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THE SPATIAL CONFIGURATION WHICH MINIMIZES THE LENGTH OF THE BLACK - WHITE BORDER

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Introduction

While attempting to model the process of residential segregation in urban areas, some authors have advanced the hypothesis that politically powerful whites will seek that spatial allocation of households which provides them with the least direct exposure to blacks [2], [3]. This has led to the posing of the following problem:

> _Let a circular city of fixed radius be given. Suppose a given population of blacks and whites is to be spread with uniform density over the city, but in such a way that complete segregation obtains. What spatial configuration of the black neighborhood (s) will accommodate the black population while providing a border with the white population of minimal length?_

Though many may view the above query to be one of dubious scientific value, all would agree that, if worth answering at all, it should be answered correctly. A recent paper by Yinger [4] addresses a considerably less general question. He studies the conditions under which locating blacks in a wedge shaped area gives a border length less than that arising from locating blacks in a circular area in the city's center. He finds in the case of uniformly distributed population that if the fraction of blacks in the population exceeds $1/\pi^2 \approx .101$, then a wedge shaped area will be preferred by racist whites.

Yinger's analysis, while quite correct as far as it goes, is unfortunately irrelevant and misleading. As this paper will show, it is never optimal to locate an undesired minority in the city's center. Moreover, the wedge-shaped area is optimal only in the case where blacks are exactly one-half the population. Below a complete solution to this problem is given under the assumption of uniform distribution of population. The optimal spatial configuration is seen to involve locating the undesired racial minority at the city's fringe, in a lense-shaped area determined by the intersection of two circles, one of which represents the city's boundary.

A Fundamental Principle:

Actually it is possible to get considerable insight into this problem through application of the following Principle of Duality:

> A spatial configuration containing a fixed area and yielding the minimal black-white border length must also contain the the maximal area over all configurations with that same border length.

While a formal proof of the above principle may be derived from classical calculus of variations theory [1], it is quite intuitive as it stands. The basic point is that if the border minimizing configuration did not maximize enclosed area, then there would exist an alternative configuration with the same border length enclosing greater area. By perturbing the boundary of this alternative configuration slightly we can reduce its border length and remain with an area at least as large as the original configuration. This contradicts the presumed optimality of the initial configuration.

Consequences of Duality:

Armed with the above observation, we can learn a number of preliminary facts about the shape of the border minimizing enclosure. First note that either the black area is bounded by a non-trival segment of the city's boundary or it is not. If not, then it is well known that the optimal configuration must be circular. Hence we may proceed by investigating the alternative case and comparing our results to those arising for circular enclosures. Henceforth we discuss only those enclosures bordered in part by the city's boundary.

Notice now that the optimal configuration must be a convex set. For suppose this were not so. If a presumed optimal non-convex configuration is connected (i.e. consists of one piece) then its convex hull encloses more area with shorter boundary -- a contradiction. If it were not connected then the above argument shows that each segment must be convex. Slide two disjoint segments along the city's border until they just touch. Now take the convex hull of the resulting configuration. (See Figure 1) Again the resulting set contains greater area with a shorter border, contradicting presumed optimality.

In addition, any straight line supporting the optimal configuration at an interior border point must have an intersection with the city of length at least as great as the length of the interior border of the configuration. (That is $|CD| \geq |AB|$ in Figure 2.) The reasoning is obvious. Moreover, these lengths can be the same only if the boundary and supporting line segment coincide.

It is easily seen that the wedge shaped area studied by Yinger could only be optimal when blacks are to have exactly half the city's area. Consult Figure 3. The supporting segment ABC has the same length as the interior boundary DBE. The segment FG is constructed so that the area above it is equal to the area of the wedge DBE. It obviously has a shorter border.

The Optimal Configuration:

What then is the border minimizing configuration? Intuition suggests that the interior boundary should be a circular arc. Moreover, transversality considerations of control theory indicate that this arc be normal to the city's boundary at the points of intersection. These observations uniquely determine the optimal configuration.

Consider Figure 4. The angle θ is arbitrary. The line segments AB and BC are perpendicular to the radii OA and OC respectively, and are therefore equal. Rotate AB about B until it coincides with BC, tracing out the dashed circular arc in the figure. The lense-shaped area with corners at A and C has the minimal border length over all configurations containing the same area. As θ varies from zero to π, the enclosed area in this construction varies from zero to one half the city's area. The problem is of course symmetric. (A point missed by Yinger.) If blacks are in the majority, then we must locate the white minority such that there is minimal border length. This construction represents a complete solution to the initially posed problem. Below we prove this configuration optimal among those using a non-trivial segment of the city's boundary. We then show that boundary using configurations dominate fully contained ones.

Proof of Optimality Among Boundary-Using Configurations:

Let us consider the problem most generally. A circle of radius $\bar{u}$ represents the exterior boundary of the city and is given parametrically by

$$(1) \qquad \tilde{x}(t) = \bar{u}(1 - \cos t) \qquad t \in [0, 2\pi]$$
$$\tilde{y}(t) = \bar{u} \sin t$$

We seek an arc connecting two points on this circle enclosing a given area A, which is of minimal length. Without loss of generality take the origin as one end point of this arc. Assume (again WLOG) that the fraction of the city's area to be reserved for the undesired racial group does not exceed one half. Represent an arbitrary differentiable arc from the origin to a point on the city's boundary lying on or above the X axis parametrically by

$$(2) \qquad x = x(t) \qquad t \in [0,1]$$
$$y = y(t)$$

where $x(0) = y(0) = 0$, and $x(1) \equiv x_1$ is arbitrary, while $y(1) \equiv y_1 = (\bar{u}^2 - (\bar{u} - x_1)^2)^{1/2}$. It is clear that the minimizing arc may be chosen from among these. The length of such an arc is given by

$$(3) \qquad L = \int_0^1 [\dot{x}(t)^2 + \dot{y}(t)^2]^{1/2} \, dt$$

where $\dot{z}(t)$ denotes the derivative with respect to t of the function z(t). Let C_1 denote the arc defined by (2). C_2 will denote the arc defined by the city's boundary from the origin to the point (x_1, y_1). It may be verified that C_2 has the parametric representation

$$(4) \quad \tilde{x}(t) = \bar{u}(1 - \cos t) \qquad t \in [0, t_1],$$
$$\tilde{y}(t) = \bar{u} \sin t \qquad t_1 \equiv \tan^{-1}\left(\frac{y_1}{\bar{u} - x_1}\right).$$

Figure 5 may be helpful.

Now the area enclosed by the arcs C_1 and C_2 must equal A. It is well known (from Green's Theorem) that this area may be expressed as the difference of two line integrals over the arcs C_1 and C_2 as follows:

$$(5)\ A = \int_{C_1} x\,dy - \int_{C_2} \tilde{x}\, d\tilde{y}.$$

Expressing these as Stieljes integrals we have the area constraint

$$(6)\qquad \int_0^1 x(t)\ \dot{y}(t)\ dt = A + \int_0^{t_1} \tilde{x}(t)\ \dot{\tilde{y}}(t)dt$$

where $\tilde{x}$, $\tilde{y}$ and t_1 are given once x_1 is.

A solution is then a pair of functions (x^*, y^*) on the unit interval minimizing L subject to (6). Form the Lagrangean

$$(7)\mathcal{L} \equiv \int_0^1 [\dot{x}(t)^2 + \dot{y}(t)^2]^{1/2} dt + \lambda\{A + \int_0^{t_1} \tilde{x}(t)\dot{\tilde{y}}(t)dt - \int_0^1 x(t)\dot{y}(t)dt\}$$

A solution pair must be extremals of (7) and hence satisfy Euler differential equations

$$(8)-\lambda\dot{y} = \frac{d}{dt}\ [\dot{x}(\dot{x}^2 + \dot{y}^2)^{-1/2}]$$

$$(9)\ \lambda\dot{x} = \frac{d}{dt}\ [\dot{y}(\dot{x}^2 + \dot{y}^2)^{-1/2}].$$

Denote by $\left(\frac{dy}{dx}\right)_t$ the slope of the arc C_1 relative to the X axis (possibly infinite) at the point $(x(t),y(t))$. Then we may rewrite the Euler equations as

$$(8')\ - \lambda\dot{y}_t = \frac{d}{dt}\ ([1 + (\frac{dy}{dx})_t^2]^{-1/2})$$

$$(9')\qquad \lambda\dot{x}_t = \frac{d}{dt}\ ([1 + (\frac{dy}{dx})_t^{-2}]^{-1/2}).$$

Now define $\theta(t) \equiv \tan^{-1}\left(\frac{dy}{dx}\right)_t$. Since the optimal area is a convex set, θ will be monotonically increasing in t. Moreover, since C_1 has been assumed differentiable, θ is either strictly increasing or constant throughout. If constant, then (8') and (9') imply $\theta \equiv 0$. This corresponds to the obviously optimal configuration of using the city's diameter when $A = \pi\bar{u}^2/2$. Except in this case, we may treat θ as strictly increasing, invert it, and again rewrite the Euler equations (with the obvious trigonometric substitutions) as

$$\lambda\frac{dy}{d\theta} = \sin\theta$$

$$\lambda\frac{dx}{d\theta} = \cos\theta$$

for $\theta\epsilon[\theta(0),\theta(1)]$. Integrating we may characterize the external arcs parametrically by

$$(10) \quad y = y(\theta) = \frac{1}{\lambda}\,[\cos\theta(0) - \cos\theta]$$

$$(11) \quad x = x(\theta) = \frac{1}{\lambda}\,[\sin\theta - \sin\theta\,(0)]$$

for $\theta\epsilon[\theta(0),\theta(1)]$. This is clearly a circular arc with radius $1/\lambda$. Moreover, the geometric interpretation of the standard transversality condition requires $\theta(0) = 0$. It is easily seen that this implies $\theta(1) = t_1$. λ is chosen so that the enclosed area just equals A. Clearly, as $A \to 0$, $\lambda \to \infty$ and $t_1 \to 0$, while as $A \to \pi\bar{u}^2/2$, $\lambda \to 0$ and $t_1 \to \pi$. This justifies completely the construction asserted in the previous section.

<u>Proof of Dominance over Interior Circular Configurations</u>:

We now show that the optimal boundary using configuration always dominates the interior circular configuration when both are enclosing strictly positive area. To do this it is sufficient (from the duality considerations above) to show that the area enclosed within a border of given length is always greater for boundary using configurations.

Now the area of a circle (A) is proportional to the square of its circumference (L) satisfying

(12) $A = L^2/4\pi$.

Consider now the relationship between area and border length for optimal boundary using configurations. Consult Figure 6. Recall from previous discussion that the optimal border is a circular arc with radius $1/\lambda$, where λ is the Lagrange multiplier from the constrained minimization problem. It may be verified from the figure that the length of the common border is given in terms of λ by

(13) $L = \frac{2}{\lambda} \tan^{-1}(\lambda\bar{u})$.

We give here also the parametric formula for enclosed area, though we shall not need it to establish the desired result. After some trigonometric manipulation it may be shown that for boundary using configurations

(14) $A = \bar{u}^2 \tan^{-1}(1/\bar{u}\lambda) + \lambda^{-2} \tan^{-1}(\bar{u}\lambda) - \bar{u}/\lambda$.

(13) and (14) give the relationship between A and L depicted in Figure 7. As $1/\lambda$ varies from zero to infinity we move along the boundary using curve from the origin to the point $(2\bar{u}, \pi\bar{u}^2/2)$.

Now since λ is the multiplier, it takes the interpretation of the additional units of common border required to enclose an additional unit of area. Hence $1/\lambda$ gives the slope of the boundary using curve in Figure 7 at any point $L = 2 \tan^{-1}(\lambda\bar{u})/\lambda$. Let $\left(\frac{dA}{dL}\right)_I$ represent the slope of the interior circular curve and $\left(\frac{dA}{dL}\right)_B$ be the slope of the boundary using curve, both evaluated at a common point $L \in [0,2\bar{u}]$. Then at $L = 2 \tan^{-1}(\lambda\bar{u})/\lambda$ we have

$$\left(\frac{dA}{dL}\right)_B = \frac{1}{\lambda}$$

while

$$\left(\frac{dA}{dL}\right)_I = \frac{L}{2\pi} = \frac{1}{\lambda}\left(\frac{\tan^{-1}(\lambda\bar{u})}{\pi}\right).$$

Since $\tan^{-1}(\lambda\bar{u}) \leq \frac{\pi}{2}$ for $\lambda \in [0,\infty)$ we have that

$$(15) \qquad \left(\frac{dA}{dL}\right)_I \leq \frac{1}{2}\left(\frac{dA}{dL}\right)_B$$

with equality only for L = 0. Integrating (15) gives the desired result.

<u>Conclusion</u>:

We have characterized the full optimum in the case of uniformly distributed population. A much more difficult problem is posed if one considers exponential population densities. Conversion of the problem to polar coordinates may be helpful in this case. One is led to speculate that hyperbolic or elliptical arcs will emerge for boundary using configurations, though I have not pursued this seriously. However, it is clear that interior configurations will not be uniformly dominated in this case. As the slope of the density gradient becomes sufficiently steep, the optimal configuration will necessarily be centrally located.

A final disclaimer should be added. The problem studied here is of mathematical interest. However, it should not be inferred that the author considers this effort to be of any particular value in our pursuit of understanding of urban racial conflict.

References:

[1] Bliss, G.A., Lectures on the Calculus of Variations, University of Chicago Press, Chicago, 1968.

[2] Courant, P., "Urban Residential Structure and Racial Prejudice," Institute of Public Policy Studies Discussion Paper No.62, University of Michigan, Ann Arbor (1974)

[3] Rose-Ackerman, S., "Racism and Urban Structure," Journal of Urban Economics, Vol. 2, 85-103 (1975)

[4] Yinger, J., "A Note on the Length of the Black-White Border," Journal of Urban Economics, Vol. 3, 370-382 (1976)

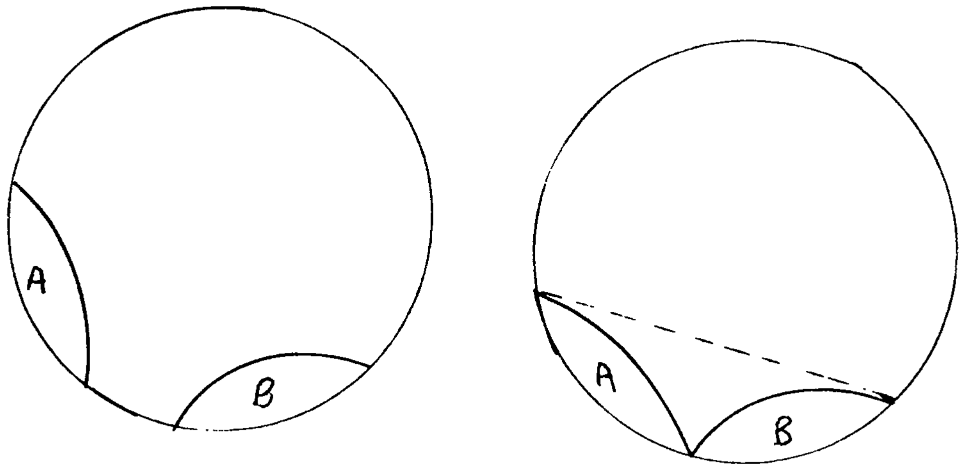


Figure 1

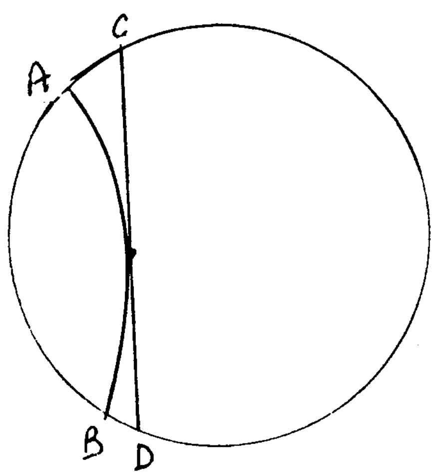


Figure 2

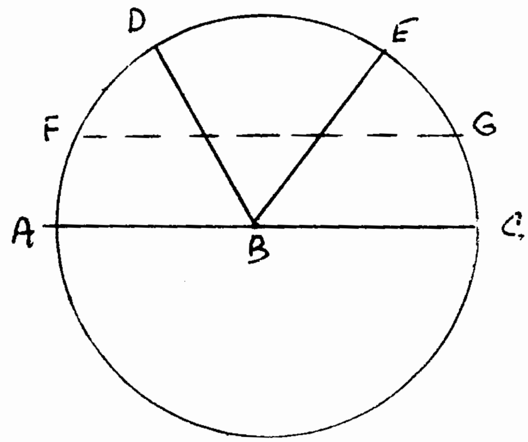


Figure 3

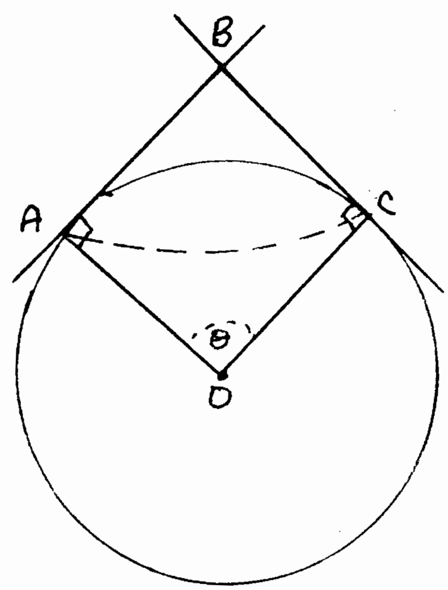
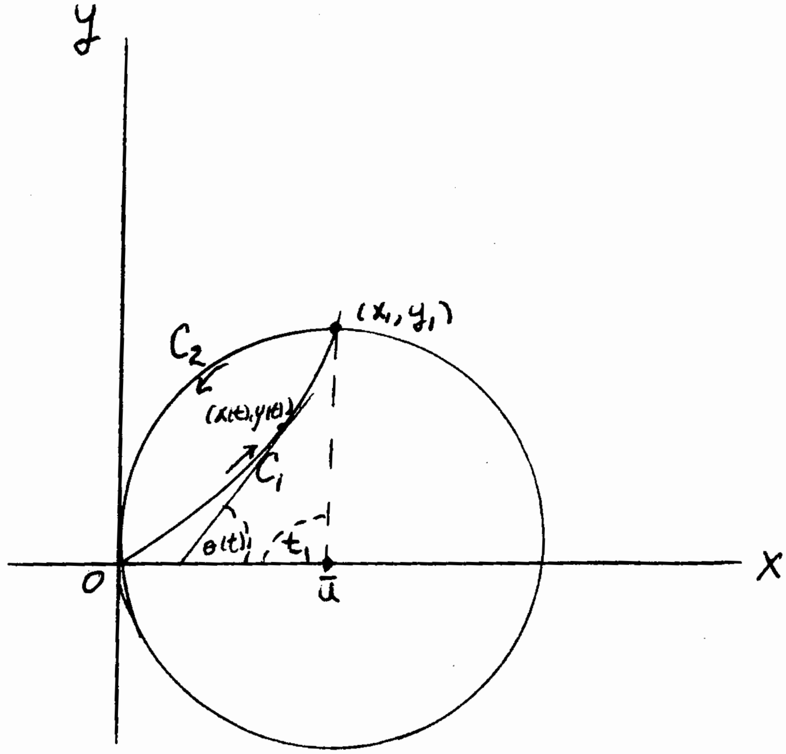


Figure 4

Figure 5



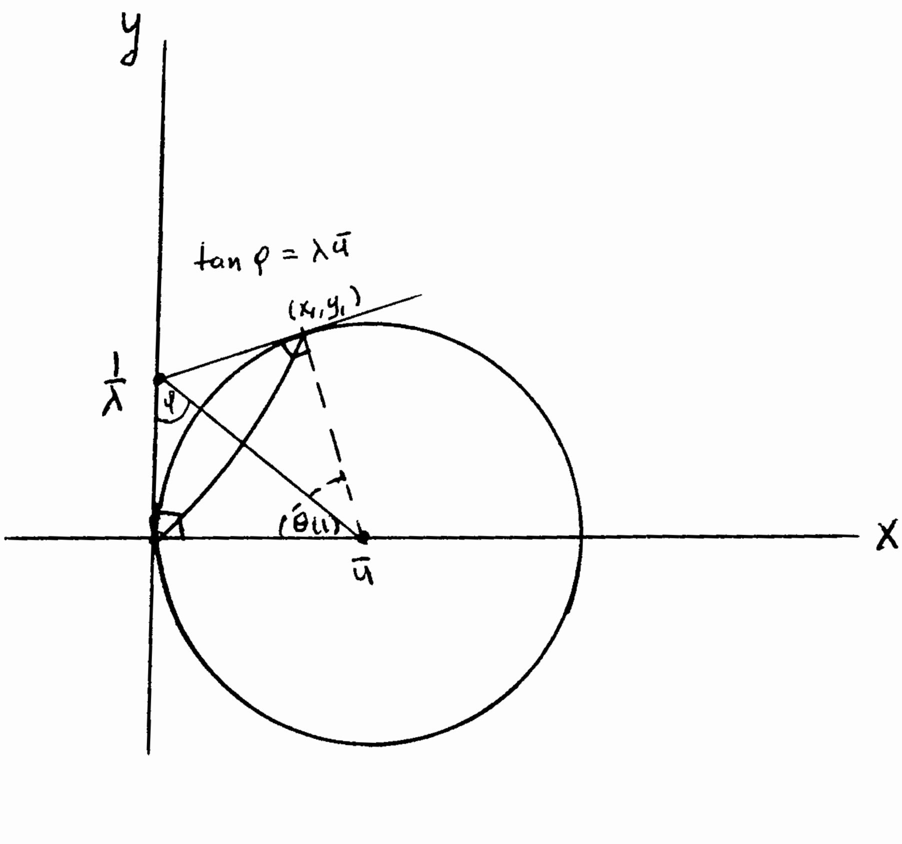
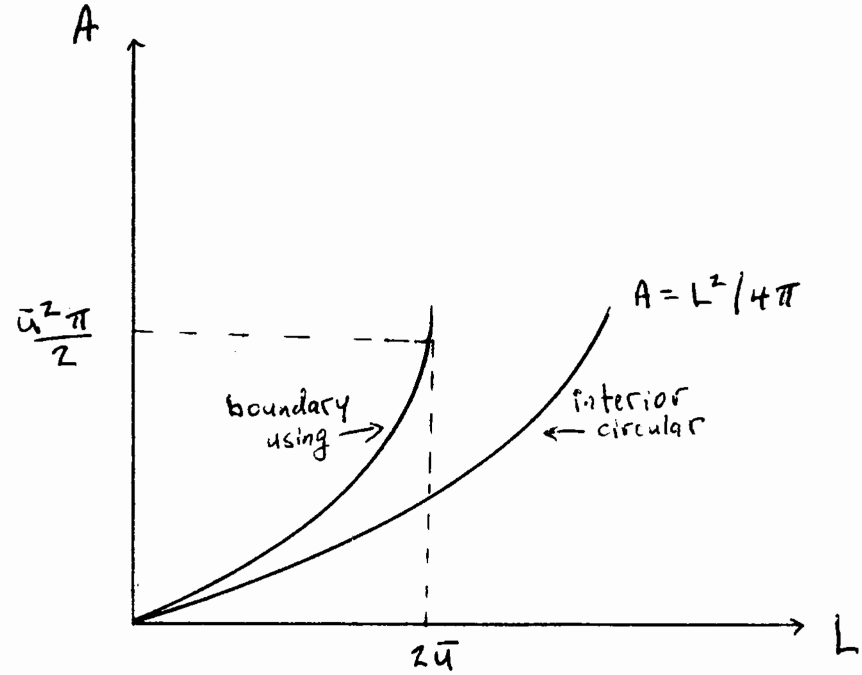


Figure 6

Figure 7



A
$\frac{\bar{u}^2\pi}{2}$
$A = L^2/4\pi$
boundary
using
interior
circular
$2\bar{u}$
L