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THE OPTIMAL EXPLOITATION OF AN UNKNOWN RESERVE

by

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I. Introduction:

The general problem of the optimal utilization of scarce natural resources has received considerable recent attention in the literature. Most study has been of the case where the level of the resource available to the economy is known with certainty. In those instances where uncertainty has been explicitly introduced, the focus has been on analyzing the effect of the existence of a technologically advanced substitute for the scarce resource which will become available at some unknown future date [3]. The more difficult problem of optimal planning when premature exhaustion is a real possibility due to a lack of precise knowledge about the total supply of the resource has received little attention. (Gilbert [5] is a notable exception.) This paper begins an investigation of that problem.

Any treatment of resource utilization with unknown reserves must confront two rather different issues. The first of these is the possibility of exhaustion. In models where the level of the resource is known with certainty, the exhaustion of an "essential" natural resource occurs only asymptotically. When the resource base is of unknown size, however, the date on which it will have been completely exploited is also a random variable. The choice of the rate at which to consume the resource must necessarily be influenced by the effect which the consumption rate has on the probability of exhaustion at subsequent points in time.

The second issue is that of learning about the distribution of reserves over time. Even though the level of total reserves may be unknown, it must certainly be the case that the activities of exploration and extraction provide information about the distribution of remaining reserves. Moreover, the nature of this information will be influenced by the exploration and extraction decisions which are made along the way. Thus, an optimal program of extraction should properly balance both consumption and information benefits against extraction and opportunity costs at each point in time.

This paper, while treating the first issue with some generality, will take a rather limited view of the way in which information about the distribution of total reserves is assimilated. We shall assume that the planner begins with a given probability distribution of possible endowments of the natural resource, and that he updates this distribution over time by conditioning on the knowledge of his cumulative consumption at each instant. Thus we will abstract from the activity of exploration. This is a regrettable omission which we hope to correct in a later paper. The treatment of this initial problem seems essential to further progress in any event.

The plan of the paper is as follows. We begin with a review of the problem of optimal depletion without production in a certain environment. Variational techniques are then employed to deduce necessary conditions for a solution to the problem of optimal exploitation under uncertainty. It is seen that the possibility of premature exhaustion considerably alters the requirements of an optimal path. We then determine a more complete characterization of an optimum and prove a number of qualitative and comparative static results, using dynamic programming arguments. An explicit solution for the problem is exhibited in a class of cases, and its properties studied. Finally, we consider the ability of the market mechanism to support the optimal allocation.

II. Optimal depletion with a known reserve.

Suppose that the level of a non-renewable non-recyclable resource is fixed at R. The flow utility of consumption per instant is given by $u(c_t)$, 1/ where c_t is the consumption flow at time t. Let the social rate of discount be ρ. Then, assuming costless extraction, socially optimal intertemporal allocation of the resource satisfies

$$(1) \qquad \max_{c_t} \int_0^{\infty} e^{-\rho t} u(c_t) dt \qquad \text{s.t.} \int_0^{\infty} c_t dt \leqq R, \; c_t \geqq 0$$

Assume $u(o) = o, u' > o, u'' < o$, and $\lim_{c \to o} u'(c) = \infty$. Then an optimum exists and the resouce constraint must hold with equality. Obviously it is a necessary condition for optimality that for some $\lambda > 0$,

$$(2) \qquad e^{-\rho t} u'(c_t^*) = \lambda, \; \int_0^{\infty} c_t^* dt = R.$$

That is, discounted marginal utility of consumption is constant. Furthermore, the optimal policy will have strictly positive consumption everywhere, with exhaustion occurring only asymptotically.

As is well known [13], this socially optimal allocation of resource use will be attained, under certain conditions, by a competitive resource market. Suppose that demand for the resouce is given by $D^{-1}(c_t) = p_t = u'(c_t)$, and that the social discount rate is equal to the rate of return to privately held capital (i.e. the interest rate). Then, as Hotelling observed [6], for owners of the resouce is a competitive economy to be willing to supply positive quantities at all times, prices must be rising at the rate of interest. Thus, $\dot{p}_t/p_t = \rho$, or $p_t = p_o e^{\rho t}$. Further, if consumers are on their demand curves, then $u'(c_t) = p_o e^{\rho t}$. Now the requirement that markets clear, that is that demand be met at each instant, implies that this market solution must be identical to that given in (2). We emphasize the necessity of perfect futures markets for this story to be a correct one. In the absence of complete markets the global demand equal supply condition which fixed p_o optimally could not be assured to hold. This problem is discussed at length in in Stiglitz [11].

In the above model exhaustion did not occur in finite time. To get an idea of the kind of conditions which must hold when the length of the consumption period is a variable of choice let us now consider the "doomsday" model studied by Koopmans [8]. Suppose that there exists some minimal level of resource consumption necessary to life. Then we require $c_t \geqq \underline{c}$. Assume further that $u(\underline{c}) = o$, $u' > o$, $u'' < o$, and (reasonably) $\lim_{c \to \underline{c}} u'(c) = \infty$. This model differs from that given earlier in that it is clear that consumption cannot be sustained beyond the date $\overline{T} = R/\underline{c}$. Thus the terminal date (the doomsday) is a choice variable as well. Koopmans proves under the assumptions given here the plausible result that the optimal terminal date is closer, the higher is the discount rate. The problem is

$$(3) \qquad \max \int_o^T e^{-\rho t} u(c_t)dt \qquad \text{s.t.} \int_o^T c_t dt \leqq R, \quad c_t \geqq \underline{c}.$$

Koopmans shows that a solution to (3) exists, and that the optimal consumption path (c_t^*) satisfies, along with the optimal terminal date, T^*,

$$(4) \qquad e^{-\rho t} u'(c_t^*) = \lambda, \; 0 \leqq t \leqq T^*, \text{ and } u(c_{T^*}^*) - c_{T^*}^* u'(c_{T^*}^*) = 0.$$

That is, marginal utility grows exponentially at the rate of discount, and marginal utility equals average utility at the terminal date.

This last requirement allows an instructive interpretation. Along an intertemporally efficient path $(\overline{c}_t)$ with terminal date (T) slightly earlier than optimal (T^*), the first part of (4) and equality of the resource constraint would imply $\bar{c}_T > c^*_{T*}$. Since the LHS of the second part of (4) increases with c we have $u(\bar{c}_T)/\bar{c}_T > u'(\bar{c}_T)$. Now efficiency implies that the transfer of a marginal unit of consumption between any two moments of positive consumption yields no gain. Suppose, however, that a unit of consumption is transferred to the end of the program and used to extend the duration of consumption. If consumption is maintained at the same rate then the additional time bought by this change is $1/\bar{c}_T$, while the additional discounted utility per instant is $e^{-\rho T}u(\bar{c}_T)$. Thus the total gain is $e^{-\rho T}u(c_T)/\bar{c}_T$. The cost of such a transfer is $e^{-\rho T}u'(\bar{c}_T)$. Clearly it will pay to make such a move. The argument is reversed for $T > T^*$. Thus, overall optimality when the terminal date is an object of choice requires both that discounted marginal utility is constant along the program, and that marginal utility equal average utility at the terminal date. We shall meet this latter condition, in a slightly different form, again.

III. Optimal depletion under uncertainty: A Variational Approach

Suppose now that total reserve $\tilde{R}$ is a random variable with CDF $F(\cdot)$ given at the initial date. Imagine further that the only information concerning the level of total reserve is that to be had by considering the distribution of remaining reserve at any moment conditional on the quantity consumed up to that moment. Then the problem of optimal depletion is equivalent to the "cake eating" problem with a cake of unknown size. [2/] It is necessary to additionally assume that consumption and extraction are identical; that is, there is no storage.

Let us take as the objective the maximization of the expected value of the integral of discounted utility. A consumption plan must be chosen without knowing when that plan will lead to the exhaustion of the resource. Thus, as in Koopmans' problem, the (probability distribution of the) terminal date of consumption is also a variable of choice.

Formally the problem is to choose a consumption plan (c_t) so as to

$$(5) \qquad \max_{c_t} E\left(\int_0^{\tau} e^{-\rho t} u(c_t)dt\right)$$

subject to the initial CDF of reserves F, the non-negativity of c_t, and

$$(6) \qquad \int_0^{\tau} c_t dt \leq \tilde{R} ,$$

where $\tilde{R}$ is the random variable representing the unknown size of the reserve at the beginning of the program. We assume $u(o) = o$, $u' > o$, $u'' < o$ and $\lim_{c\to 0} u'(c) = \infty$. Concerning the random endowment, we assume $F(\cdot)$ continuously differentiable, with $f(\cdot)$ as the PDF, and that the expected reserve is finite:

$$(7) \qquad \int_0^{\infty} x dF(x) < \infty .$$

Moreover, we assume the support of $F, \{x \mid f(x) > o\}$, is a connected set. The need for this assumption will become clear in the analysis below.

The constraint in (6), by virtue of the assumptions on the utility function, will necessarily hold with equality. The expectation in (5) is taken over the distribution of terminal dates τ induced by (6), the consumption plan (c_t), and the CDF $F(\cdot)$. Define the CDF $G_c(\tau)$ as the probability that the total reserve is exhausted on or before τ, given the consumption plan (c_t). Then

$$(8) \qquad G_c(\tau) = \Pr(\tilde{R} \leqq \int_0^\tau c_t dt) = F(\int_0^\tau c_t dt) \equiv F(X_\tau)$$

where $X_\tau \equiv \int_0^\tau c_t dt$ is the cumulative consumption of the resource at time τ. Our problem is then

$$(9) \qquad \max \int_0^\infty \int_0^\tau e^{-\rho t} u(c_t) f(X_\tau) c_\tau \, dt d\tau$$

where c_t is non-negative, but otherwise unconstrained.

For now we employ a variational technique to deduce necessary conditions which an optimal plan must satisfy. This method enables us to make useful comparisons with previous results in the theory of optimal capital accumulation. The following proposition illustrates the kinship of this problem of optimal depletion under uncertainty to Koopmans' doomsday problem mentioned above.

<u>Proposition I</u>: At any point along an optimal program (c_t^*) for (9) discounted marginal utility of consumption must equal the expected value of discounted average utility at the end of the program, where the expectation is taken over the induced distribution of terminal dates τ, conditional on $\tau \geqq t$. That is

$$(10) \qquad e^{-\rho t} u'(c_t^*) = E(e^{-\rho\tau} u(c_\tau^*)/c_\tau^* \mid \tau \geqq t)$$

where the conditional expectation is taken using the CDF $G_{c^*}(\cdot)$.

The proof of Proposition I follows directly from the following lemma which extends the classical Euler necessary conditions for optimality to a broader class of problems of which (9) is a special case. This lemma is of independent interest as it gives easily the results of Dasgupta and Heal [3] for the problem of uncertain availability of a "backstop" technology.

Lemma 1: Let $\psi(x,\dot{x},t)$ and $\Phi(x,\dot{x},t)$ be continuously differentiable functions of their arguments, and suppose $x^*(t)$ is an interior solution to the problem

$$(11) \qquad \max \int_0^\infty \psi(x,\dot{x},t) \int_0^t \Phi(x,\dot{x},\tau)d\tau dt, \quad x(o) = x_o.$$

Then $x^*(t)$ satisfies the integro-differential equation [3]

$$(12) \qquad \frac{\partial \psi^*}{\partial x}(t) \int_0^t \Phi^*(\tau)d\tau + \frac{\partial \Phi^*}{\partial x}(t)\int_t^\infty \psi^*(\tau)d\tau$$

$$- \frac{d}{dt}\left[\frac{\partial \psi^*}{\partial \dot{x}} \int_0^t \Phi^*(\tau)d\tau + \frac{\partial \Phi^*}{\partial \dot{x}} \int_t^\infty \psi^*(\tau)d\tau\right] \equiv 0$$

and the boundary condition [4]

$$(13) \qquad \lim_{t\to\infty} \left[\frac{\partial \psi^*}{\partial \dot{x}}(t) \int_0^t \Phi^*(\tau)d\tau + \frac{\partial \Phi^*}{\partial \dot{x}}(t) \int_t^\infty \Phi^*(\tau)d\tau\right] = 0$$

Proof: Let $\eta(t)$ be a bounded, continuously differentiable but otherwise arbitrary real valued function on $[0,\infty)$, with the property $\eta(0) = 0$. For η fixed and ϵ any positive number define $H(\epsilon)$ as the value of the maximand in (11) when x^* is replaced by $x^* + \epsilon\eta$. Since x^* is optimal, $H(\cdot)$ must attain a local maximum at zero for all functions η. Now it is readily seen that

$$H'(0) = \int_0^\infty \psi^*(t)\int_0^t \left[\frac{\partial \Phi^*}{\partial x}(t)\eta(t) + \frac{\partial \Phi^*}{\partial \dot{x}}(t)\, \dot{\eta}(t)\right]dtd\tau$$

$$+ \int_0^\infty \left[\frac{\partial \psi^*}{\partial x}(t)\eta(t) + \frac{\partial \psi^*}{\partial \dot{x}}(t)\dot{\eta}(t)\right]\int_0^t \Phi^*(\tau)d\tau dt$$

$$\equiv I_1 + I_2.$$

Integrating by parts the second integral (I_2) becomes

$$I_2 = \int_0^\infty \eta(t)\{\frac{\partial\psi^*}{\partial x}(t)\int_0^t \Phi^*(\tau)d\tau - \frac{d}{dt}[\frac{\partial\psi^*}{\partial\dot{x}}(t)\int_0^t \Phi^*(\tau)d\tau]\}dt$$

$$+ \lim_{t\to\infty} \eta(t)\frac{\partial\psi^*}{\partial\dot{x}}(t)\int_0^t \Phi^*(\tau)d\tau .$$

Moreover, by inverting the order of integration in I_1 and then integrating by parts it follows that

$$I_1 = \int_0^\infty \eta(t)\{\frac{\partial\Phi^*}{\partial x}(t)\int_t^\infty \psi^*(\tau)d\tau - \frac{d}{dt}[\frac{\partial\Phi^*}{\partial\dot{x}}(t)\int_t^\infty \psi^*(\tau)d\tau]\}dt$$

$$+ \lim_{t\to\infty} \eta(t)\frac{\partial\Phi^*}{\partial\dot{x}}(t)\int_t^\infty \psi^*(\tau)d\tau .$$

Hence we have

$$H'(0) = I_1 + I_2 = \int_0^\infty \eta(t)\{\frac{\partial\psi^*}{\partial x}(t)\int_0^t \Phi^*(\tau)d\tau + \frac{\partial\Phi^*}{\partial x}(t)\int_t^\infty \psi^*(\tau)d\tau$$

$$- \frac{d}{dt}[\frac{\partial\psi^*}{\partial\dot{x}}(t)\int_0^t \Phi^*(\tau)d\tau + \frac{\partial\Phi^*}{\partial\dot{x}}(t)\int_t^\infty \psi^*(\tau)d\tau]\}dt$$

$$+ \lim_{t\to\infty}\{\eta(t)[\frac{\partial\psi^*}{\partial\dot{x}}(t)\int_0^t \Phi^*(\tau)d\tau + \frac{\partial\Phi^*}{\partial\dot{x}}(t)\int_t^\infty \psi^*(\tau)d\tau]\}.$$

Since $H(\cdot)$ has a local maximum at zero, the RHS above must vanish for all admissible η. Thus (13) follows immediately, and (12) is a consequence of the Fundamental Lemma of the Calculus of Variations [2].

Q.E.D.

Proof of Proposition I: Applying Lemma 1 with $\Phi(x,\dot{x},t) = e^{-\rho t}u(\dot{x}_t)$ and $\psi(x,\dot{x},t) = f(x_t)\dot{x}_t$ one finds the following necessary conditions for an optimal allocation:

$$(14) \qquad f'(x_t^*)\dot{x}_t\int_0^t e^{-\rho\tau}u(\dot{x}_\tau^*)d\tau \equiv \frac{d}{dt}[f(x_t^*)\int_0^t e^{-\rho\tau}u(\dot{x}_\tau^*)d\tau$$

$$+ e^{-\rho t}u'(\dot{x}_t^*)\int_t^\infty f(x_\tau^*)\dot{x}_\tau^* d\tau]$$

$$(15) \qquad \lim_{t\to\infty} [f(x_t^*)\int_0^t e^{-\rho\tau}u(\dot{x}_\tau^*)d\tau + e^{-\rho\tau}u'(\dot{x}_t^*)\int_t^\infty f(x_\tau^*)\dot{x}_\tau^*] = 0.$$

Since each term in (15) is non-negative, each must be going to zero. Now by integrating each side of (14) from t to ∞, integrating the resulting LHS by parts and using the just noted implications of (15) one finds

$$(16) \qquad \int_t^\infty e^{-\rho\tau}u(\dot{x}_\tau^*)f(x_\tau^*)d\tau = e^{-\rho t}u'(\dot{x}_t^*)\int_t^\infty f(x_\tau^*)\dot{x}_\tau^* d\tau.$$

Define $M \equiv \sup\{x \mid f(x) > 0\}$. When $M < \infty$ we may replace (15) with the requirement $\lim_{t\to\infty} x_t^* = M$. When $M = \infty$, note that (15) implies $\lim_{t\to\infty} f(x_t^*) = 0$. Since the support of F is connected it again follows that $\lim_{t\to\infty} x_t^* = M$. Finally, since $F(M) = 1$, a simple change of variables reveals

$$\int_t^\infty f(x_\tau^*)\dot{x}_\tau^* d\tau = 1 - F(x_t^*).$$

Thus (16) implies (recalling that $\dot{x}_t^* \equiv c_t^*$)

$$(17) \qquad e^{-\rho t}u'(c_t^*) = \int_t^\infty [e^{-\rho\tau} \frac{u(c_\tau^*)}{c_\tau^*}] [\frac{f(x_\tau^*)c_\tau^*}{(1 - F(x_t^*))}]d\tau$$

which is easily seen to imply (10).

Q.E.D.

We will comment on Proposition I in a moment. First observe that lemma 1 in principle enables us to weaken the assumptions of additively separable utility and of a distribution of remaining reserves independent of the rate of resource extraction. However, the resulting first order conditions do not lend themselves to easy interpretation. More importantly, notice that if $\psi(x,\dot{x},t)$ in lemma 1 is taken to be independent of x and $\dot{x}$, and integrates to one, then we may use the lemma to generate the Dasgupta-Heal [3] results concerning when the uncertain arrival of a superior technology may be treated as merely increasing the (now time dependent) social rate of discount. 5/

An intuitive justification for Proposition I may be readily had. At any time t along an optimal program a marginal increase in the rate of consumption has the payoff $e^{-\rho t}u'(c_t^*)$. The cost of marginally increasing consumption at t is to make earlier the time at which the reserve will be exhausted, given that the optimal path is followed subsequently. Suppose that exhaustion would have occurred at time $s \geq t$. Then the reduction in the duration of the program due to the unit increase in earlier consumption is $1/c_s^*$. Hence the resulting utility loss is $e^{-\rho s}u(c_s^*)/c_s^*$. Since at t, s is random, (10) simply expresses the requirement that at any point along the optimal program, the marginal gain to increased consumption is just balanced by the expected marginal cost.

Differentiation of (17) yields the following Keynes-Ramsey type equation:

$$(18) \qquad \frac{\dot{c}_t^*}{c_t^*} = \frac{[(u(c_t^*)/c_t^* - u'(c_t^*))/u'(c_t^*)][f(x_t^*)c_t^*/(1-F(x_t^*))] - \rho}{-c_t^* u''(c_t^*)/u'(c_t^*)}$$

This equation is quite similar to that which would be derived in a one sector optimal accumulation model with stationary population and discounted utility, except that the first term in the numerator of (18) would be replaced by the marginal efficiency of investment. In our problem this term is precisely the marginal rate of return to deferred consumption (investment). Because there is no production in this model, returns to deferred consumption are measured completely by the gains to extending the duration of the program. If the reserve were just being exhausted at time t, an event whose conditional probability is $f(x_t^*)c_t^*/(1-F(x_t^*))$, then the net return to a unit of deferred consumption is (by the reasoning of the previous paragraph) $u(c_t^*)/c_t^* - u'(c_t^*)$. Thus, the first term in the numerator of (18) gives the expected net rate of return per unit of "investment" at time t.

Furthermore, (18) points out the counter-intuitive fact that consumption will be increasing when the probability of exhaustion in the next instant is "very high", and decreasing when that probability is "very low". Analogizing again with optimal capital accumulation, we may give this a natural economic justification. It is well known (from Samuelson and Solow [10] for example) that the optimal accumulation of a productive asset requires that the sum of the rental rate and the instantaneous rate of capital gains on the asset be equal to the discount rate everywhere along an optimal path. Here, if the probability of exhaustion in the next instant is "very high," then the rental rate, or return to deferred consumption, will be greater than the discount rate, requiring negative "capital gains", or a declining shadow price (i.e. marginal utility) of consumption. This means that consumption must be rising. Similarly, if it is reasonably certain that exhaustion will not occur in the next instant, then the rate of "capital gains" must be positive and hence consumption must be declining. Indeed, in the certainty problem where exhaustion is known to be impossible, marginal utility must be growing at the discount rate.

IV. Further qualitative properties of the optimal program.

In this section we employ dynamic programming techniques to deduce some comparative static results. We shall also derive an explicit solution to the problem for a particular utility function (iso-elastic) and arbitrary CDF of reserves. Consider first the following problem :

$$(19) \qquad \max_{c_t} \int_o^\infty e^{-\rho t} u(c_t)(1 - F(x_t))dt \quad \text{s.t.} \ \dot{x}_t = c_t, x(o) = x_o, \ c_t \geqq 0.$$

As the reader may have guessed, (19) is just a much simpler way of writing (9).

Lemma 2: A solution to (19) with $x_o = 0$ is a solution for (9).

Proof: Reverse the order of integration in (9) and note (from the proof of Proposition 1) that for an optimal policy we must have $\lim_{t \to \infty} F(x_t) = 1$.

Q.E.D.

While problem (19) seems a straightforward control problem, the existence question is non-trivial since the integrand need not be concave in the state variable x. We have not surmounted this difficulty to date, though this is not as troublesome as it might have been. The reason is that one readily derives an upper bound on the maximand in (9). It seems then that existence can only fail for pathological $F(\cdot)$.

<u>Lemma 3</u>: Under the assumptions already adopted on $u(\cdot)$ and $F(\cdot)$, (9) is uniformly bounded over all consumption plans.

<u>Proof</u>: Let $\hat{J}(R)$ be the value of optimal behavior in (1), and let $\bar{R}$ be the mean reserve. Then $\hat{J}(\cdot)$, like $u(\cdot)$ is concave. Let (c_t) be an arbitrary consumption plan. Clearly

$$\int_0^{\tau} e^{-\rho t}u(c_t)dt \leqq \hat{J}(x_\tau), \qquad \forall\ \tau \geqq 0.$$

Then

$$E(\int_0^{\tau} e^{-\rho t}u(c_t)dt) \leqq E\hat{J}(x_\tau) \leqq \hat{J}(Ex_\tau) = \hat{J}(\bar{R}) < \infty$$

by (7) and Jensen's inequality.

Q.E.D.

We proceed in the conventional way to define $J(x)$ as the value in (19) of an optimal policy, given that $x_o = x$. It is well known and readily seen that $J(\cdot)$ must satisfy the following functional equation:

$$(20) \qquad \rho J(x) = \max_{c \geq 0} [u(c)(1 - F(x)) + cJ'(x)].$$

Furthermore, since $F(\cdot)$ is a CDF and consumption is non-negative it is also clear that

$$(21) \qquad \lim_{x \to \infty} J(x) = 0.$$

Equation (21) is our analogue of the standard transversality condition. Note here that the maximum in (20) will be interior, given our assumptions on $u(\cdot)$. Furthermore it will be unique. Hence we have that

$$(22) \qquad u'(c)(1 - F(x)) = - J'(x) \text{ or } c = u'^{-1}(-J'(x)/(1 - F(x)))$$

Using (20) and (22) it is easily seen that $J(\cdot)$ must solve the following differential equation:

$$(23) \qquad \rho J(x) = (1 - F(x))u(u'^{-1}(-J'(x)/1 - F(x))) + J'(x)u'^{-1}(-J'(x)/(1-F(x)))$$

with boundary condition (21). A solution to (23) may then be used to determine the optimal initial consumption flow $c_o(x)$ whenever $x_o = x$, through (22). Since (19) and (9) are only equivalent for $x_o = 0$, one might conclude that only $c_o(0)$ is relevant. In fact, however, a solution for (23) gives immediately a complete solution for (9).

<u>Proposition</u> <u>II</u>: Let $c_o(x)$ be the optimal initial consumption flow derived from (20) - (23). Define $T(x) \equiv \int_o^x dv/c_o(v)$. Then $T(\cdot)$ is invertible. Let $x^{-1}(\cdot)$ be its inverse. Then $c^*(t) = c_o(x^{-1}(t))$ is the <u>unique</u> solution to (9).

<u>Proof</u>: Let $c^*(\cdot)$ be a solution to (9), and $x^*(\cdot)$ defined in the usual way as the cumulative consumption function along the solution path $c^*(\cdot)$. Now, by Bellman's Principle of Optimality [1] it follows that c_t^* must satisfy

$$c_t^* = c_o(\int_o^t c_s^* ds) = c_o(x_t^*), \forall\ t \geq 0.$$

Recalling the definition of $T(\cdot)$ we have

$$T(x_t^*) = \int_o^{x_t^*} dv/c_o(v).$$

Consider the change of variables $v = x_s^*$. Then $dv = c_s^* ds$ and $c_o(v) = c_o(x_s^*) = c_s^*$. It follows then that

$$T(x_t^*) = \int_o^t ds = t, \forall\ t \geq 0.$$

Hence

$$x^{-1}(T(x_t^*)) = x_t^* = x^{-1}(t),\ \forall\ t \geq 0.$$

Therefore $c_t^* = c_o(x_t^*) = c_o(x^{-1}(t))$. Uniqueness follows immediately from the definition of $x^{-1}(\cdot)$ and the uniqueness of $c_o(x)$.

Q.E.D.

Proposition II says that the optimal control $c_o(x)$ for (15) with initial condition $x_o = x$ is identical to the optimal consumption flow c_t^* for (9) when cumulative consumption (and hence extraction) by time t is equal to x. However, J(x) does not represent the expected discounted value of the utility flow arising from c^* in (9), viewed from the point at which a total of x has already been consumed. This is because the distribution function $F(\cdot)$ in (19) has not been renormalized to take account of the fact that total initial reserves could not have been less than x_o. This could be done by multiplying (19) by $(1 - F(x_o))^{-1}$. If $J^*(x)$ denotes the current value of the expected discounted integral of utility along an optimal program, then $J^*(x) = (1 - F(x))^{-1}J(x)$.

Hence $J^*(x)$ may be unbounded even though $J(x) \to 0$. The "information" which the planner gains by appropriately conditioning the distribution of remaining reserves after observing the total quantity of resources consumed by any point serves only to make him "feel" better off, but has no effect whatever on the policy actually adopted. 6/ Our boundary condition (21) may thus be interpreted as saying that asymptotically, without the benefit of this "information", the value of the program must be going to zero. This condition appears to have no analogue in the theory of optimal growth.

The following characterization of $J^*(\cdot)$ will prove extremely useful in deriving some results on the effects of changes in the distribution of reserves on the optimal consumption policy.

Lemma 4: Whenever cumulative consumption along an optimal program has reached the level x, then the current value of the remainder of the program is given by

(24) $$J^*(x) = \frac{1}{\rho}(u(c_o(x)) - c_o(x)u'(c_o(x)))$$

<u>Proof:</u> From (22), (23) and the immediately preceding discussion,

$$\begin{aligned} J^*(x) &= (1 - F(x))^{-1}J(x) \\ &= (1 - F(x))^{-1}\frac{1}{\rho}((1 - F(x))u(c_o(x)) - u'(c_o(x))(1 - F(x))c_o(x)) \\ &= \frac{1}{\rho}(u(c_o(x)) - c_o(x)u'(c_o(x))). \end{aligned}$$

Q.E.D.

That is, the current expected value of an optimal program is equivalent to the constant utility flow $u(c_o) - c_o u'(c_o)$ in perpetuity. This flow equals the net consumer's surplus when c_o is being consumed in the competitive resource market under certainty discussed in Section II. Lemma 4 is mathematically trivial, but economically very important. 7/ Because the RHS of (24) is strictly increasing in c_o, a strictly monotonic relationship between the value of optimal behavior and the optimal level of the control variable has been established. Any change in the distribution of reserves will increase (decrease) current optimal consumption if and only if it also increases (decreases) the current value of an optimal policy. We have then the following surprising result:

<u>Corollary 1</u>: An increase in the riskiness of the distribution of remaining reserves in the sense of Diamond and Stiglitz [4] (i.e. a mean utility preserving increase in risk) will not affect current optimal consumption.

Without much difficulty, we also have the result:

Corollary 2: If an uncertain distribution of reserves is replaced by a certain level which gives the same total well being, then initial consumption will not change. If a nondegenerate distribution of reserves is replaced by its mean then initial consumption will rise.

Proof: Recall the definitions of $\hat{J}(R)$ and $\bar{R}$ from the proof of Lemma 3. It follows from the argument given there that $J^*(0) < \hat{J}(\bar{R})$. Repeating the proof of Lemma 4 for the certainty case one finds

$$\hat{J}(R) = \frac{1}{\rho}(u(\hat{c}(R)) - \hat{c}(R)u'(\hat{c}(R)))$$

where $\hat{c}(\cdot)$ is the optimal certainty consumption function. Both statements in the corollary then follow immediately.

Q.E.D.

Thus, if a previously known holding of the resource becomes suddenly subjected to noise, an optimal consumption policy will be immediately more conservative. Clearly it cannot remain more conservative always, as that would imply some of the resource not being used with positive probability. It is not true however that the addition of noise to an arbitrary distribution will lead to an initially more conservative policy. Essentially this is because if the initial distribution is such that its right tail is very "desirable" (that is, $J^*(x)$ is increasing rapidly), then the addition of noise can make that tail even more desirable. It could then be optimal to consume the resource even more quickly, as the cost in terms of increased probability of premature exhaustion is outweighed by the gain of making the desirable tail available at an earlier date. The following result gives a characterization of when this perverse effect can happen.

Corollary 3: Let an initial distribution of resources $F(\cdot)$ be given, and let c_t^* be the corresponding optimal consumption program. If $e^{-\rho t}u(c_t^*)$ is strictly increasing everywhere, but remains bounded, then a mean preserving increase in the riskiness of $F(\cdot)$ will increase initial consumption.

Proof: Let $G(\cdot)$ differ from $F(\cdot)$ by a mean preserving spread (MPS). Then (see Rothschild and Stiglitz [9])

$$\forall y \geqq 0, \int_o^y (G(x) - F(x))dx \geqq 0, \int_o^\infty (G(x) - F(x))dx = 0. \quad \text{Thus}$$

$$\int_o^\infty e^{-\rho t}u(c_t^*)(1 - G(x_t^*))dt = \int_o^\infty e^{-\rho t}u(c_t^*)(1 - F(x_t^*))dt - \int_o^\infty e^{-\rho t}u(c_t^*)(G(x_t^*) - F(x_t^*))dt$$

$$= \int_o^\infty e^{-\rho t}u(c_t^*)(1 - F(x_t^*))dt + \int_o^\infty \frac{d}{dt}(e^{-\rho t}u(c_t^*)) \int_o^t (G(x_s^*) - F(x_s^*))dsdt$$

$$- \lim_{t\to\infty} ((e^{-\rho t}u(c_t^*)) \int_o^t (G(x_t^*) - F(x_s^*))ds.$$

The last term vanishes by the definition of a MPS. Thus

$$\int_o^\infty e^{-\rho t}u(c_t^*)(1 - G(x_t^*))dt > \int_o^\infty e^{-\rho t}u(c_t^*)(1 - F(x_s^*))ds$$

Since c_t^* was the optimal path for $F(\cdot)$, the result follows from Lemma 4.

Q.E.D.

An example of a situation where Corollary 3 applies is given when $u(c) = \frac{1}{\gamma}c^\gamma$, $(o < \gamma < 1)$, and $f(x) = (n - 1)(1 + x)^{-n}$, $(n > 2)$. In this case (as may be checked by the explicit solution given below) $(2\gamma - 1)/(1 - \gamma) > n$ implies that the hypotheses of the corollary are met.

In spite of the difficulty in obtaining general results about the effects of increased uncertainty in the mean preserving sense, it is possible to show under rather general circumstances that randomization, appropriately defined, causes a more conservative exploitation of an unknown resource base. Toward this end let us define a <u>mixing</u> of distributions as follows:

<u>Definition:</u> For any positive integer n, let $\{F_i\}_{i=1}^{n}$ be a family of CDF's on $[0,\infty)$. Then the CDF $G(\cdot)$ is a mixing of the $\{F_i\}$ if there exist $\alpha_i, (i=1,n)$ such that $\alpha_i > 0$, $\sum_1^n \alpha_i = 1$, and $G(x) = \sum_1^n \alpha_i F_i(x)$, for all $x \in [0,\infty]$. A mixing of CDF's is thus a randomization of those distributions. Moreover, if $F(\cdot)$ is an arbitrary CDF which is continuously differentiable with finite mean, define $J^{**}(F)$ as the value of an optimal policy for (9) under $F(\cdot)$ at time zero. That is $J^{**}(F) \equiv J_F^*(0)$. Similarly, define $\hat{c}(F)$ as $c_F^*(0)$, the optimal initial consumption flow under $F(\cdot)$. We may now state

<u>Proposition III</u>: Assume $cu''' + u'' > 0, \forall\, c \geq 0$. Let $\{F_i\}, (i=1,n)$ be a family of CDF's, not all identical, and let $G(\ )$ be a mixing of $\{F_i\}$ with weights α_i. Then

$$\hat{c}(G) < \sum_1^n \alpha_i \hat{c}(F_i).$$

<u>Proof</u>: The proof procedes by induction. Define $P(c) \equiv u(c) - cu'(c)$. Then under our assumptions, $P' > 0$, $-P'' > 0$. Hence P has an inverse, $P,^{-1}$ which is strictly increasing and strictly convex. Let $n=2$. Then

$$J^{**}(F_1 + (1-\alpha)F_2) \equiv \max_{c_t} \int_0^\infty e^{-\rho t} u(c_t)(1 - \alpha F_1(x_t) - (1-\alpha)F_2(x_t))dt$$

$$= \max \int_0^\infty e^{-\rho t} u(c_t)((\alpha(1 - F_1(x_t)) + (1-\alpha)(1 - F_2(x_t)))dt$$

$$\leqq \alpha \max_{c_t} \int_0^\infty e^{-\rho t} u(c_t)(1 - F_1(x_t))dt + (1-\alpha)\max_{c_t} \int_0^\infty e^{-\rho t} u(c_t)(1-F_2(x_t))dt$$

$$= \alpha J^{**}(F_1) + (1-\alpha)J^{**}(F_2)$$

with equality if and only if $F_1 \equiv F_2$, since(by (22) for example) this is the only way that the optimal consumption function will be everywhere identical for F_1 and F_2. Thus

$$\hat{c}(G) = \hat{c}(\alpha F_1 + (1-\alpha)F_2) = P^{-1}(\rho J^{**}(\alpha F_1 + (1-\alpha)F_2)) < P^{-1}(\alpha\rho J^{**}(F_1) + (1-\alpha)\rho J^{**}(F_2))$$

$$< \alpha P^{-1}(\rho J^{**}(F_1)) + (1-\alpha)P^{-1}(\rho J^{**}(F_2)) = \alpha\hat{c}(F_1) + (1-\alpha)\hat{c}(F_2).$$

This establishes the proposition for $n = 2$. Assume the proposition true for $n = m - 1$ and consider $n = m$. Then

$$\hat{c}(G) = \hat{c}(\sum_1^m \alpha_i F_i) = \hat{c}(\alpha_m F_m + (1-\alpha_m) \sum_1^{m-1} (\alpha_i/(1-\alpha_m))F_i)$$

$$= P^{-1}(\rho J^{**}(\alpha_m F_m + (1-\alpha_m) \sum_1^{m-1} (\alpha_i/(1-\alpha_m))F_i))$$

$$< \alpha_m P^{-1}(\rho J^{**}(F_m)) + (1-\alpha_m)P^{-1}(\rho J^{**}(\sum_1^{m-1} (\alpha_i/(1-\alpha_m))F_i))$$

$$< \alpha_m \hat{c}(F_m) + (1-\alpha_m) \Sigma (\alpha_i/(1-\alpha_m))\hat{c}(F_i)$$

$$= \sum_1^m \alpha_i \hat{c}(F_i).$$

Q.E.D.

We note that the assumption $u'' + cu''' > 0$ is reasonably strong, though it is consistent with the standard assumptions of increasing relative and decreasing absolute risk aversion.

Proposition III tells us that if the planner faces an array of possible distributions of reserves F_i, each of which having probability α_i of being the true distribution, then his optimal extraction policy must be more conservative than the expected value of the extraction policy, knowing the true distribution. Moreover, if any sequence of distributions has the property that each distribution in the sequence leads to the same initial consumption, then a mixing of the distributions will lower initial consumption. In this sense, less uncertainty is always preferred to more uncertainty.

Observe now that the role of discounting in determining the qualitative character of the intertemporal consumption program is significantly mitigated by the fact that the rate of consumption also affects the probability distribution of the duration of the program. Thus, as we discussed in the previous section, consumption may be rising or falling over time, depending on the probability that the reserve is exhausted in the next instant. Curiously, there is a case (noted by Gilbert [5]) where consumption will be constant over time no matter what the utility function or how high the rate of discount. This curcumstance is limited to a specific class of probability distributions, however.

<u>Corollary 4</u>: Optimal consumption will be constant over time if and only if $(1 - F(x)) = e^{-\lambda x}$, for some $\lambda > 0$.

<u>Proof</u>: Suppose $1 - F(x) = e^{-\lambda x}$. It is clear from (19) that for all $x_o \geqq 0$,

$J(x_o) = e^{-\lambda x_o} J(0)$. Furthermore, $J^*(x_o) = (1 - F(x_o))^{-1} J(x_o) = J(0), \forall x_o \geq 0$. Hence J^* is constant. It follows from lemma 4 that $c_o(x)$ is constant, and then Proposition II implies c_t^* is constant.

Suppose conversely that c_t^* is constant. Then $c_o(x)$ must be constant, and by lemma 4, $J^*(x)$ is constant as well. Let c^* and J^* denote these constant values. Now $J(x) = (1 - F(x))J^*$, and by (18)

$$u'(c^*)(1 - F(x)) = - J^* \frac{d}{dx} (1 - F(x)) = J^* f(x)$$

or

$$- u'(c^*)/J^* = \frac{d}{dx}(1 - F(x))/(1 - F(x)).$$

The result follows immeditately from integrating the above equation with

$\lambda = u'(c^*)/J^*$.

Q.E.D.

We turn now to the task of solving (9) for a class of special cases. Accordingly, assume that the utility function is iso-elastic, or equivalently, possesses constant relative risk aversion. Thus we write

$$u(c) = \frac{1}{\gamma} c^{\gamma}, \ 0 < \gamma \leqq 1, \ c \geqq 0.$$

In this case (23) becomes

$$(25) \qquad J(x) = \frac{1}{\rho\gamma}(-J'(x)/(1 - F(x))^{\gamma/\gamma-1}(1 - F(x)) + \frac{J'(x)}{\rho}(-J'(x)/(1 - F(x)))^{1/\gamma-1}.$$

Together with (21), (25) gives a non-linear first order differential equation for $J(x)$ which may be readily solved. The solution for (21) and (25), and the implied solution $J^*(x)$ are given below:

(26) $$J^*(x) = (\frac{1-\gamma}{\rho})^{1-\gamma} \frac{1}{\gamma}(\int_x^\infty ((1 - F(s))/(1 - F(x)))^{1/\gamma} ds)^\gamma.$$

Now, since $P(c) = ((1-\gamma)/\rho\gamma)c^\gamma$, it follows from (24) that

(27) $$c_o(x) = \frac{\rho}{1-\gamma}\int_x^\infty ((1 - F(s))/(1 - F(x)))^{1/\gamma} ds.$$

Equation (27) and Proposition 2 can then be used to determine c_t^*, once $F(\cdot)$ has been specified.

Equation (26) reveals a remarkable fact. If utility is iso-elastic and the initial distribution of reserves is $F(\ )$, then the present value of that distribution is proportional to the distance of $F(\cdot)$ from the constant function 1 in the L^p norm, where $p = 1/\gamma$! That is, $J^{**}(F) = k\|1 - F\|_p$ though k and p of course are not independent. The "further away" from unity (which represents no reserves with certainty) in this specific sense is the distribution function, the more valuable it is. Similarly, the optimal rate of consumption at any instant is a continuous, increasing function of the distance from unity of the normalized distribution of remaining reserves.

This observation is very useful as it enables us to bring the well known properties of the L^p norms to bear on our problem. For example, the heart of the proof of Proposition III becomes a simple consequence of the triangle inequality. We can also make economic use of the properties of $\|\ \|_p$ for p close to 1 and ∞ (i.e. γ close to 1 and 0). As $p \to 1$, $\|1 - F\|_p \to \int_o^\infty |1 - F(s)| ds = \bar{x}$, the mean reserve. This is very natural; as consumers become risk neutral the optimal value of a distribution function converges to a constant multiple of its mean. On the other hand, as $p \to \infty$, $\|1 - F\|_p \to \sup|1 - F(x)|$. This requires careful interpretation because the constant k is unbounded as $\gamma \to 0$. What it does imply is that for very risk averse consumers with γ near zero, a small increase in the probability that the resource endowment is low must be balanced by a very large increase in the probability of a high endowment if the value of the resulting distribution function is to remain unchanged. Indeed, we can use the convergence of the L^p norms to the sup norm as $p \to \infty$ to see that the following corollary must be true. (The tedious proof is omitted.)

Corollary 5: Let F and G be two CDF's satisfying

$$\inf \{x|G(x) > F(x)\} < \inf \{x|F(x) > G(x)\}$$

where both sets above are assumed to be non-empty. [That is, G is bigger than F "before F is bigger than G.] Then there exists $\bar{\gamma} > 0$ such that $0 < \gamma < \bar{\gamma}$ implies

$$J_\gamma^{**}(F) > J_\gamma^{**}(G)$$

V. Competitive allocations with unknown reserves.

In this section we would like to consider the question: "Do competitive markets exploit an unknown resource base optimally?" As mentioned in Section II, it is widely recognized that when the set of futures markets is complete and the resource base is known, competitive allocations will be intertemporally efficient. Below we prove that if in addition there are complete contingent claims markets (in a sense to be made precise), then this classical result may be extended to stochastic environments.

Before beginning the analysis it should be stressed that the result stated above is of limited practical significance due to the exclusion from the model of the activity of exploration. For it is precisely in this area of generating information about the endowment that one most readily expect that private and social returns do differ. It is nonetheless of interest to note the (ideal) conditions under which competition leads to efficiency.

Imagine a world in which there are a large number of identical, infinitely lived consumers and identical, competitive firms. Each consumer has the same initial endowment of income which can be spent either on the natural resource or on "other goods." The marginal utility of "other goods" is constant (unity). Each firm has the same initial (unknown) holding of the natural resource.[8] Firms are profit maximizers and consumers maximize their expected utility of consumption of the natural resource and "other goods." The instantaneous utility derived from consuming a flow of c units of the natural resource is $u(c)$ where $u(\cdot)$ has the properties assumed in the planning model earlier. Utility of "other goods" is measured by the income remaining after the purchase of the natural resource. No storage is possible, so firms learn their holdings only at the point that there is no more left. All agents know the objective distribution of firm holdings represented by the CDF $F(\cdot)$.

The usual starting point for analysis of this kind is the Arrow-Debreu model of general equilibrium. One imagines a complete set of markets open for trading at the initial date on which agents may purchase promises for delivery of a desired commodity at any future date, conditional on the occurence of a particular "state of nature." Of course, such contracts are meaningful only if the agents involved can ascertain just which state of nature has occured. In our model uncertainty derives solely from not knowing the level of holdings of any one firm, $\tilde{R}$. Thus, the true state of nature may be identified with these holdings. But the level of these holdings become known only at the moment of exhaustion, and hence cannot be the basis for forward contracts which mature before that moment.

We seek then a set of events the occurence of which can be verified by market participants at each date for which forward contracting occurs. Suppose that for each future date t, the maximal quantity extracted (and hence sold) by any firm, X_t, is known to all agents. Then every agent can verify the occurence of the event $\{\tilde{R} \varepsilon (X_t, \infty]\}$. Moreover, given our assumptions, this exhausts what any agent can say about $\tilde{R}$. Hence, the relevant states of nature upon which to base contingent claims are the events $\{\tilde{R} \geq X\}$ for non-negative X. For reasons explained in footnote 8, we restrict ourselves to symmetric behavior by our identical firms. That is, we seek an equilibrium in which each firm extracts at the same rate, finding, it optimal to do so given the extraction rates of all other firms. Then X_t will equal each firm's cumulative extraction at date t.

Imagine then that at the initial date consumers can purchase contracts of the form "one unit of resource at date t given that cumulative sales as of that date have been x, and that exhaustion has not yet occured." That is, a promise to deliver is distinguished both by its date and its "quality" (i.e., the number of units the firm will have committed prior to its date). This enables consumers to buy contingent on their position in the firm's "queue." Let the present price of such a contract be $p(x,t)$. The probability that delivery on such a contract will actually be made is $1 - F(x)$. Competitive behavior in this world means that all agents take $p(x,t)$ as given. Consumers choose their demand for consumption (i.e. quantity and quality) on each date to maximize expected discounted utility. Firms choose their supply of the resource at each date (and hence the "quality" of this supply) to maximize profits. An equilibrium is a price function $p^*(x,t)$ which allows these optimizing decisions to be consistent. Existence is not discussed, though it follows from the argument below and the uniqueness result of Proposition II that an equilibrium exists if and only if a planning optimum exists.

Consider first the behavior of firms. They must choose a quantity of resource to extract and supply to the market at each date t so as to maximize their revenues. With $p(x,t)$ given they must

$$\max_{c_t} \int_0^\infty p(x_t,t)c_t dt \quad \text{s.t. } x_o = 0, \ \dot{x}_t = c_t, \ c_t \geqq 0.$$

If an interior solution exists (i.e., finite positive supply at all dates) then we must have

$$\frac{\partial p}{\partial x}(x_t,t)c_t = \frac{d}{dt} p(x_t,t) . \tag{28}$$

This is readily seen to imply $\frac{\partial p}{\partial t} = 0$, or that <u>contingent on cumulative consumption the price for future delivery of the resource must be constant across dates</u>, generalizing Hotelling [6]. Moreover, using this fact and changing variables in the integrand one gets

$$\int_0^\infty p(x_t,t)c_t dt \equiv \int_0^\infty \hat{p}(x_t)c_t dt = \int_0^{\bar{x}} \hat{p}(x)dx \tag{29}$$

where $\bar{x} \equiv \lim_{t\to\infty} x_t$. Thus, when present prices are constant across market dates, firm revenues depend only on the total contracts sold. Recall $M \equiv \sup \{x | F(x) < 1\}$. If M is finite then no price function could provide an equilibrium unless $\hat{p}(x) \equiv 0$, $x > M$. Similarly, whether or not M is finite, we must in equilibrium have $\hat{p}(x) > 0$, $x < M$. Thus, firms maximize profits if and only if $\bar{x} = M$.

On the demand side we may formulate the consumers' lifetime allocation problem as

$$\max_{c_t, x_t} \int_0^\infty [e^{-\rho t} u(c_t) \left(1-F(x_t)\right) - p(x_t,t)c_t]dt.$$

Here c_t is the quantity of resource demanded for future delivery at date t and x_t is the "quality" of resource demanded. That is, at the prevailing prices, x_t is the cumulative extraction which consumers would find most attractive in the firm from which they buy their date t contracts. Necessary conditions are

$$e^{-\rho t} u'(c_t)(1-F(x_t)) = p(x_t,t) \tag{30}$$

$$e^{-\rho t} u(c_t) f(x_t) = -\frac{\partial p}{\partial x}(x_t,t)c_t \tag{31}$$

Combining (30) and (31) with (28) and the discussion below (29) we learn that in a market equilibrium the following conditions must hold:

$$\frac{d}{dt}\left[e^{-\rho t} u'(c_t)(1-F(x_t))\right] = -e^{-\rho t} u(c_t) f(x_t) \tag{32}$$

and

$$\lim_{t\to\infty} x_t = M \tag{33}$$

which by the proof of Proposition I imply

<u>Proposition IV</u>: If a solution for the planning problem (9) exists then a competitive equilibrium also exists. Moreover the equilibrium and optimal allocations coincide.

VI. Conclusion

We have seen that the problem of uncertain reserves may be treated in a rather satisfactory way, so long as the "information" available about the level of reserves is not changing in any interesting way over time. It would seem to be straight-forward and interesting to extend the model given here to the case of production and capital accumulation. In such a model, the effect of changing uncertainty on the rate of substitution of capital for the natural resource in production could be studied. In a growth model of this sort, the complex question of decentralized allocations and efficiency might be approached through a combination of the techniques employed here and those used by Stiglitz [11] in his study of the market mechanism in deterministic models with natural resources. However, it seems that the problem most in need of elucidation concerns the modeling of the process of exploration.

FOOTNOTES

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1/ Throughout the paper a subscript denotes the time at which the given function is to be evaluated.

2/ Murray Kemp has recently treated this problem [7], deriving some results similar to those presented below.

3/ By $\psi^*(t)$ we mean $\psi(x^*(t),\dot{x}^*(t),t)$, etc.

4/ Strictly speaking, this condition only applies when there is no predetermined terminal condition on x.

5/ The results of Yaari [15] concerning when uncertain lifetime merely raises consumer's discount rate are also a special case of Lemma 1.

6/ If "reconditioning" is thought of as changing the discount rate, then this result follows from Strotz' well known sufficient condition for dynamic consistency [12]. I am indebted to Mort Kamien for pointing this out to me.

7/ This result is the analogue in our problem of the theorem justifying NNP as a welfare measure proved by Weitzman in [14].

8/ There is a basic problem here of decentralizing ownership of the resource while maintaining informational symmetry with the planning problem. If $\tilde{R}$ is society's random endowment and $\tilde{R}_i$, $i = 1,2,\ldots,n$, are n random variables representing separate pools and satisfying $\Sigma\tilde{R}_i = \tilde{R}$, then it is apparent that the planner would rather face the n random variables than their sum. The reason is that he can learn about his holdings as individual pools are exhausted. This is true even if the $\tilde{R}_i$ are i.i.d. Only in the limit as $n \to \infty$ with $(\tilde{R}_i)i=1,n$ as i.i.d. random variables summing to $\tilde{R}$ is complete symmetry attained. However, this operation is possible only for a very small class of distribution functions of $\tilde{R}$, the so-called "infinitely divisible" distributions (see Feller [16]). In general this method of solution would not be feasible. (E.g., if $\tilde{R}$ has compact support then it is <u>not</u> infinitely divisible.) In the text $\tilde{R}_i = 1/n\tilde{R}$, the optimal behavior would be to exploit one firm's holdings completely, thereby learning all the other endowments. If one constrains the planner to extract equally from each pool, however, then equivalence with his original problem is again obtained. In the market equilibrium described here, all firms will end up extracting at the same (constrained optimal) rate. These problems should be borne in mind when interpreting the result.

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