots,y_{nn})$ is a smooth mapping from $\mathbb{R}^n \times \mathbb{R}^{n^2}$ into $\mathbb{R}^n$. Let $\frac{d\underline{q}}{dt} = f(\underline{q})$ be a differential equation on U arising from an excess demand function ζ as above. Consider the new differential equation

$$(1.1) \qquad \frac{d\underline{q}}{dt} = M(f(\underline{q});\frac{\partial f_1}{\partial q_1}(\underline{q}),\ldots,\frac{\partial f_1}{\partial q_n}(\underline{q}),\ldots,\frac{\partial f_n}{\partial q_n}(\underline{q}))$$

$$\equiv M(f(\underline{q}),\frac{\partial f}{\partial q}(\underline{q})) \equiv M_f(\underline{q}) \ .$$

We shall investigate the existence of functions M with <u>ignorable coordinates</u> y_{ij} (that is, with coordinates y_{ij} such that $\frac{\partial M}{\partial y_{ij}} \equiv 0$) which serve as effective price mechanisms in the sense described above. So, we will want almost every orbit of (1.1) to tend asymptotically to some stationary point of (1.1). These "sinks" of (1.1) should be zeroes of f.

Each ignorable coordinate corresponds to a direction or a bit of information which is not necessary for the effectiveness of a mechanism -- information which is differential in nature and both expensive and difficult to obtain. Consequently the hope is that some mechanism M with several ignorable coordinates exists. However, as we stated earlier, our results show that the information content of the G.N. methods, that is, knowledge of $f(\underline{q})$ and all $\frac{\partial f_i}{\partial q_j}(\underline{q})$'s, cannot be substantially relaxed for effective adaptive price mechanisms. Thus, price mechanisms require a considerable amount of information to become effective!

We would like to thank H. Sonnenschein for originally suggesting the question to us and for several subsequent helpful discussions. We would also like to thank G. Debreu, who invited us to present this paper at the N.B.E.R. Conference on the Theory of General Economic Equilibrium in February, 1976. We benefited considerably from discussions and conversations at this meeting.

2. BACKGROUND ON DIFFERENTIAL EQUATIONS

Let U be an open subset of $\mathbb{R}^n$. Let $f: U \to \mathbb{R}^n$ be a C^1 (continuously differentiable) function, and consider the differential equation on U

$$(2.1) \qquad \frac{d\underline{q}}{dt} = f(\underline{q}) \quad .$$

We remind the reader of some basic facts and definitions concerning solutions of (2.1). We suggest Brauer-Nohel (1969), Arnold (1973), or Hirsch-Smale (1974) as references in this area.

For $\underline{q} \in U$, let $\varphi(t;\underline{q})$ be the (unique) solution of (2.1) which passes through $\underline{q}$ at $t = 0$, i.e., $\varphi(0,\underline{q}) = \underline{q}$ and $\frac{d}{dt}\varphi(t;\underline{q}) = f(\varphi(t;\underline{q}))$. If $f(\underline{q}^\circ) = \underline{0}$, then $\varphi(t,\underline{q}^\circ) = \underline{q}^\circ$ for all $t \in \mathbb{R}$, i.e., $\underline{q}^\circ$ is a stationary or equilibrium point of (2.1). For such a $\underline{q}^\circ$, the <u>stable set</u> of $\underline{q}^\circ$, $W^s(\underline{q}^\circ)$, is the union of all points in U whose orbit tends to $\underline{q}^\circ$ as $t \to +\infty$, that is,

$$W^s(\underline{q}^\circ) = \{\underline{q} \in U \mid \varphi(t,\underline{q}) \to \underline{q}^\circ \text{ as } t \to +\infty\}.$$

Similarly, the unstable set of $\underline{q}^\circ$, $W^u(\underline{q}^\circ)$, is

$$\{\underline{q} \in U \mid \varphi(t,\underline{q}) \to \underline{q}^\circ \text{ as } t \to -\infty\}.$$

One calls $\underline{q}^\circ$ a <u>sink</u>, an attractor, or an asymptotically stable zero of (2.1) if $W^s(\underline{q}^\circ)$ contains an open neighborhood of $\underline{q}^\circ$ in U. Similarly, one calls $\underline{q}^\circ$ a <u>source</u> if $W^u(\underline{q}^\circ)$ contains an open neighborhood of $\underline{q}^\circ$ in U. If $\underline{q}^\circ$ has non-trivial stable sets and non-trivial unstable sets, $\underline{q}^\circ$ is called a <u>saddle point</u>.

There are effective algebraic tests for determining whether a zero of f is a sink, source, or saddle. Let $f(\underline{q}^\circ) = \underline{0}$ and let $Df(\underline{q}^\circ)$ be the Jacobian matrix $\left(\left(\frac{\partial f_i}{\partial q_j}(\underline{q}^\circ)\right)\right)$.

a) If every eigenvalue of $Df(\underline{q}^\circ)$ is negative, or has a negative real part, then $\underline{q}^\circ$ is a sink.

b) If every eigenvalue of $Df(\underline{q}^\circ)$ is positive, or has a positive real part, then $\underline{q}^\circ$ is a source.

c) If some eigenvalue of $Df(\underline{q}^\circ)$ is positive or has positive real part, then there is an open neighborhood V_1 of $\underline{q}^\circ$ and a dense open subset V_2 of V_1 so that the orbits of all $\underline{q} \in V_2$ leave V_1 as $t \to +\infty$, i.e., $\underline{q}^\circ$ is not a sink.

d) If $Df(\underline{q}^\circ)$ has j eigenvalues with negative real part and k eigenvalues with positive real part with $j + k = n$, then $W^s(\underline{q}^\circ)$ is a smooth j-dimensional disk through $\underline{q}^\circ$ (stable manifold) and $W^u(\underline{q}^\circ)$ is a smooth k-dimensional disk through $\underline{q}^\circ$ (unstable manifold). $W^s(\underline{q}^\circ)$ and $W^u(\underline{q}^\circ)$ are tangent to the appropriate eigenspaces and intersect transversally at $\underline{q}^\circ$. In this case, $\underline{q}^\circ$ is called a <u>hyperbolic</u> zero of f.

FIGURE 1 -- See page 57.

e) If $\underline{q}^\circ$ is a non-hyperbolic zero of f with $Df(\underline{q}^\circ)$ having some zero or pure imaginary eigenvalues and having no eigenvalues with positive real part, then one cannot determine whether or not $\underline{q}^\circ$ is an attractor simply by examining $Df(\underline{q}^\circ)$. In this situation, the stability or non-stability of $\underline{q}^\circ$ is usually determined by derivatives of f of order $\geqq 2$.

Definition. We will call $\underline{q}^\circ$ a non-degenerate or non-singular zero of f if $f(\underline{q}^\circ) = \underline{0}$ and $\det Df(\underline{q}^\circ) \neq 0$.

One can associate a number $I(\underline{x}^\circ, f)$, called the index, to every isolated zero $\underline{x}^\circ$ of the differential equation $\dot{\underline{x}} = f(\underline{x})$ on $\mathbb{R}^n$. To define the index, let

$$S = \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x}\| = 1\}$$

be the unit (n - 1)-dimensional sphere in $\mathbb{R}^n$. Let $B_r(\underline{x}^\circ)$ be the ball of radius r about $\underline{x}^\circ$, with r chosen so that $\underline{x}^\circ$ is the only zero of f in $B_r(\underline{x}^\circ)$. Consider the induced map $\hat{f}: S \to S$, defined by

$$\hat{f}(\underline{x}) = \frac{f(\underline{x}^\circ + r\underline{x})}{\|f(\underline{x}^\circ + r\underline{x})\|}, \quad \|\underline{x}\| = 1.$$

As a map between spheres, $\hat{f}$ has a degree which is an integer defined via algebraic topology and which measures how mahy times $\hat{f}(S)$ "wraps around" S. This integer is the index (or degree) of $\underline{x}^\circ$ as a zero of f, $I(\underline{x}^\circ, f)$. See Milnor (1965), Smale (1967), Arnold (1973), or Dierker (1974) for the formal definition and further discussion.

We now list some properties of $I(\underline{x}^\circ, f)$ which will be needed in later sections of this paper:

f) Let U be any neighborhood of $\underline{x}^\circ$ with the property that the boundary of U, ∂U, is homeomorphic to S. One can use U instead of $B_r(\underline{x}^\circ)$ to define $I(\underline{x}^\circ, f)$, provided $\underline{x}^\circ$ is the only zero of f in U.

g) If $\partial U \approx S$ and if $U \cap f^{-1}(\underline{0}) = \{\underline{x}^\circ, \ldots, \underline{x}^k\}$, then the degree of the map $\tilde{f}: \partial U \approx X \to X$, where

$$\tilde{f}(\underline{x}) = \frac{f(x)}{\|f(\underline{x})\|}$$

is well-defined and equals $\sum_{i=0}^{k} I(\underline{x}^i, f)$.

h) In particular, if the vector field f points into U along ∂U, then $\tilde{f}$ is homotopic to the antipodal map of S, which has degree $(-1)^n$. The sum of the zeroes of f in U must equal $(-1)^n$.

i) If $\underline{x}^\circ$ is a hyperbolic zero of f with an s-dimensional stable manifold and a u-dimensional unstable manifold, then

$$I(\underline{x}^\circ, f) = (-1)^s = \text{sign det } Df(\underline{x}^\circ).$$

From statements h) and i), one sees that if $\underline{x}^\circ$ is the only zero of a vector field f on an n-dimensional disk U and if $(-1)^n \det Df(x^\circ) < 0$, then f cannot point into U all along the boundary of U. Later, we will prove the converse of this statement, i.e., if B is an $n \times n$ matrix such that $(-1)^n \det B > 0$ and $\underline{p}^\circ$ is a point in the n-disk U, then there exists a vectorfield f on U such that f points into U on ∂U, $\underline{p}^\circ$ is the only zero of f in U, and $Df(\underline{p}^\circ) = B$.

3. DEFINITIONS

As we mentioned in the introduction, one price mechanism which does work is the generalized Newton method (G.N.), which near a zero of $f(\underline{q})$ assumes a form close to $\frac{d\underline{q}}{dt} = -(Df(q))^{-1}f(q)$, a form we shall call Newton's method (N). Let q^* be a zero of f. Then, q^* is also a zero of $q \longmapsto -(df(q))^{-1}f(q) \equiv N(q)$. To see why this method works, we perform an eigenvalue analysis of the latter equation at $\underline{q}^*$ as described in section 2, assuming for convenience that $(Df(\underline{q}^*))^{-1}$ exists. Taking derivatives, we find

$$DN(\underline{q}^*)\underline{v} = -D^2f^{-1}(f(\underline{q}^*))(f(\underline{q}^*),\underline{v}) - (Df(\underline{q}^*))^{-1} \cdot Df(\underline{q}^*)\underline{v}$$

$$= \underline{0} - I\underline{v},$$

since $f(\underline{q}^*) = \underline{0}$. So, $DN(\underline{q}^*) = -I$ and every eigenvalue of $DN(\underline{q}^*)$ is -1. By a) of section 2, $\underline{q}^*$ is a sink of $\frac{d\underline{q}}{dt} = N(\underline{q})$, and all orbits starting near $\underline{q}^*$ tend toward $\underline{q}^*$. Thus, the price mechanisms G.N. and N. have the effect of converting a zero of $f(\underline{q})$ into a sink or attractor of the new differential equation.

This motivates the following definitions.

Definition 1. $M(\underline{x};y_{11},\ldots,y_{nn})$ is a local effective price mechanism (LEPM) if for any smooth excess demand function f and for any $\underline{q}$ such that $f(\underline{q}) = \underline{0}$, the point $\underline{q}$ is an attractor for Equation (1.1). Furthermore, we require that there exist a smooth excess demand function f^* and a zero $\underline{q}^*$ of f^* such that $DM_{f^*}(\underline{q}^*)$ is non-singular.

Definition 2. M is an effective price mechanism (EPM) if for any smooth excess demand function f, the following are satisfied:

a) if $f(\underline{q}) = 0$, then $M(f(\underline{q});Df(\underline{q})) = \underline{0}$; b) for almost all q in some open subset V of U_1, the solution of (1.1) through $\underline{q}$ tends asymptotically to a zero of f as $t \to +\infty$ (the choice of V is fixed and, hence, independent of the choice of f); and c) for some $\underline{q}$ and f for which $f(\underline{q}) = \underline{0}$ and $Df(\underline{q})$ is non-singular, $\underline{q}$ is a non-singular zero of $\dot{\underline{q}} = M_f(\underline{q})$.

An LEPM can start at any price near an equilibrium point and the price mechanism (Equation (1.1)) will adjust prices so that they will tend asymptotically to the equilibrium point. However, this may not happen for initial prices outside some designated neighborhood of a specified equilibrium point. Consequently the limitation of a LEPM is that you must start near an equilibrium point.

For an EPM, we require only that it converges to some zero of $f(\underline{q})$, but we have no control over which one it may be. For example, it may turn out that the adjusted prices (solutions of equation (1.1)) pass arbitrarily close to one zero of $f(\underline{q})$, only to leave this neighborhood and converge to a second zero of f. Examples of this phenomenon can be found in the proof of our main theorem in Section 6. (Of course, a combination of such an EPM and an LEPM would eliminate this behavior.) On the other hand, the initial condition need not be near an equilibrium point. A mechanism can be an EPM and not an LEPM, and vice versa.

Intuitively, before a price mechanism is labeled effective, we want most of the solutions to converge to an equilibrium point. It turns out that this is a very strong condition; for example, such a requirement excludes the G.N. methods. Without a condition on the initial conditions, it is easy to construct examples of economies for which the G.N. methods will not converge to equilibria. Therefore we relax this restriction by merely requiring (condition b) that most solutions starting in some predetermined open set V should converge to some equilibria. (In the G.N. methods, V is some open neighborhood of the boundary of U.) Technically, condition b) means that M depends on V. However, since we use only the fact that V is an open set, we ignore this dependency. Thus the main purpose of condition b is to admit a larger class of potential EPM's.

The last parts of the above two definitions are natural non-degeneracy conditions. They are natural for a number of reasons. First of all, an open dense set of all vector fields on U have only a finite number of zeroes and these zeroes are not only non-degenerate but also hyperbolic. (See, for example, Smale (1967).) There are two additional reasons one would want an EPM to have a hyperbolic attractor. First of all, as stated in assertion e) of section 2, if $\underline{q}^*$ is a non-hyperbolic attractor of M_f, its stability is very fragile. In particular, one could drastically alter the stability of $\underline{q}^*$ as a zero of M_f by changing some higher order derivatives of f at $\underline{q}^*$: Secondly, in order to be truly effective, our price mechanism should converge rapidly to a zero of M_f. To guarantee that orbits near a sink $\underline{q}$ of (1.1) tend reasonably quickly to $\underline{q}$, $\underline{q}$ must be a hyperbolic sink of (1.1) (in which case, such orbits move exponentially toward $\underline{q}$).

A price mechanism could certainly not be considered effective if it transformed every zero of every excess demand function, including all the strongly attracting hyperbolic sinks, to weak degenerate zeroes. In fact, every concrete example of a price mechanism transforms every non-singular zero to another non-singular zero and often to a hyperbolic zero.

Nevertheless we shall see in the proof of our theorem that this regularity condition can be considerably weakened. We shall indicate how this condition can be relaxed, primarily to illustrate the role of the regularity condition.

The following lemma will be useful in working with price mechanisms.

Lemma 1. Let $M(\underline{x};y)$ be a price mechanism with the property that $M(f(\underline{q}^*);Df(\underline{q}^*)) = \underline{0}$ whenever $f(\underline{q}^*) = \underline{0}$, e.g., M may be an EPM or LEPM. Suppose that $M:\mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}^n$ is C^k, where $k \geq 1$. Then, there exists a C^k mapping $A:\mathbb{R}^n \times \mathbb{R}^{n^2} \to L(\mathbb{R}^n,\mathbb{R}^n) \cong \mathbb{R}^{n^2}$ (actually defined only on some neighborhood of $\underline{x} = 0$) such that

(3.1) $$M(\underline{x};\underline{y}) = A(\underline{x};\underline{y}) \cdot \underline{x}$$

Proof: This is a standard result in singularity theory, and is sometimes called Hadamard's Lemma.

By hypothesis, $M(\underline{0};\underline{y}) = \underline{0}$ for all $\underline{y}$.

Let $M = (M_1,\ldots,M_n)$. Then,

$$M_j(\underline{x};\underline{y}) = M_j(t\underline{x};\underline{y}) \Big|_{t=0}^{t=1} = \int_0^1 \frac{d}{dt} M_j(t\underline{x};\underline{y})dt$$

$$= \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial M_j}{\partial x_i}(t\underline{x};\underline{y})dt \quad .$$

So, one can let $A(\underline{x};\underline{y})$ be the matrix $\left(\left(a_{ij}(\underline{x};\underline{y})\right)\right)$

where $$a_{ij} = \int_0^1 \frac{\partial M_j}{\partial x_i}(t\underline{x};\underline{y})dt \ .$$

For example, in the Newton method,

$$A(\underline{x};\underline{y}) = -(\underline{y})^{-1},$$

where $\underline{y}$ is viewed as an $n \times n$ matrix.

Note that if y_{hk} is an ignorable coordinate for M, it is also an ignorable coordinate for A in that

$$\frac{\partial A}{\partial y_{hk}}(\underline{x};\underline{y}) = \underline{0}, \text{ for all } (\underline{x};\underline{y}).$$

However, in the statement and proof of our theorem, we actually use a weaker concept of an ignorable coordinate in that we only require that

$$\frac{\partial A}{\partial y_{hk}}(\underline{0};\underline{y}) = \underline{0} \quad \text{for all} \quad \underline{y} \in \mathbb{R}^{n^2} \quad .$$

We are now ready to state our main result.

Theorem: Let ζ be an excess-demand function for a standard economy with c commodities. Let $n = c - 1$ and let $f: U \to \mathbb{R}^n$ be the corresponding vector field as constructed in Section 1. Let $M: \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}^n$ be a price mechanism.

a) If $c = 2$, $\frac{d\underline{p}}{dt} = \zeta(\underline{p})$ is an EPM;

b) If $c \geq 3$ $(n \geq 2)$ and if M has some ignorable coordinate y_{ij} in some neighborhood of $\underline{x} = \underline{0}$, then M cannot be an LEPM;

c) If $c \geq 3$ $(n \geq 2)$, if y_{ij} and y_{hk} are ignorable coordinates for M in some neighborhood of $\underline{x} = \underline{0}$, where $i \neq h$ and $j \neq k$, and if a_{ji} or a_{kh} is not identically zero on $\{0\} \times \mathbb{R}^{n^2}$, then M cannot be an EPM. Here $a_{ij}(\underline{x};\underline{y})$ is an entry in the matrix $A(\underline{x};\underline{y})$, where A is as in (3.1).

d) If the vector $\underline{y}_j$ and some y_{ik} for $k \neq j$ are ignorable coordinates for $M(\underline{x};\underline{y}_1,\ldots,\underline{y}_n)$, then M cannot be an EPM.

e) If $n = 2$ or 3 and M has two ignorable coordinates y_{ij} and y_{hk} with $i \neq h$ and $j \neq k$, then M cannot be an EPM. If $n = 4$ and M has three ignorable coordinates $y_{h_1 k_1}, y_{h_2, k_2}$, and y_{h_3, k_3} with the h_i's not all equal and the k_j's not all equal, then M cannot be an EPM.

REMARKS:

As we stated earlier, the regularity condition in the definition of an EPM can be considerably weakened. For the proof of part c) it can be dropped altogether. For the proof of part d), it can be weakened to the regularity requirement of a LEPM, namely, that the mechanism transforms some zero of some excess demand function to a non-degenerate zero. For the proof of part e), a similar relaxation of this requirement is possible; there need only exist a zero of a special type that is transformed to a non-degenerate zero by M. See the remark at the end of Section 7.

Recall that the Newton method,

$$\frac{d\underline{q}}{dt} = - (Df(\underline{q}))^{-1} f(\underline{q}),$$

is an LEPM. Part b) of the above theorem states that the information content required by the Newton method cannot be relaxed for price mechanisms of the type we are considering. The other parts indicate that an EPM cannot be devised which substantially reduced the information requirements of the generalized Newton methods.

Part d) states that, if a proposed price mechanism does not take into consideration the rate of change of the demand for two commodities, the mechanism will not be effective. In fact, if the mechanism ignores this rate for one commodity and also ignores one component of this rate for another commodity, it will not be effective.

If one writes the y_{ij} coordinates as an $n \times n$ matrix $((y_{ij}))$, part d) states than an effective M cannot ignore a column and one more entry of $((y_{ij}))$. The same result (with the same proof) holds if M has a row of $((y_{ij}))$ and one other entry as ignorable coordinates. As a result, an effective price mechanism cannot fail to take into consideration how two commodities affect the rate of change of the demand for all the commodities.

Part e) of the Theorem attempts to sharpen the result of part d), at least for economies with less than six commodities. In answer to the question: just how many entires of $((y_{ij}))$ can be ignorable coordinates for an effective M, we show that in economies with three or four commodities $(n = 2 \text{ or } 3)$, if M has two ignorable coordinates in different rows and columns of $((y_{ij}))$, then M cannot be effective. Presumably, this pattern continues for any number of commodities. (See Remarks 1 and 2 in section 8.) In the last section of this paper, we will indicate that these results are about as sharp as one can expect by discussing examples of effective mechanisms.

4. PRELIMINARIES ON EXCESS DEMAND FUNCTIONS AND PRICE MECHANISMS

Before proving the theorem, we need some results from the literature.

Definition. An excess-demand type function $F:P \to \mathbb{R}^c$ is a continuous function such that

a) $\underline{p}\cdot F(\underline{p}) = 0$ for every $\underline{p} \in P$,

b) $\|F(\underline{p})\| \to \infty$ as $\underline{p} \to \partial P$,

c) there exists $k \in \mathbb{R}$ such that for every $\underline{p} \in P$, $F_i(\underline{p}) > k$,

where $F = (F_1, F_2, \ldots, F_c)$.

If F is an excess demand type function, let its set of equilibria be denoted by E_F. So,

$$E_F = \{\underline{p} \in P \mid F(\underline{p}) = \underline{0}\}.$$

Recall that we are working with pure exchange economies which have continuous, monotone, strictly convex preference relations. For such economies, it is known that every aggregate excess demand function ζ is an excess demand type function. (For example, see Arrow and Hahn (1972), chapter four.) Let $P_\epsilon = \{\underline{p} \in P \mid p_i \geq \epsilon \; \forall\, i\}$. Let F be an excess demand type function. It follows from the work of Sonnenschein (1972;1973), Mantel (1974), and Debreu (1974) that F need only satisfy condition a) in order to guarantee that there exist an economy whose (aggregate) excess demand function agrees with F on P_ϵ. However, it does not necessarily follow that the equilibria for this economy all lie inside the trimmed price simplex P_ϵ. To ensure this result, we need the following stronger statement proved by Mas-Collel (to appear).

Lemma 2. Let F be an excess demand type function and let $\epsilon > 0$. Then, there is a μ with $0 < \mu < \epsilon$ and an economy with excess demand function ζ such that $\zeta = F|P_\mu$ and $E_F = E_\zeta \subset P_\mu$.

Since our main concern is the set E_ζ, it follows from Lemma 2 that we can view any excess demand type function as an excess demand function for some standard economy.

REMARK:

Let us return to our chart U for the pure simplex P. Recall that Q is the n-dimensional hyperplane in $\mathbb{R}^c = \mathbb{R}^{n+1}$ which is perpendicular to the vector $(1,1,\ldots,1)$ and contains the origin, that $g: P \to Q$ is the projection along the vector $(1,1,\ldots,1)$ in ^{c}, and that $U = g(P)$. Let $h: U \to P$ be the inverse of g. If ξ is any vector field on U, then $Z = dh(\xi)$ will be a vector field on P and thus will automatically satisfy condition a) for an excess demand type function.

Suppose that ξ is a continuous vector field on $\overline{U}$ that is non-zero on ∂U and points inward into U at all points of ∂U. Let V be an open n-disk lying in U so that $\overline{V} \subset U$, ξ has no zeroes in $U-\overline{V}$, and ξ points into V at all points of ∂V. The vectorfield $Z = h(\xi)$ will point "into" P on the boundary of P. In particular, at the intersection of ∂P with the x_j - axis, Z_i will be positive for $i \neq j$ and zero for $i = j$. One can now modify Z to vector field $\hat{Z}$ on P so that: 1) $\hat{Z}$ agrees with Z on $h(V)$, 2) $\hat{Z}$ is non-zero on $\overline{P}-h(V)$, and 3) if $\{\underline{p}_k\}$ is a sequence in P tending to $\underline{p}_o \in \partial\overline{P}$, then $|\hat{Z}(\underline{p}_k)| \to \infty$ while the negative components of $\hat{Z}(p_k)$ (in $\mathbb{R}^{n+1}$) are bounded below. Thus, $\hat{Z}$ will be an excess demand type function on P with all its zeroes in $h(V)$ where it agrees with $Z = h(\xi)$.

Consequently, whenever ξ is a continuous vector field on $\overline{U}$ that is non-zero and points into U at all points of ∂U, there is an excess demand function $\hat{Z}$ on P such that $\hat{Z} = h(\xi)$ except on some thin band around ∂P and $E_{\hat{Z}} = E_{h(\xi)}$.

We will call such a vectorfield ξ on $\bar{U}$ an <u>excess demand type vector field</u> on $\bar{U}$.

Let us now return to our price mechanism $M(\underline{x};y)$. For any given f on U,

$$\underline{q} \to M(f(\underline{q}), Df(q) \equiv M_f(\underline{q})$$

is a mapping from $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$. Thinking of M_f as a vectorfield on U, it makes sense in view of section 2 to discuss its eigenvalues at a zero point $\underline{q}^*$. The following lemma collects some of the invariant properties of these eigenvalues.

<u>Lemma 3</u>. If $\underline{q}^*$ is a zero for f and an attractor for M_f, then i) the eigenvalues of $DM_f(\underline{q}^*)$ must have non-positive real parts, ii) the trace of $DM_f(\underline{q}^*)$ must be non-positive,

iii) $(-1)^n \det DM_f(\underline{q}^*) \geq 0$, and

iv) $(-1)^n \det A(\underline{0}; Df(\underline{q}^*)) \det Df(\underline{q}^*) \geq 0$.

<u>Proof</u>: Statement i) is standard and was discussed in section 2. Since the trace and determinant of a matrix are invariant under linear coordinate changes, it follows from the Jordan cannonical representation of a matrix that the trace of $DM_f(\underline{q}^*)$ equals the sum of its eigenvalues and the determinant of $DM_f(\underline{q}^*)$ equals the product of its eigenvalues. Thus, ii) and iii) follow directly from i). Recall from Lemma 1 that

$$M_f(\underline{q}) = M(f(\underline{q}); Df(\underline{q})) = A(f(\underline{q}), Df(\underline{q}))\cdot f(\underline{q}).$$

But, $$DM_f(\underline{q}^*) = A(\underline{0}, Df(\underline{q}^*))\cdot Df(\underline{q}^*)$$

by the chain rule since $f(\underline{q}^*) = \underline{0}$, and iv) follows from iii).

5. PROOF OF THE MAIN THEOREM: PARTS A AND B

Proof of Part a): This well known result follows immediately from the fact that U can be viewed as a line segment and f(q) can be viewed as a continuous real valued function on the interior of U which is positive near the left hand end point and negative near the right hand endpoint. Let q^* be some initial condition in predetermined open set $V \subset U$. If $f(q^*) > 0$, then the solution of $\dot{q} = f(q)$ moves to the right until either it encounters a zero of f, or it reaches the right-hand endpoint of U. The intermediate value theorem shows that the former case will occur. Thus the solution tends to an equilibrium point of f. If $f(q^*) < 0$, a similar argument a applies. Notice, the zero need not be a sink, i.e. our EPM is not an LEPM. For example, If $U = (-2,2)$, $f(q) = (q+1)^2(1-q)\frac{1}{(4-q^2)}$ and $V = (-2,-\frac{3}{2})$, then the solution tends to -1; but $q = -1$ is not a sink.

Proof of Part b): We now show that a local effective price mechanism cannot have any ignorable coordinates. Assume then that $M(\underline{x};\underline{y})$ is an LEPM and that y_{hk} is an ignorable coordinate for M. By Lemma 1, $M(\underline{x};\underline{y}) = A(\underline{x};\underline{y})\,\underline{x}$ for some smooth $A:\mathbb{R}^n \times L(\mathbb{R}^n,\mathbb{R}^n) \to L(\mathbb{R}^n,\mathbb{R}^n)$, since $M(\underline{0};\underline{y}) = \underline{0}$; and y_{hk} is also an ignorable coordinate for A. According to the second part of the definition of a LEPM, there is an excess demand function $f^*: U \to \mathbb{R}^n$ and a point $\underline{q}^*$ such that $f^*(\underline{q}^*) = \underline{0}$, $M_{f^*}(\underline{q}^*) = \underline{0}$, and $DM_{f^*}(\underline{q}^*)$ is non-singular.

Denote $Df^*(\underline{q}^*)$ by B. Recall from the calculations of Lemma 3 that $DM_{f^*}(\underline{q}^*) = A(0;B)\cdot B$. Since

$$0 \neq \det DM_{f^*}(\underline{q}^*) = \det A(0;B)\cdot\det B,$$

B is also non-singular. By the inverse function theorem, there is a neighborhood V of $\underline{q}^*$ in U such that $\underline{q}^*$ is the only zero of f^* in V. Since M is an LEPM, we know from Lemma 3 that

$$(-1)^n \det A(0;B)\cdot\det B > 0.$$

There is a standard identification of $n \times n$ matrices with $\mathbb{R}^{n^2}$. With this identification, the determinant can be viewed as a continuous mapping from $\mathbb{R}^{n^2} \to \mathbb{R}$. By using the definition of a continuous function, it follows that there is a non-empty open set in $\mathbb{R}^{n^2}$ such that the determinant of any element has the same sign as $\det A(0;B)$. But $A(0;\_)$ is a smooth mapping from $\mathbb{R}^{n^2}$ into $\mathbb{R}^{n^2}$. Consequently there is an open set $G \subset \mathbb{R}^{n^2}$ containing B such that if $C \in G$, then the sign $\det A(0;C)$ is the same as the sign of $\det A(0;B)$.

Let $B' \in G$ be such that B'_{hk}, the co-factor of the hk^{th} entry of B', is non-zero. Since the set of points where this cofactor is equal to zero is a lower dimensional algebraic set, such a B' can always be found. From elementary matrix theory,

$$\det B' = \sum_{j=1}^{n} (-1)^{h+j}\, b'_{hj}\, B'_{hj},$$

where $B' = ((b'_{ij}))$ and B'_{ij} is the cofactor of b'_{ij}. Since B'_{hk} is non-zero, we can change the sign of this determinant by changing the hk^{th} entry of B' appropriately while leaving all the other entries of B' fixed. Call this new matrix B''. Since y_{hk} is an ignorable coordinate for A,

$$\det A(0,B') = \det A(0,B'').$$

Since $\det B'$ and $\det B''$ have opposite signs ,

$$(-1)^n \det[A(0,B'')\cdot B''] = (-1)^n \det A(0,B')\cdot\det B'' < 0.$$

Finally, we find an excess demand function g on U such that $g(\underline{q}^*) = \underline{0}$ and $Dg(\underline{q}^*) = B''$. Let $h: U \to \mathbb{R}^n$ be an excess demand type vector field on $\bar{U}$ so that h points into U on ∂U and h is non-zero on ∂U. Let W be an open subset of U such that

$$\underline{q}^* \in V \subset \bar{V} \subset W \subset \bar{W} \subset U \ .$$

(as usual, $\bar{V}$ means the closure of V.) Let $\varphi: U \to \mathbb{R}$ be a C^∞ function such that φ is identically 1 on $\bar{V}$ and φ is identically zero outside W. Define $g: U \to \mathbb{R}^n$ by

$$g(\underline{x}) = \varphi(\underline{x})B''(\underline{x}-\underline{q}^*) + (1-\varphi(\underline{x}))h(\underline{x}).$$

On V, $g(\underline{x}) = B''(\underline{x}-\underline{q}^*)$; so, $g(\underline{q}^*)=\underline{0}$ and $Dg(\underline{q}^*) = B''$. Near the boundary of U, g is h. Consequently, g is an excess demand type function; and by Lemma 2 and the Remark below it there is a standard economy whose excess demand function (when projected into U) is g.

However, since

$$(-1)^n \det DM_g(\underline{q}^*) = (-1)^n \det A(0,B'') \det B'' < 0 \ .$$

we know from Lemma 3 that $\underline{q}^*$ cannot be an attractor for M_g. This contradicts our assumption that M was an LEPM and finishes the proof of part b) of the Theorem.

The construction leading to B'' shows there exists an <u>open</u> set of counter-examples to the efficiency of M. This will be true for most of what follows. This means that the counter-examples are not isolated occurances, but they are, in fact, stable in the implied sense of perturbations.

As we remarked in section three, the non-degeneracy requirement in the definition of an LEPM is a natural one for a number of reasons: the rareness of vector fields with degenerate zeroes (i.e., zeroes where the Jacobian is singular), the instability of any degenerate zero of M_f, and the need for a guarantee that the orbits of M_f will tend quickly to the zero of f. Nevertheless, we will demonstrate now that one can weaken this requirement considerably and still show that a mechanism which is locally effective cannot have ignorable coordinates.

First, suppose $M: U \times \mathbb{R}^{n^2} \to \mathbb{R}^n$ is a smooth mapping with $M(\underline{x};\underline{y}) = 0$ whenever $\underline{x} = \underline{0}$. By Lemma 1, there is a smooth map A such that $M(\underline{x};\underline{y}) = A(\underline{x};\underline{y})\underline{x}$. Suppose that M (and A) have y_{hk} as an ignorable coordinate and that $a_{kh}(0;\underline{y})$ is not identically zero. We shall show that M cannot be locally effective, i.e., there exists a smooth excess demand function f and a zero q^k of f such that q^* is not an attractor for $M_f(\underline{x}) \equiv M(f(\underline{x});Df(\underline{x}))$.

To see this, let $B = ((b_{ij}))$ be a matrix such that $a_{kh}(0;B) \neq 0$. Clearly,

$$\text{Trace } A(\underline{0};B) \circ B = \sum_{i,j=1}^{n} a_{ij}(0,B)\, b_{ji} .$$

Since y_{hk} is an ignorable coordinate for A, if we change b_{hk} to b'_{hk} and leave all the other b_{ij} fixed, we will not change any $a_{ij}(0,B)$. Since $a_{kh}(0,B) \neq 0$, we can find a b'_{hk} such that the above trace is positive. Let B' be this new matrix, i.e., B and B' differ only in entry (h,k) but

$$\text{Trace } A(0,B') \circ B' > 0.$$

By the methods used earlier in this section, construct an excess demand function g on U so that $g(\underline{q}^*) = \underline{0}$ and $Dg(\underline{q}^*) = B'$ for some $\underline{q}^* \in U$. Since

$$\text{Trace } DM_g(\underline{q}^*) = \text{Trace } A(0,B') \circ B' > 0,$$ $\underline{q}^*$ cannot be an attractor for M_g by Lemma 3. Consequently, M cannot be locally effective in any sense.

Both of the proofs in this section used first order techniques to show that a mechanism is not locally effective, i.e., they used $DM_g(\underline{q}^*)$ and Lemma 3 to show that $\underline{q}^*$ is not an attractor for M_g. On the other hand, if a mechanism M has y_{hk} as an ignorable coordinate and column h of the corresponding matrix $A(\underline{0};\underline{y})$ is identically zero, then y_{hk} will also be an ignorable coordinate for the first derivative of M, i.e., $A(\underline{0};\underline{y}) \circ \underline{y}$ will be independent of y_{hk}. It is certainly improbable that a mechanism with so many zeroes built into it can turn any zero of any f to an attractor of the corresponding M_f. However, our first order techniques will not work here and one would have to examine the higher order derivatives of M_f to study these very fragile zeroes of M_f. We conclude this section by studying what we feel is the weakest regularity requirement of an LEPM for which first order techniques will work.

As we indicated earlier, we feel that the stronger regularity conditions given above handle any problem of interest to economics. Nevertheless we include the following result to illustrate how weak the regularity condition can be without it adversely affecting our conclusion.

Suppose $M:U \times \mathbb{R}^{n^2} \to \mathbb{R}^n$ is a price mechanism in that $M(\underline{0};\underline{y}) = \underline{0}$, so that $M(\underline{x};\underline{y}) = A(\underline{x},\underline{y})\underline{x}$ for some A as in Lemma 1. Suppose that y_{hk} is an ignorable coordinate for M(and A). Suppose there is a matrix B such that b_{hk} is the only non-zero entry in row h of B and such that A(0,B) has rank n-m, but if one replaces column h of A(0,B) with a column of zeroes then the resulting matrix has rank less than n-m. (Thus, column h contributes to the rank of A(0;B).) For example, A(0,B) may have full rank--the case treated at the beginning of this section. We will show now that M cannot be locally effective.

To prove this statement and to prove other parts of our main Theorem, we will use the following lemma frequently. Although it is a well-known result of modern algebra, at the end of this section we will sketch a proof of it for the sake of completeness.

Lemma 4. Let $p(u) = u^n + q_1 u^{n-1} + \ldots + q_{n-1} u + q_n$ be a polynomial with real coefficients $q_1, \ldots, q_n$. Suppose that all the zeroes of p are non-positive or have non-positive real part. Then, each q_i is non-negative. If all the zeroes of p are negative or have negative real part, then each q_i is positive.

To prove the statement above the Lemma, let $M(\underline{x};\underline{y})$ be a price mechanism with ignorable coordinate y_{hk}, which satisfies the hypothesis of the statement. Let $B = ((b_{ij}))$ be a matrix with $b_{hj} = 0$ for $j \neq k$ and for which the corresponding $A(0;B)$ has rank $n-m$; but if one replaces column h by a column of zeroes, then the resulting matrix has rank less than $n-m$. Since these rank conditions are defined by inequalities and since non-singular matrices are dense in the set of all matrices, we can choose such a B that is non-singular. Denote by $B(\lambda)$ or B_λ the matrix B with λ replacing b_{hk}. So, $B(b_{hk}) = B$; and $A(\underline{0};B_\lambda) = A(\underline{0};B)$ for all $\lambda \in \mathbb{R}$.

Write a_{ij} for the i,j^{th} entry of $A(\underline{0};B)$ and c_{ij} for the i,j^{th} entry of $C \equiv A(\underline{0};B) \circ B(0)$. For simplicity, we will take $h = k = 1$. Therefore,

$$A(0;B(\lambda))\circ B(\lambda) = \begin{pmatrix} a_{11} & \cdots & a_{in} \\ a_{21} & \cdots & a_{2n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda a_{11} + c_{11} & c_{12} & \cdots & c_{1n} \\ \lambda a_{21} + c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ \lambda a_{n1} + c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} .$$

The characteristic polynomial of $A(0;B(\lambda) \circ B(\lambda)$ is

$$p_\lambda(u) = \det\left(uI - A(0;B) \circ B(\lambda)\right)$$

$$= \det\begin{pmatrix} -\lambda a_{11} & -c_{12} & \cdots & -c_{1n} \\ -\lambda a_{21} & u-c_{22} & \cdots & -c_{2n} \\ \vdots & \vdots & & \vdots \\ -\lambda a_{n1} & -c_{n2} & \cdots & u-c_{nn} \end{pmatrix} + \det(uI-C)$$

$$= \lambda\,\{a_{11}p^{(1)}(u) + a_{21}\,p^{(2)}(u) + a_{31}\,p^{(3)}(u) + \ldots + a_{n1}\,p^{(n)}(u)\}$$

$$+ u^n + \gamma_1 u^{n-1} + \gamma_2 u^{n-2} + \ldots + \gamma_n$$

(5.1) $$= u^n + (\gamma_1 + \gamma\alpha_1)u^{n-1} + (\gamma_2 + \lambda\alpha_2)\,u^{n-2} + \ldots + (\gamma_n + \lambda\alpha_n),$$

where $\gamma_1, \ldots, \gamma_n, \alpha_1, \ldots, \alpha_n$ are real constants independent of λ and each $p^{(j)}(u)$ is the $(j,1)$-th cofactor of $(uI-C)$ and is thus a polynomial of degree $\leq n-1$, independent of choice of λ.

Suppose that $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, i.e., that $\{a_{11}p^{(i)}(u) + \ldots + a_{n1}p^{(n)}(u)\}$ is identically zero. Then,

$$p_\lambda(u) = u^n + \gamma_1 u^{n-1} + \ldots + \gamma_n \quad \text{for all } \lambda\,;$$

and rank $A(0;B)$ = rank $A(0;B) \circ B$, since B is non-singular,

= number of non-zero roots of $p_{b_{11}}(u)$

= largest j such that γ_j is non-zero.

However, the matrix C does not use the elements $a_{11}, a_{21}, \ldots, a_{n1}$ and the γ_j are computed from $\det(uI-C)$. Therefore, the γ_j's are independent of $a_{11}, a_{21}, \ldots, a_{n1}$; and the rank of $A(0;B)$ is not changed if one sets the first column equal to zero. This contradiction to our initial hypothesis that column one contributes to the rank of $A(0;B)$ implies that some $\alpha_{jo} \neq 0$ in (5.1).

Now, choose λ, say equal to λ^*, so that $(\gamma_{jo} + \lambda^*\alpha_{jo})$ is negative. By Lemma 4, $p_{\lambda^*}(u)$ will have a positive root or a root with positive real part. Then, $A(0;B(\lambda^*)) \circ B(\lambda^*)$ has an eigenvalue with positive real part. As before, construct an excess demand function g so that $g(\underline{q}^*) = 0$ and $Dg(\underline{q}^*) = B(\lambda^*)$ for some $\underline{q}^* \in U$. Since $DM_g(\underline{q}^*) = A(0;B(\lambda^*) \circ B(\lambda^*)$ has an eigenvalue with positive real part, $\underline{q}^*$ is not an attractor for M_g, although it is a zero for g. Consequently, M cannot be considered locally effective.

As we promised earlier, we finish this section by sketching a proof of Lemma 4. See Mostowski and Stark [1964] for a complete proof and discussion. Lemma 4 may be considered as a special case of Descartes' Rule of Signs and also of the Routh-Hurwicz Theorems.

<u>Proof of Lemma 4</u>: Let $a_1,\ldots,a_r$, b_1, $\bar{b}_1,\ldots$, b_s, $\bar{b}_s$ be the complete set of zeroes of $p(u) = u^n + q_1 u^{n-1} +\ldots+ q_n$, where $a_1,\ldots,a_r$ are real and non-positive and $b_1, \bar{b}_1,\ldots,b_s,\bar{b}_s$ are complex with non-positive real part. Then,

$$p(u) = u^n + q_1 u^{n-1} +\ldots+ q_n = (u-a_1)\ldots(u-a_r)\ (u-b_1)\ (u-\bar{b}_1)\ldots(u-b_s)(u-\bar{b}_s)$$

$$= (u + |a_1|)\ \ldots\ (u + |a_r|)\ (u^2 + 2|\mathrm{Re}b_1|u + |b_1|^2)\ldots(u^2 + 2|\mathrm{Re}b_s| + |b_2|^2)\ .$$

Clearly, the product of polynomials with non-negative coefficients is a polynomial with non-negative coefficients. The first part of Lemma 4 now follows from the fact that two polynomials are equal if and only if corresponding coefficients are equal.

If all the a_i are negative and all the b_j have negative real part, each $|a_i|, |Re b_j|$, and $|b_j^2|$ is positive in (5.2). To prove the second part of Lemma 4, one uses induction and the fact that, if p_1 is a polynomial of degree d_1 with all its $d_1 + 1$ coefficients positive and if p_2 is a polynomial of degree d_2 with all its $d_2 + 1$ coefficients positive, then $p_1 p_2$ is a polynomial of degree $d_1 + d_2$ with all its $d_1 + d_2 + 1$ coefficients positive.

6. Proof of Part C

Assume now that $M(\underline{x};\underline{y})$ is an effective price mechanism with ignorable coordinates y_{ij} and y_{hk}. We shall reach a contradiction by constructing an excess demand function for which M is not effective. It should be clear that, because of the global constraints on the vector field ζ, we will need more intricate arguments than those of the previous section. For example, consider an excess demand vector field ζ with a single rest point and with all vectors pointing in on the boundary of our simplex P. The index of the rest point must be $(-1)^n$ by statement h) in section 2. If we modify ζ to $\hat{\zeta}$ as we did in Section 5 by changing the sign of an eigenvalue at the rest point, then we change the index of the rest point by statement i) in section 2. Since $\hat{\zeta}$ must still point inward on ∂P and the sum of the indices of its zeroes is $(-1)^n$, it follows that $\hat{\zeta}$ has more than one rest point. To show that M is not effective for $\hat{\zeta}$, we will have to examine all the zeroes of $\hat{\zeta}$, including the newly created ones. Consequently, it is to our advantage if we can construct our new excess demand vector field $\hat{Z}$ so that it points into U on the boundary of U and has only one zero in U -- necessarily a zero of index $(-1)^n$. Lemma 5 below will be a valuable tool in such constructions.

In fact, the following argument shows that it makes sense to choose our original Z so that it has only one rest point. Suppose that Z has more than one zero but that Z is generic in that all its zeroes are hyperbolic and therefore have index ± 1. (See Abraham-Robbin [1967] or Smale [1967].) It follows that Z has a zero, p_o, of index $(-1)^{n+1}$, i.e., $(-1)^{n+1} \det DZ(p_o) > 0$. Suppose that M is a price mechanism with ignorable coordinate y_{11} and that p_o is a hyperbolic attractor for M_Z, i.e., $(-1)^n \det DM_Z(p_o) > 0$. Use the argument in the first proof of section 5 to change $B \equiv DZ(p_o)$ to B' by changing only b_{11} so that det B and det B' have opposite signs. Since $A(0,B) = A(0,B')$ where $M(\underline{x};\underline{y}) = A(\underline{x};\underline{y})x$, $\det A(0,B')\circ B'$ and $\det A(0,B)\circ B$ will have opposite signs. Lemma 5 below states that we can find an excess demand field Z' such that $DZ'(p_o) = B'$ and p_o is the only zero of Z' in U. Since

$$(-1)^n \det DM_{Z'}(p_o) = (-1)^n \det A(0,B') \circ B' < 0 ,$$

p_o cannot be an attractor for $M_{Z'}$ (Lemma 3), and consequently M cannot be an EPM.

This argument shows that if p_o is a zero of index $(-1)^{n+1}$ (and therefore not an attractor) for some excess demand function Z and if M is an effective price mechanism with an ignorable coordinate, then p_o will not be a hyperbolic attractor for M_Z. If M_z is "generic", then some other zero of Z will be the attractor for M_Z. In particular, M cannot be an LEPM. Since we want to concentrate on the zeroes of Z which can become attractors for M_Z, we will work only with the zeroes of Z of index $(-1)^n$. If all the zeroes of Z have this index, Z can have only one zero. Thus, it makes sense to emphasize excess demand functions with a single zero.

We now indicate how to extend a linear map with an isolated zero of index $(-1)^n$ to an excess demand function with a single zero. We will use this Lemma a number of times in this paper.

Lemma 5: Let $U \subset \mathbb{R}^n$ be our chart for the price simplex P with $\underline{0} \in U$. Let C be an $n \times n$ hyperbolic matrix such that $(-1)^n \det C > 0$. Then there exists a vector field Z on $\bar{U}$ such that i) $\underline{0}$ is the only rest point of U, ii) $DZ(\underline{0}) = C$, and iii) Z points into U on the boundary of U.

Note that the Z of Lemma 5 is what we called "an excess demand type vector field on U" in the remark following Lemma 2. If one projects Z onto the price simplex P, then there exists an excess demand function which equals this projection -- except possibly on some narrow band near ∂P where both functions are non-zero.

Proof: Since C is hyperbolic $\mathbb{R}^n = E_1 \dotplus E_2$ where E_1 is the maximal invariant subspace corresponding to the eigenvalues of C with negative real part and E_2 is the maximal invariant subspace corresponding to eigenvalues of C with positive real part. In the notation of Section 2, $E_1 = W^s(\underline{0};C)$ and $E_2 = W^u(\underline{0};C)$. Let $s = \dim E_1$ and $u = \dim E_2$. Since $\det C$ is the product of the eigenvalues of C, sign det $C = (+1)^u(-1)^s = (-1)^s$. However, by construction, sign det $C = (-1)^n$. Therefore, $(-1)^n = (-1)^s$, n and s have the same parity, and $u = n-s$ is even.

First, consider the even dimensional subspace E_2. Let $C_2 = C|E_2$. By b) of Section 2, $\underline{0}$ is a source for $\dot{\underline{x}} = C_2(\underline{x})$ on E_2. By standard linear algebra arguments (see p. 149 of Hirsch-Smale [1974]), there is an inner product $<\,,\,>$ on E_2 such that $\langle \underline{x}, C_2\underline{x}\rangle > 0$ for all non-zero $\underline{x}$ in E_2. This means that the vector field $C_2(\underline{x})$ points out of each sphere about $\underline{0}$. Choose $r > 0$ so that the ball of radius $2r$ (in this metric) lies in $E_2 \cap U$. Let $\tilde{g}(\underline{x})$ be a non-zero vector field on the (odd dimensional) unit sphere in E_2. Let $g(\underline{x}) = \tilde{g}\left(\frac{\underline{x}}{<\underline{x},\underline{x}>^{\frac{1}{2}}}\right)$ on $E_2 - \{\underline{0}\}$.

Let $\lambda:[0,\infty) \to \mathbb{R}_+$ be a smooth function so that

$$\lambda(t) = \begin{cases} 0 \text{ for } t \ll \frac{1}{2}r , \\ 1 \text{ for } t = r , \end{cases}$$

and $\lambda'(t) > 0$ for all $t > \frac{1}{2}r$. Let $\mu(\underline{x}) = \lambda(<\underline{x}, \underline{x}>^{\frac{1}{2}})$; and let $f(\underline{x}) = \mu(\underline{x})\, g(x) + (1-\mu(\underline{x}))\, C_2(\underline{x})$, a smooth vector field on E_2 which equals C_2 for $|\underline{x}| < \frac{r}{2}$ and equals the non-zero vector field $\hat{g}$ for $|\underline{x}| = r$. For $0 < |\underline{x}| < r$, $f(\underline{x}) \neq 0$ since $\langle \underline{x}, f(\underline{x})\rangle = (1-\mu(\underline{x})\, \langle \underline{x}, C_2(\underline{x})\rangle$ which is positive. For $|\underline{x}| > r$, $\langle \underline{x}, f(\underline{x})\rangle < 0$ and f points into all balls whose radius is greater than r. Since $B_{2r}(\underline{0}) \cap E_2 \subset U \cap E_2$, we can construct our f on $U \cap E_2$ so that it points inward on $\partial[U \cap E_2]$ and is only zero at $\underline{0}$.

Now, let $C_1 = C|E_1$. Since $\underline{0}$ is a global sink for $\dot{\underline{x}} = C_1\underline{x}$ on E_1, we can readily modify C_1 to h so that $\underline{0}$ is a global sink for h on $U \cap E_1$ and h points into U on $\partial[U \cap E_1]$. Finally, writing $\mathbb{R}^n$ as $E_1 \oplus E_2$, the vector field

$(x_1, x_2)' = (f(x_1), h(x_2)) \equiv H(x_1, x_2)$ has the property that $\underline{0}$ is the only rest point for H, H points into U on ∂U, and $DH(\underline{0}) = C$. (If $\dim E_2 = \dim \mathbb{R}^n = 2$, the above construction yields the phase portrait of figure 2 -- the phase portrait of Scarf's original example. Thus, the proof of Lemma 5 describes a higher order "limit cycle" construction.)

FIGURE 2. -- See page 57.

We now return to the proof of part c) of the Theorem. Assume that $M(\underline{x};\underline{y})$ is an effective price mechanism with ignorable coordinates y_{ij} and y_{hk} where $i \neq h$ and $j \neq k$. For simplicity of notation, we will assume that $(i,j) = (1,1)$ and $(h,k) = (2,2)$. The modifications of the following proof that are needed in the general case are straightforward and will be left to the reader.

By Lemma 1, $M(\underline{x};\underline{y}) = A(\underline{x},\underline{y})\underline{x}$ for some smooth mapping A. By hypothesis, there is a matrix B such that $a_{11}(0;B)$ or $a_{22}(0;B)$ is non-zero, where $A = ((a_{ij}))$. Without loss of generality, we will assume that $a_{11}(\underline{0};B) \neq 0$. Since the non-singular hyperbolic matrices form an open, dense subset of the set of all matrices (see section 7.3 in Hirsch-Smale [1974]) and since a_{11} is continuous, we can find a (non-singular) hyperbolic matrix $\hat{C} = ((\hat{c}_{ij}))$ such that $a_{11}(0, \hat{C}) \neq 0$ and

$$\hat{C}_{33} \neq 0, \text{ where } \hat{C}_{33} = \det \begin{vmatrix} \hat{c}_{33} & \dots & \hat{c}_{3n} \\ \\ \hat{c}_{n3} & \dots & \hat{c}_{nn} \end{vmatrix} .$$

Now using the fact that

$$\frac{\partial a_{ij}}{\partial y_{kk}} (\underline{0};\hat{C}) = 0 \quad \text{for all } i, j; \text{ for } k = 1,2,$$

we will construct a new matrix C by varying c_{11} and c_{22} so that $a_{11}(\underline{0};C) \neq 0$, the trace of $A(\underline{0};C) \circ C$ is positive, and $\det C = (-1)^n$.

First choose $\lambda \in \mathbb{R}$ so that

$$\operatorname{sign} \lambda \hat{C}_{33} = \operatorname{sign} (-1)^n \quad \text{and}$$

$$a_{11} (0;\hat{C}) + \lambda\, a_{22} (0;\hat{C}) \neq 0.$$

Let γ be a real parameter and let $C_\gamma = ((c_{ij}))$ where $c_{11} = \gamma$, $c_{22} = \lambda\gamma$, $c_{ij} = \hat{c}_{ij}$ otherwise.

By a simple calculation,

det $C_\gamma = \lambda\gamma^2 \hat{C}_{33} + \gamma F(\lambda, c_{ij}) + G(c_{ij})$ and trace $A(0;C)C = \gamma [a_{11} + \lambda a_{22}] + H(\underline{a}, \underline{c})$ where F, G and H are independent of c_{11} and c_{22}. Now, let $|\gamma| \to +\infty$ while $\gamma(a_{11} + \lambda a_{22})$ remains positive. Because of the domination by the γ^2-term, det C_γ has the same sign as $\lambda \hat{C}_{33}$ (which has the same sign as $(-1)^n$), and the trace $A(\underline{0};C_\gamma)C_\gamma > 0$, for $|\gamma| \geq |\gamma_o|$. Let C denote C_γ for such a γ. Again, by the density of hyperbolic matrices, we can perturb C to a hyperbolic matrix C^* keeping the properties that det C^* has the same sign as $(-1)^n$ and that the trace of $A(0;C^*) \circ C^*$ is positive. Now use Lemma 5 to construct a vector field Z on U such that i) $\underline{0}$ is the only rest point of Z on U, ii) $DZ(\underline{0}) = C^*$, and iii) Z points into U on the boundary ∂U. Finally, use Lemma 2 and the remarks below it to realize Z as the excess demand function of some standard economy.

To show that our price mechanism M is not effective, we need only show that $\underline{0}$ is not an attractor for M_Z. However, by Lemma 3,

$$DM_Z(\underline{0}) = A(\underline{0};DZ(\underline{0})) \circ DZ(\underline{0}) = A(\underline{0},C^*) \circ C^* .$$

By construction, the trace of $DM_Z(\underline{0})$ is positive. By Lemma 3, $\underline{0}$ cannot be an attractor for M_Z.

It should be clear that the choice of y_{11} and y_{22} as the two ignorable coordinates of M was made only for the sake of simplicity of notion. A very similar construction works if y_{ij} and y_{hk} are the ignorable coordinates of M, provided that $i \neq h$ and $j \neq k$.

7. Proof of Part d)

Part d) of the Theorem states that if $\underline{y}_i$ and some y_{jk} for $i \neq k$ are ignorable coordinates for $M(\underline{x};\underline{y}_1,..,\underline{y}_n)$, then M cannot be an EPM. For this part of the theorem, we can further relax the regularity requirement in the definition of an EPM by requiring for each effective price mechanism M only that there exist an excess demand function Z and a zero p_o of Z such that p_o is a non-degenerate attractor for M_Z, i.e., det $DM_Z(p_o) \neq 0$.

Suppose there is an effective price mechanism $M(\underline{x};\underline{y}_1,..,\underline{y}_n)$ with $y_{11}, y_{21}..,y_{n1}$, and y_{12} as ignorable coordinates. (Again, we have chosen these subindices only to simplify the notation.) Let Z be an excess demand function such that p_o is a zero of Z and a non-degenerate attractor of M_Z. As usual, define $A(\underline{x};\underline{y})$ by $M(\underline{x};\underline{y}) = A(\underline{x};\underline{y})\underline{x}$ and let $B = DZ(p_o)$. By Lemma 3,

$$DM_Z(p_o) = A(\underline{0};B) \circ B.$$

Our goal will be to change B to B' without affecting $A(\underline{0};B')$ so that $(-1)^n \det B' > 0$ and the trace $A(\underline{0};B')^\circ B'$ is positive, just as we did in Section 6.

Since $\det DM_Z(p_o) \neq 0$, $\det A(\underline{0};B) \neq 0$.

First, perturb B to B' so that $A(0;B')$ is still non-singular and

$$B'_{33} = \det \begin{bmatrix} b'_{33} & \cdots & b'_{3n} \\ \vdots & & \vdots \\ b'_{n3} & \cdots & b'_{nn} \end{bmatrix}$$

is non-zero. By part c) of our Theorem, we can assume that $a_{21}(0,B') = 0$ and $a_{1k}(0,B') = 0$ for $k \neq 1$. Since $\det A(0,B') \neq 0$, $a_{11}(0,B') \neq 0$. Change b'_{11} to b''_{11} so that

$$a_{11}(0,B')\, b''_{11} + \sum_{\substack{i,j=1 \\ (i,j) \neq (1,1)}}^{n} a_{ij}(0,B') b'_{ji} > 0. \qquad (7.1)$$

Since $B'_{33} \neq 0$, we can change b'_{21} and b'_{12} to b''_{21} and b''_{12} so that det B" has the same sign as $(-1)^n$ where B" is B' with b''_{11}, b''_{21} and b''_{12} replacing b'_{11}, b'_{21} and b'_{12} respectively. Since y_{21} and y_{12} are ignorable coordinates for A, $A(0,B'') = A(0,B') = A(0,B)$. Since $a_{12}(0,B) = a_{21}(0,B) = 0$, the choice of b''_{12} and b''_{21} will not affect inequality (7.1).

(7.1) Summarizing, we have constructed a matrix B" so that $(-1)^n \det B'' > 0$ and trace $A(0,B'') \circ B'' > 0$. As in Section 6, one uses Lemma 5 to construct an excess demand function Z with a single rest point at $\underline{0}$ so that $DZ(\underline{0}) = B''$. Since

$$\text{trace } DM_Z(\underline{0}) = \text{trace } A(0,B'') \circ B'' > 0,$$

$\underline{0}$ cannot be an attractor for M_Z by Lemma 3. Consequently, M cannot be an EPM.

8. Proof of Part e)

In this section, we will try to derive the sharpest possible estimates on just how many ignorable coordinates a price mechanism can have and still be effective -- at least for economies with not many commodities. Part a) of the theorem took care of the problem for economies with two commodities. We examine next the situation for three commodities, i.e., n = 2.

Suppose that $M: \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^2$ is an EPM with two ignorable coordinates: y_{ij} and y_{hk} where $i \neq h$ and $j \neq k$. Again, for simplicity of notation, we will choose (i,j) to be (1,1) and (h,k) to be (2,2), without loss of generality. Since $M(\underline{0};\underline{y}) = \underline{0}$, $M(\underline{x};\underline{y}) = A(\underline{x};\underline{y})\underline{x}$ for some $A = ((a_{ij}))$, by Lemma 1. Since M is an EPM, we can assume that $a_{11}(\underline{0},\underline{y})$ and $a_{22}(\underline{0},\underline{y})$ are identically zero by part c). Let Z be an excess demand function on U such that Z has a unique zero $p_o \in U$ and $DZ(p_o)$ is the diagonal matrix

$$B = DZ(p_o) = \begin{pmatrix} \lambda_1^o & 0 \\ 0 & \lambda_2^o \end{pmatrix} .$$

By our regularity requirement on M, $DM_Z(p_o)$ is a non-singular matrix. (Actually, the regularity requirement can be weakened considerably in this case. See the remark at the end of this section.)

By Lemma 3,

$$DM_Z(p_o) = A(\underline{0},B) \circ B = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \circ \begin{pmatrix} \lambda_1^o & 0 \\ 0 & \lambda_2^o \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \lambda_2^o a_{12} \\ \lambda_1^o a_{21} & 0 \end{pmatrix} .$$

Since $DM_Z(p_o)$ is non-singular, $A(\underline{0},B)$ is non-singular. Let $B(\lambda_1,\lambda_2,b)$ denote the matrix

$$\begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix} ;$$

so, $B(\lambda_1^o,\lambda_2^o,0) = B$. By hypothesis, $A(\underline{0};B(\lambda_1,\lambda_2,b))$ is independent of changes in λ_1 and λ_2. Furthermore,

$$A(\underline{0};B(\lambda_1,\lambda_2,b)) \circ B(\lambda_1,\lambda_2,b) = \begin{pmatrix} 0 & \lambda_2 a_{12} \\ \lambda_1 a_{21} & b\, a_{21} \end{pmatrix} ,$$

where a_{12} and a_{21} are functions only of b.

Since $A(\underline{0},B)$ is non-singular, $a_{21}(0) \neq 0$, say $a_{21}(0) < 0$. Then, by the continuity of a_{21}, there exists positive ε such that for all $|b| < \varepsilon$, $a_{21}(b) < 0$. Choose $b^o = -\frac{\varepsilon}{2}$ and let $B^o = B(-1,-1,b^o)$. Since det $B^o = 1$ and B^o is hyperbolic, there is an excess demand function Z^o such that $\underline{0}$ is the only zero of Z^o and $DZ^o(\underline{0}) = B^o$. On the other hand, since

$$\text{trace } DM_{Z^o}(0) = \text{trace } A(0,B^o) \circ B^o = b^o a_{21}(b^o) > 0,$$

$\underline{0}$ cannot be an attactor for M_{Z^o}. Since $\underline{0}$ was the only zero of Z^o, M cannot be an EPM. Thus, in economies with three commodities, an EPM cannot have two ignorable coordinates in different rows and columns of $((y_{ij}))$.

Next, we examine economies with four commodities, that is, $n = 3$. Suppose that $M:\mathbb{R}^3 \times L(\mathbb{R}^3,\mathbb{R}^3) \to \mathbb{R}^3$ is an EPM with ignorable coordinates y_{ij} and y_{hk}, $i \neq h$, $j \neq k$. As in the previous arguments, we will assume for notation's sake that $(i,j) = (1,1)$ and $(h,k) = (2,2)$. Once again, $M(\underline{x};\underline{y}) = A(\underline{x};\underline{y})\underline{x}$ for some $A = ((a_{ij}))$; and $a_{11}(\underline{0},\underline{y}) = a_{22}(\underline{0};\underline{y}) = 0$ for all $\underline{y}$.

Let Z be an excess demand function with a unique zero p_o and with $DZ(p_o)$ a non-singular diagonal matrix:

$$DZ(p_o) = \begin{pmatrix} \lambda_1^o & 0 & 0 \\ 0 & \lambda_2^o & 0 \\ 0 & 0 & b_{33} \end{pmatrix} = C.$$

[If y_{13} and y_{21} were the ignorable coordinates, for example, we would choose Z so that the corresponding Jacobian matrix C has non-zero entries only in the $(1,3)$, $(2,1)$, and $(3,2)$ positions.] Since M is effective, $DM_Z(p_o)$ is non-singular. [Once again, the reader is referred to the remark at the end of this section to see how the regularity requirement for an EPM can be relazed in this situation.]

Keeping b_{33} fixed, let $B(\lambda_1,\lambda_2,b_{12},b_{21})$ denote the matrix

$$\begin{pmatrix} \lambda_1 & b_{12} & 0 \\ b_{21} & \lambda_2 & 0 \\ 0 & 0 & b_{33} \end{pmatrix} .$$

In particular, $B(\lambda_1^o,\lambda_2^o,0,0) = C$. Since y_{11} and y_{22} are ignorable coordinates for M (and A), $A(\underline{0};B(\lambda_1,\lambda_2,b_{12},b_{21}))$ is independent of λ_1 and λ_2. Denote this matrix by $A(b_{12},b_{21})$. Recall that $a_{12}(b_{12},b_{21}) = a_{22}(b_{12},b_{21}) = 0$ for all b_{12} and b_{21}.

Since C is non-singular, b_{33} is non-zero. For simplicity of notation, we will take b_{33} to be negative. Throughout the argument for $n = 3$, we will always choose $\lambda_1,\lambda_2,b_{12},b_{21}$ so that $(\lambda_1\lambda_2 - b_{12}b_{21}) > 0$, i.e., so that $(-1)^3 \det B(\lambda_1,\lambda_2,b_{12},b_{21}) > 0$. For each such choice, we can find an excess demand function Z such that Z has a unique zero at $\underline{0}$ and $DZ(\underline{0})$ is $B(\lambda_1,\lambda_2,b_{12},b_{21})$ (Lemma 5). Since M is an EPM, $\underline{0}$ will have to be an attractor for M_Z and the characteristic polynomial of

$$DM_Z(\underline{0}) = A(b_{12},b_{21}) \circ B(\lambda_1,\lambda_2,b_{12},b_{21})$$

must have all its coefficients non-negative by Lemmas 3 and 4.

One computes easily that $A(b_{12},b_{21}) \circ B(\lambda_1,\lambda_2,b_{12},b_{21})$ equals

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \circ \begin{pmatrix} \lambda_1 & b_{12} & 0 \\ b_{21} & \lambda_2 & 0 \\ 0 & 0 & b_{33} \end{pmatrix}$$

$$(8.1) \qquad = \begin{pmatrix} a_{12}b_{21} & a_{12}\lambda_2 & a_{13}b_{33} \\ a_{21}\lambda_1 & a_{21}b_{12} & a_{23}b_{33} \\ a_{31}\lambda_1 + a_{32}b_{21} & a_{31}b_{12} + a_{32}\lambda_2 & a_{33}b_{33} \end{pmatrix} .$$

Write the characteristic polynomial of (8.1) as

$$p(\lambda_1,\lambda_2,b_{12},b_{21})(u) = u^3 + q_1u^2 + q_2u + q_3.$$

From linear algebra, q_j is $(-1)^j$ times the sum of the determinants of the $(j \times j)$ <u>principal</u> minors of (8.1). Consequently,

$$q_1 = - a_{12}b_{21} - a_{21}b_{12} - a_{33}b_{33},$$

$$q_2 = (b_{21}b_{12} - \lambda_1\lambda_2)\, A_{33} + b_{21}b_{33}A_{21} - \lambda_1b_{33}A_{22} - \lambda_2b_{33}A_{11} + b_{12}b_{33}A_{12},$$

$$q_3 = - \det A(b_{12},\, b_{21}) \cdot \det B(\lambda_1,\lambda_2,b_{12},b_{21}),$$

where $A_{ij} = A_{ij}(b_{12},b_{21})$ is the (i,j)th cofactor of $A(b_{12},b_{21})$, i.e., the determinant of the 2×2 submatrix one obtains by deleting row i and column j from $A(b_{12},b_{21})$.

As remarked above, since M is an EPM,

(8.2) each $q_j(\lambda_1,\lambda_2,b_{12},b_{21})$ must be non-negative for all $\lambda_1,\lambda_2,b_{12},b_{21}$ such that $(\lambda_1\lambda_2 - b_{12}b_{21}) > 0$.

We first claim that $A_{11}(b_{12},b_{21})$ and $A_{22}(b_{12},b_{21})$ must be identically zero. Suppose there is a b'_{12},b'_{21} such that $A_{11}(b'_{12},b'_{21}) \neq 0$. We will work with $b_{33}A_{11}(b'_{12},b'_{21})$ positive, but a similar argument is valid if $b_{33}A_{11}(b'_{12},b'_{21}) < 0$. Choose

$$\lambda_1 = \begin{cases} \dfrac{b'_{12}b'_{21}}{2\lambda_2}, & \text{if } b'_{12}b'_{21} < 0 , \\[2ex] \dfrac{2b'_{12}b'_{21}}{\lambda_2}, & \text{if } b'_{12}b'_{21} > 0 , \\[2ex] -\dfrac{1}{\lambda_2} , & \text{if } b'_{12}b'_{21} = 0 . \end{cases}$$

Let λ_2 tend to $+\infty$. As it does, $\lambda_1\lambda_2 - b'_{12}b'_{21}$ will remain positive. By (8.2),$q_2(\lambda_1,\lambda_2,b'_{12},b'_{21})$ must be non-negative. However, as $\lambda_2 \to +\infty$, the $(-\lambda_2 b_{33}A_{11})$ - term in q_2 will tend to $-\infty$, the $(-\lambda_1 b_{33}A_{22})$-term will tend to zero, and the other terms will remain constant. As a result, q_2 will eventually become negative. (If $b_{33}A_{11}(b'_{12},b'_{21}) < 0$, let $\lambda_2 \to -\infty$.) This contradiction to (8.2) demonstrates that $A_{11}(b_{12},b_{21})$

must be identically zero.

Similarly, if one lets $|\lambda_1| \to +\infty$ and choose λ_2 so that $(\lambda_1\lambda_2 - b'_{12}b'_{21}) > 0$ and $\lambda_2 \to 0$, one sees that $A_{22}(b_{12},b_{21})$ must be identically zero.

Next, we claim that $A_{12}(0,0) = A_{21}(0,0) = 0$. To prove this, one argues as in the $n = 2$ case at the beginning of this section. Suppose $A_{21}(0,0) > 0$. Then, there is an $\varepsilon > 0$ such that $A_{21}(b_{12},b_{21}) > 0$ for all $|b_{12}| < \varepsilon$. Choose $b_{12} = 0$ and $b_{21} = \pm \frac{\varepsilon}{2}$ so that

$b_{21}\, b_{33}\, A_{21}\, (0,b_{21}) = \delta < 0$. Let λ_1 and λ_2 go to zero so that $\lambda_1 \lambda_2$ stay positive but the $(-\lambda_1\lambda_2A_{33})$ term in q_2 become less than δ in absolute value. Since $A_{11} = A_{22} = 0$ in q_2, q_2 will be negative when λ_1 and λ_2 are small enough. This contradiction to (8.2) shows that $A_{21}(0,0) = 0$. Similarly, $A_{12}(0,0) = 0$.

Finally, we claim that

$$A_{11}(0,0) = A_{22}(0,0) = A_{12}(0,0) = A_{21}(0,0) = 0$$

implies that the determinant of $A(0,0)$ must be zero. Expanding along the first row, we have

$$\text{(8.3)} \qquad \det A(0,0) = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} = a_{13}A_{13} = a_{13}a_{21}a_{32}\ .$$

Expanding along the first column, we find

$$\det A(0,0) = a_{11}A_{11} - a_{21}A_{21} + a_{31}A_{31}$$

$$\text{(8.4)} \qquad = a_{31}A_{31} = a_{31}a_{12}a_{23}\ .$$

Since $0 = A_{22} = -a_{31}a_{13}$, (8.3) or (8.4) must be zero; and therefore, $\det A(0,0) = 0$.

The fact that det $A(0,0)$ is zero contradicts our original choice of C so that $A(\underline{0};C) = A(0,0)$ is non-singular. This contradiction shows that M with its two ignorable coordinates cannot be an EPM.

Last of all, we consider the case $n = 4$ which corresponds to economies with five commodities. We will demonstrate that a price mechanism which has three ignorable coordinates (not all in the same row or column of the matrix $((y_{ij}))$) cannot be considered effective. There are three cases to consider, each with a different proof: i) each ignorable coordinate is in a different row and different column of $((y_{ij}))$, ii) two ignorable coordinates are in the same column (row) of $((y_{ij}))$ but they all lie in different rows (columns), and iii) two ignorable coordinates lie in the same column of $((y_{ij}))$ and two ignorable coordinates lie in the same row of $((y_{ij}))$.

First, consider case i), the simplest case. We will assume again, for simplicity only, that the ignorable coordinates are y_{11}, y_{22}, and y_{33}. So, suppose that M is an EPM with y_{11}, y_{22}, and y_{33} as ignorable coordinates. Let Z be an excess demand function with a unique zero at p_o such that $DZ(p_o)$ is non-singular and diagonal. Let

$$DZ(p_o) = C = \begin{pmatrix} \overset{\circ}{\lambda}_1 & 0 & 0 & 0 \\ 0 & \overset{\circ}{\lambda}_2 & 0 & 0 \\ 0 & 0 & \overset{\circ}{\lambda}_3 & 0 \\ 0 & 0 & 0 & b_4 \end{pmatrix}$$

By our regularity requirement for M, $DM_Z(p_o)$ must be non-singular. (See the remark at the end of this section.) By Lemma 1, $M(\underline{x};\underline{y}) = A(\underline{x},\underline{y})\underline{x}$ for some $A = ((a_{ij}))$; by part c), $a_{11}(0,\underline{y}) = a_{22}(0;\underline{y}) = a_{33}(0;\underline{y}) = 0$ for all $\underline{y}$.

By Lemma 3, we can write

$$DM_Z(p_o) = A(\underline{0};C) \circ C$$

$$= \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} \lambda_1^o & 0 & 0 & 0 \\ 0 & \lambda_2^o & 0 & 0 \\ 0 & 0 & \lambda_3^o & 0 \\ 0 & 0 & 0 & b_4 \end{pmatrix} .$$

Now, let λ_1, λ_2, and λ_3 vary in C, keeping the non-zero b_4 fixed. Call the resulting diagonal matrix $B(\lambda_1,\lambda_2,\lambda_3)$. Since y_{11}, y_{22}, and y_{33} are ignorable coordinates for M and A, $A(\underline{0};B(\lambda_1,\lambda_2,\lambda_3)) = A(\underline{0};C)$, which we will write simply as A. Our goal is to derive a contradiction to the non-singularity of A.

One computes quickly that $A \circ B(\lambda_1,\lambda_2,\lambda_3)$ equals

$$\begin{pmatrix} 0 & \lambda_2 a_{12} & \lambda_3 a_{13} & b_4 a_{14} \\ \lambda_1 a_{21} & 0 & \lambda_3 a_{23} & b_4 a_{24} \\ \lambda_1 a_{31} & \lambda_2 a_{32} & 0 & b_4 a_{34} \\ \lambda_1 a_{41} & \lambda_2 a_{42} & \lambda_3 a_{43} & b_4 a_{44} \end{pmatrix}$$

and that its characteristic polynomial $u^4 + q_1 u^3 + q_2 u^2 + q_3 u + q_4$ has

$$(8.5) \quad q_2 = - \lambda_1\lambda_2 a_{12}a_{21} - \lambda_1\lambda_3 a_{31}a_{13} - \lambda_1 b_4 a_{41}a_{14} - \lambda_2\lambda_3 a_{23}a_{32} - \lambda_2 b_4 a_{42}a_{24} - \lambda_3 b_4 a_{34}a_{43} .$$

Arguing as we did in establishing statement (8.2) above, we claim that

(8.6) $q_2(\lambda_1,\lambda_2,\lambda_3)$ must be positive for all $(\lambda_1,\lambda_2,\lambda_3)$ such that $(-1)^4 \det B(\lambda_1,\lambda_2,\lambda_3) = \lambda_1\lambda_2\lambda_3 b_4 > 0$.

Suppose (8.6) is not true, i.e., there is a $(\lambda_1^*,\lambda_2^*,\lambda_3^*)$ such that $q_2(\lambda_1^*,\lambda_2^*,\lambda_3^*) \leq 0$ and $\lambda_1^*\lambda_2^*\lambda_3^* b_4 > 0$. By Lemma 5, there is an excess demand function Z^* such that $\underline{0}$ is the only zero of Z^* and $DZ^*(\underline{0}) = B(\lambda_1^*,\lambda_2^*,\lambda_3^*)$. Since M is an EPM, all the eigenvalues of $DM_{Z*}(\underline{0})$ must be non-positive by Lemma 3. Consequently, the q_2-coefficient of the characteristic polynomial of $DM_{Z*}(\underline{0})$ must be non-negative by Lemma 4. Since this q_2 is the above $q_2(\lambda_1^*,\lambda_2^*,\lambda_3^*)$, we have $q_2(\lambda_1^*,\lambda_2^*,\lambda_3^*) = 0$. By the proof of Lemma 4, q_2 is the sum of all products of pairs of eigenvalues of $DM_{Z*}(\underline{0})$. One computes easily that if each of these four eigenvalues is non-positive or has non-positive real part, q_2 can only be zero if one of these eigenvalues is zero, i.e., if $DM_{Z*}(\underline{0})$ is singular. Since $DM_{Z*}(\underline{0}) = A \circ B(\lambda_1^*,\lambda_2^*,\lambda_3^*)$ and $B(\lambda_1^*,\lambda_2^*,\lambda_3^*)$ is non-singular, A must be singular. This contradiction to our choice of Z and C shows that if M is an EPM, (8.6) must hold.

However, a careful examination of (8.5) shows that (8.6) cannot hold. For example, let us show that $a_{12}a_{21}$ must be zero. Choose $\lambda_1 = \lambda_2$ and $\lambda_3 = \pm \frac{1}{\lambda_1}$ with the sign chosen so that $\lambda_1\lambda_2\lambda_3 b_4 > 0$. Let $\lambda_1 \to +\infty$. Since $-\lambda_1\lambda_2 a_{12}a_{21} = -\lambda_1^2 a_{12}a_{21}$ will eventually dominate q_2 in (8.5), $q_2 \geq 0$ implies that $-a_{21}a_{12} \geq 0$. Now, let $\lambda_1 = -\lambda_2$ and $\lambda_3 = \mp \frac{1}{\lambda_1}$, and let $\lambda_1 \to +\infty$ again. This time, we see that $-a_{21}a_{12} \leq 0$. Consequently, $a_{21}a_{12} = 0$. The same type of argument shows that each term in (8.5) must be zero and, thus, that $q_2 = 0$. This contradiction to (8.6) shows that M cannot be an EPM.

Remark 1. The same proof will work for any n. In other words, if $M: \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}^n$ has (n-1) ignorable coordinates - all in different rows and columns of the matrix $((y_{ij}))$, then M cannot be an EPM.

We now examine cases ii) and iii) listed above for $n = 4$, i.e., where two of the three ignorable coordinates lie in the same row or same column of $((y_{ij}))$. Once more, for simplicity of notation, we will assume that the ignorable coordinates are y_{11}, y_{22}, and y_{12} or y_{13}. Suppose that M is an EPM having these y_{ij}'s as ignorable coordinates. Let Z be an excess demand function such that Z has a single zero p_o and $DZ(p_o)$ is non-singular and diagonal. Write

$$DZ(p_o) = C = \begin{pmatrix} \overset{\circ}{\lambda}_1 & 0 & 0 & 0 \\ 0 & \overset{\circ}{\lambda}_2 & 0 & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{pmatrix}.$$

Since M is an EPM, $DM_Z(p_o)$ must be non-singular. Let $A(\underline{x},\underline{y})$ be the matrix such that $M(\underline{x},\underline{y}) = A(\underline{x},\underline{y})\,\underline{x}$ (Lemma 1). By part C', we can suppose that $a_{11}(\underline{0};\underline{y})$ and $a_{22}(\underline{0};\underline{y})$ are identically zero entries of $A(\underline{0};\underline{y})$.

Let $B(\lambda_1,\lambda_2,b_{12},b_{21},b_{13})$ denote the matrix

$$\begin{pmatrix} \lambda_1 & b_{12} & b_{13} & 0 \\ b_{21} & \lambda_2 & 0 & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{pmatrix},$$

where b_{33} and b_{44} will remain fixed (non-zero) entries throughout this proof. Let $A(b_{12},b_{21},b_{13}) = A(\underline{0};B(\lambda_1,\lambda_2,b_{12},b_{21},b_{13}))$. Then, $C = B(\overset{\circ}{\lambda}_1,\overset{\circ}{\lambda}_2,0,0,0)$ and $A(\underline{0};C) = A(0,0,0)$.

First, set $b_{13} = 0$. Let $A_{ij}(b_{12}, b_{21}, b_{13})$ denote the $(i,j)^{th}$ cofactor of $A(b_{12}, b_{21}, b_{13})$. By arguing exactly as in the $n=3$ case above and working with q_3, the coefficient of the u-term in the character polynomial of $A(b_{12}, b_{21}, 0) \circ B(\lambda_1, \lambda_2, b_{12}, b_{21}, 0)$, one computes that

$$A_{11}(b_{12}, b_{21}, 0) = 0 ,$$

$$A_{22}(b_{12}, b_{21}, 0) = 0 ,$$

$$A_{12}(0,0,0) = 0 ,$$

$$A_{21}(0,0,0) = 0 .$$

We now concentrate on case ii) where y_{13} (and not y_{12}) is the third ignorable coordinate for M. We will keep $b_{12} = b_{21} = 0$ in this part of the proof. Let $u^4 + q_1 u^3 + q_2 u^2 + q_3 u + q_4$ denote the characteristic polynomial of $A(0,0,b_{13}) \circ B(\lambda_1, \lambda_2, 0, 0, b_{13})$. One computes, as above, that

(8.7)
$$\begin{aligned} q_1 &= - b_{13}a_{31} - b_{33}a_{33} - b_{44}a_{44} , \\ q_2 &= - \lambda_1\lambda_2 a_{12}a_{21} - \lambda_1(b_{33}a_{13}a_{31} + b_{44}a_{14}a_{41}) \\ &\quad - \lambda_2 b_{13}a_{21}a_{32} - \lambda_2(b_{33}a_{23}a_{32} + b_{44}a_{24}a_{42}) \\ &\quad + b_{13}b_{44}(a_{31}a_{44} - a_{41}a_{34}) + b_{33}b_{44}(a_{33}a_{44} - a_{34}a_{43}) \\ q_3 &= - \lambda_1\lambda_2(b_{33}A_{44} + b_{44}A_{33}) - \lambda_1 b_{33}b_{44}A_{22} \\ &\quad + \lambda_2 b_{13}b_{44}A_{13} - \lambda_2 b_{33}b_{44}A_{11}, \quad \text{and} \\ q_4 &= \det A(0,0,b_{13}) \cdot \det B(\lambda_1, \lambda_2, 0, 0, b_{13}) . \end{aligned}$$

Once again, one argues as in the proofs of statements (8.2) and (8.6) to demonstrate that

(8.8) each q_j in (8.7) must be non-negative for all $\lambda_1, \lambda_2, b_{13}$ such that $\lambda_1\lambda_2 b_{33}b_{44} = (-1)^4 \det B\ (\lambda_1, \lambda_2, 0, 0, b_{13})$ is positive.

Since y_{11}, y_{22} and y_{13} are ignorable coordinates for A, the a_{ij} and A_{ij} in (8.7) are independent of our choice of λ_1, λ_2 and b_{13}. Furthermore, det $B(\lambda_1,\lambda_2,0,0,b_{13})$ is independent of b_{13}. One can use part c) or one can notice that a_{31} is the coefficient of b_{13} in q_1 in (8.7) to demonstrate that a_{31} must be zero.

Next, examine q_3 in (8.7). As we noted above, $A_{11}(0,0,0) = A_{22}(0,0,0) = 0$. One can easily verify this again, by setting $b_{13} = 0$ and then letting $|\lambda_1| \to \infty$ and $\lambda_1\lambda_2 \to 0$ in q_3. If $A_{22}(0,0,0) \neq 0$, $-\lambda_1 b_{33} b_{44} A_{22}$ will dominate q_3 and will force it to take on both signs, contradicting (8.8). We now show that A_{13} must be zero by the same argument. This time, fix λ_2 and let $\lambda_1 \to 0$ so that $\lambda_1\lambda_2 b_{33} b_{44}$ stays positive but $\lambda_1\lambda_2 \to 0$. The $\lambda_2 b_{13} b_{44} A_{13}$-term will dominate q_3. By first letting b_{13} go to $+\infty$ and then to $-\infty$, we can make q_3 take on both signs if $A_{13} \neq 0$. Since q_3 cannot be negative by (8.8), A_{13} must be zero.

Next, by examining q_2 in (8.7), we claim that $a_{14}a_{41}$ must be zero. Fix $b_{13} = 0$; let $|\lambda_1| \to \infty$ and $\lambda_2 \to 0$ so that $\lambda_1\lambda_2 \to 0$. Then, $-\lambda_1(b_{33}a_{13}a_{31} + b_{44}a_{14}a_{41})$ will dominate q_2. Recall that $a_{31} = 0$. So if $a_{14}a_{41} \neq 0$, we can make this term go to $+\infty$ and to $-\infty$, forcing q_2 to take on both signs. Since q_2 cannot be negative by (8.8), $a_{14}a_{41}$ must be zero.

Let us bring together all our information. At $(\overset{\circ}{\lambda}_1,\overset{\circ}{\lambda}_2,0,0,0)$, we have $A_{11} = A_{12} = A_{21} = A_{22} = A_{13} = 0$ and $a_{11} = a_{22} = a_{31} = a_{14}a_{41} = 0$. By computing det $A(0,0,0)$ along the first row and also along the first column as in (8.3) and (8.4), one finds that det $A(\underline{0};C) = 0$. This contradiction to our hypothesis that

$$DM_Z(p_o) = A(\underline{0};C) \circ C$$

is non-singular shows that an EPM cannot have y_{11}, y_{22} and y_{13} as ignorable coordinates.

Finally, we examine case iii) for $n = 4$, where y_{11}, y_{22} and y_{12} are the ignorable coordinates for the mechanism M which we assume to be an EPM. For this part of the proof, we will set $b_{13} = 0$ and work with $B(\lambda_1, \lambda_2, b_{12}, b_{21}, 0)$. Let Z be as in the proof of part ii) so that p_o is the only zero of Z, $DZ(\underline{0}) = B(\overset{\circ}{\lambda}_1, \overset{\circ}{\lambda}_2, 0,0,0) = C$ and $DM_Z(\underline{0}) = A(\underline{0};C) \circ C$ is non-singular.

Recall that we have already computed that

$A_{11}(0,0,0) = A_{22}(0,0,0) = A_{12}(0,0,0) = 0$

and $a_{11}(0,0,0) = a_{22}(0,0,0) = 0$.

Let $u^4 + q_1 u^3 + q_2 u^2 + q_3 u + q_4$ denote the characteristic polynomial of $A \circ B(\lambda_1, \lambda_2, b_{12}, 0, 0)$, where we have written A for $A(\underline{0}; B(\lambda_1, \lambda_2, b_{12}, 0, 0))$ since the latter is a constant matrix because y_{11}, y_{22}, y_{12} are ignorable coordinates for $A(\underline{0};\underline{y})$. Once again, the hypothesis that M is an EPM requires that

each q_j must be non-negative for all $\lambda_1, \lambda_2, b_{12}$ such that

(8.9) $(-1)^4 \det B(\lambda_1, \lambda_2, b_{12}, 0, 0) = \lambda_1 \lambda_2 b_{33} b_{44} > 0$.

(Compare (8.9) with (8.2), (8.6) and (8.8).)

First, compute that $q_1 = -b_{12}a_{21} - b_{33}a_{33} - b_{44}a_{44}$. Since we can vary b_{12} without affecting $\det B(\lambda_1, \lambda_2, b_{12}, 0, 0)$, if $a_{21} \neq 0$, we can make q_1 negative by proper choice of b_{12} and, thus, violate (8.9). Consequently, $a_{21} = 0$.

Next, compute det A using $a_{11} = a_{22} = a_{21} = 0$. One finds that

$$\text{(8.10)} \qquad \det A = \det \begin{pmatrix} a_{31} & a_{22} \\ a_{41} & a_{42} \end{pmatrix} \det \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}$$

Now, compute q_2 using $a_{11} = a_{22} = a_{21} = 0$:

$$\begin{aligned} q_2 = &- \lambda_1(b_{33}a_{13}a_{31} + b_{44}a_{14}a_{41}) \\ &- \lambda_2(b_{33}a_{23}a_{32} + b_{44}a_{24}a_{42}) \\ &- b_{12}(b_{33}a_{23}a_{31} + b_{44}a_{24}a_{41}) \\ &+ b_{33}b_{44}(a_{33}a_{44} - a_{34}a_{43}) . \end{aligned}$$

Fix λ_1 and λ_2 so that $\lambda_1\lambda_2 b_{33}b_{44} > 0$ and let $|b_{12}| \to \infty$. If $(b_{33}a_{23}a_{31} + b_{44}a_{24}a_{41}) \neq 0$, we can make q_2 negative by proper choice of b_{12} and, as a result, violate (8.9). If $(b_{33}a_{13}a_{31} + b_{44}a_{14}a_{41}) \neq 0$, we can fix b_{12}, let $|\lambda_1| \to \infty$ and $\lambda_2 \to 0$ so that $\lambda_1\lambda_2 b_{33}b_{44}$ stays positive. Again, proper choice of λ_1 will force q_2 to be negative and violate (8.9). Consequently,

$$(b_{33}a_{31}, b_{44}a_{41}) \cdot (a_{13}, a_{14}) = 0 \quad \text{and}$$

$$(b_{33}a_{31}, b_{44}a_{41}) \cdot (a_{23}, a_{24}) = 0.$$

By hypothesis, b_{33} and b_{44} are non-zero. Thus, in order for these two equalities to hold, either $a_{31} = a_{41} = 0$ or

$$\det \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} = 0 .$$

However, if we substitute these zeroes into (8.10), we see that det A must be zero.

This contraction to the non-singularity of

$$DM_Z(p_o) = A \circ C$$

shows that M with its three ignorable coordinates cannot be an EPM.

Remark 2. This type of argument works for values of n greater than 4. However, there is a quantum increase in the number of cases that must be examined and in the complexity of the arguments for $n > 4$. We feel that we have proven our point: A price mechanism which can be considered effective in any sense of the word must take into consideration just about all of the first partial derivatives of the excess demand function.

Remark 3. In section 3, we required that in order to be effective a price mechanism $M:\mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}^n$ should have the property that if an excess demand function Z has a non-degenerate zero, then the corresponding M_Z should have a non-degenerate zero at some zero of Z. Actually we have used a much weaker regularity requirement in this section. We have proven that:

If each zero of an excess demand function Z is also a zero of M_Z, if M has ignorable coordinates $y_{h_1k_1}, \ldots, y_{h_mk_m}$, and if there exists an excess demand function Z such that:

i) Z has a unique zero p_o,

ii) the matrix of $DZ(p_o)$ has exactly one non-zero entry in each row and in each column; and if $j = h_i$ for some i, then the non-zero entry in row j occurs at one of the (h_r,k_r)'s (and similarly for the columns),

iii) $DM_Z(p_o)$ is non-singular,

then if m is large enough (e.g., $m \geq n - 1$) and if the $y_{h_ik_i}$ do not lie in the same column or same row of $((y_{ij}))$, there is an excess demand function Z' such that no zero of Z' is an attractor of $M_{Z'}$.

9. SOME EXAMPLES

In the last three sections we examined the following algebraic problem: Let $\mathcal{M}_n$ denote the space of all $n \times n$ matrices, a space naturally isomorphic to $\mathbb{R}^{n^2}$. Let $A:\mathcal{M}_n \to \mathcal{M}_n$ be a continuous (non-linear) mapping. Suppose

i) there exist a $B \in \mathcal{M}$ such that $A(B)$ is non-singular,

ii) for all $B \in \mathcal{M}$ with $(-1)^n \det B > 0$, all the eigenvalues of $A(B) \circ B$ have non-positive real part.

Can A have ignorable coordinates; and if so, how many?

Actually, we are willing to let the domain of A be some open (and preferably dense) subset of $\mathcal{M}$. For example, let $\mathcal{N}$ be the set of all non-singular matrices in $\mathcal{M}$, and define $A^*:\mathcal{N} \to \mathcal{M}$ by $A^*(B) = -B^{-1}$. Thus, A^* works since $A^*(B)\circ B = -I$, all of whose eigenvalues are -1. Of course, A^* corresponds to Newton's Method and has no ignorable coordinates.

In Section 6, we saw that if A has two ignorable coordinates b_{ij} and b_{hk} with $i \neq h$ and $j \neq k$ and if components a_{ji} and a_{kh} of A are not identically zero, then ii) will not hold for A. In ection 7, we saw that if $b_{1i},..,b_{ni}$, b_{jk} are ignorable coordinates for A with $k \neq i$, then A will not satisfy both i) and ii). In Section 8, we saw that for $n = 3, 4$, if A has (n-1) ignorable coordinates not all in the same row or column of B and if A satisfies i) for some special B, then A will not satisfy ii).

To see that these results are as strong as possible, we will construct an $\tilde{A} : \mathcal{M}_n \to \mathcal{M}_n$ which satisfies i) and ii) and which has a <u>complete column of entries of B as ignorable coordinates</u>. If $\tilde{A}$ had one more ignorable coordinate, then the above mentioned results show that $\tilde{A}$ could not satisfy both i) and ii).

For $n = 2$, let $\tilde{A}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} -b_{22} & b_{12} \\ -b_{12} & -b_{22} \end{pmatrix}$. For $n = 3$, let $\tilde{A}(B) = ((a_{ij}(B)))$, where

$$a_{1j}(B) = (-1)^{j+1} B_{j1};\ j = 1,2,3,$$

$$a_{2j}(B) = \frac{\sum_{i=1}^{3} b_{i2} b_{i3}}{\sum_{i=1}^{3} b_{i3}^2}\, b_{j3} - b_{j2};\ j = 1,2,3,$$

$$a_{3j}(B) = -\, b_{j3};\ j = 1,2,3.$$

Here, b_{jk} is the (j,k)th entry of B and B_{jk} is its cofactor. One computes easily that

1) $\tilde{A}(B)$ is well-defined, provided the third column of B has a non-zero entry,

2) $$\tilde{A}(B) \circ B = \begin{pmatrix} \det B & 0 & 0 \\ * & \frac{(\underline{b}_{.2} \circ \underline{b}_{.3})^2}{|\underline{b}_{.3}|^2} - |\underline{b}_{.2}|^2 & 0 \\ * & * & -|\underline{b}_{.3}|^2 \end{pmatrix},$$

   where $\underline{b}_{.k}$ is the 3-vector corresponding to the k-th column of B,

3) if $\det B \neq 0$, $\tilde{A}(B)$ is well-defined and hyperbolic,

4) if $(-1)^3 \det B > 0$, i.e., $\det B < 0$, then all the eigenvalues of $\tilde{A}(B) \circ B$ are negative,

5) b_{11}, b_{21}, and b_{31} are ignorable coordinates for $\tilde{A}$.

One can construct such a mapping $A: \mathcal{M}_n \to \mathcal{M}_n$ for each n. The first row of A(B) will contain the cofactors of the elements in the first column of B, with proper choice of signs. For $j > 1$, the j^{th} row $A_{.j}$ of A(B) will be constructed from elements in the last $n - j+1$ columns of B so that as an n-vector it is perpendicular to columns $j+1, j+2, \ldots, n$ of B and has a negative inner product with the j^{th} column of B. The resulting $A(B)$ will be well-defined and non-singular (even hyperbolic) for non-singular B and will be independent of the etnries in the first column of B. The resulting $A(B) \circ B$ will be a lower triangular matrix which will have all its eigenvalues negative if $(-1)^n \det B > 0$.

The construction of such an $\tilde{A}: \mathcal{M}_n \to \mathcal{M}_n$ shows that the results of parts c), d), and e) are about as strong as possible -- at least algebraically. We conclude this paper with an introductory report on the corresponding price mechanisms -- a report which we hope to continue in a later paper. We will work with the simplest situation, n = 2, although the following remarks hold for all n.

For n = 2, recall that

$$A\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} -b_{22} & b_{12} \\ -b_{12} & -b_{22} \end{pmatrix} \quad \text{and}$$

$$A(B) \circ B = \begin{pmatrix} -\det B & 0 \\ -b_{11}b_{12} - b_{21}b_{22} & -b_{12}^2 - b_{22}^2 \end{pmatrix}$$

The simplest corresponding M is

$$M(x_1, x_2; y_{11}, y_{12}, y_{21}, y_{22}) = \begin{pmatrix} -y_{22} & y_{12} \\ -y_{12} & -y_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

If $f = (f^1, f^2): U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is an excess demand function, then $M_f(x_1, x_2) = M(f(\underline{x}); Df(\underline{x})) =$

$$\begin{pmatrix} -f_y^2(\underline{x}) & f_y^1(\underline{x}) \\ -f_y^1(\underline{x}) & -f_y^2(\underline{x}) \end{pmatrix} \begin{pmatrix} f^1(\underline{x}) \\ f^2(\underline{x}) \end{pmatrix} = \begin{pmatrix} (f^2 f_y^1 - f^1 f_y^2)\ (\underline{x}) \\ -(f^1 f_y^1 + f^2 f_y^2)\ (\underline{x}) \end{pmatrix} .$$

If $f(\underline{x}^o) = \underline{0}$, $M_f(\underline{x}^o) = \underline{0}$. If $M_f(\underline{x}^o) = \underline{0}$ and det $Df(\underline{x}^o) \neq \underline{0}$, $f(\underline{x}^o) = \underline{0}$.

If $f(\underline{x}^o) = \underline{0}$ and det $Df(\underline{x}^o) > 0$, $\underline{x}^o$ is a hyperbolic attractor of $\dot{\underline{x}} = M_f(\underline{x})$. (However, if $f(\underline{x}^o) = \underline{0}$ and det $Df(\underline{x}^o) < 0$, $\underline{x}^o$ will not be an attractor for M_f by c) in section 2. Hence, M is not an LEPM.)

Suppose that f is an excess demand function from a regular economy, as discussed in Debreu (1970) and Dierker (1974). (Basically, this means that det $Df(\underline{x}^o) \neq 0$ whenever $f(\underline{x}^o) = \underline{0}$. Debreu (1970) shows that these are the typical excess demand functions.) Then, by index arguments, there is an $\underline{x}^o$ such that det $Df(\underline{x}^o) > 0$ and therefore $\underline{x}^o$ is an attractor of M_f. If f is an excess demand function for a regular economy with a unique price equilibrium $\underline{p}^o$ and if det $Df(\underline{x}) \neq 0$ for all $\underline{x}$, then $\dot{\underline{x}} = M_f(\underline{x})$ will be a vector field on the 2-disk whose only rest point is the attractor $\underline{p}^o$. If one starts near $\underline{p}^o$ and uses $\dot{\underline{x}} = M_f(\underline{x})$ instead of Newton's method to compute the equilibrium, this mechanism will converge rapidly to $\underline{p}^o$ -- even though there are ignorable coordinates and no matrix need be inverted. What happens globally will be the subject of a future report.

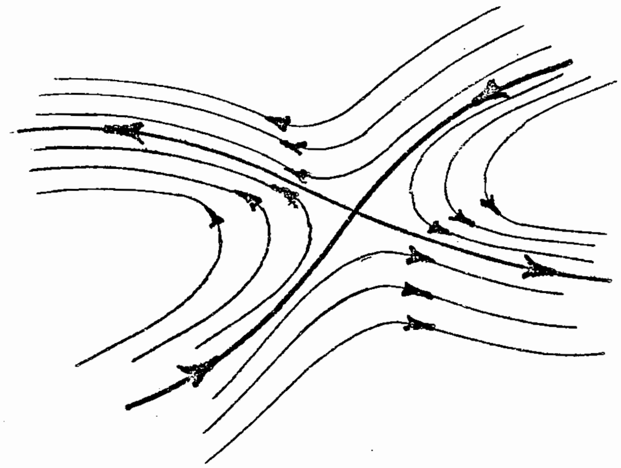

FIGURE 1.

Saddle point. The heavy line with the arrows pointing inward is the stable manifold. The other heavy line is the unstable manifold. Only those solutions on the stable manifold converge to the equilibrium point.

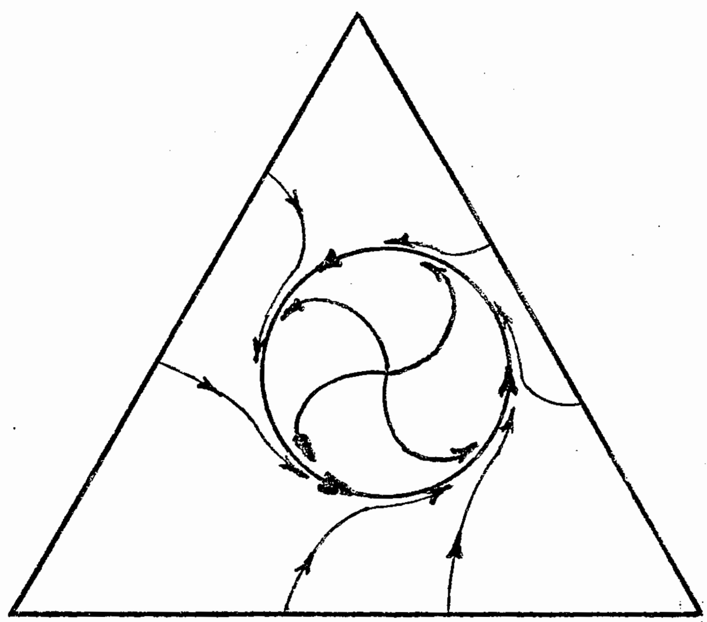

FIGURE 2.

Limit cycle. With the exception of the equilibrium position, all solutions tend toward the limit cycle.

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