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FENCHEL'S DUALITY THEOREM IN GENERALIZED GEOMETRIC PROGRAMMING

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Abstract. Fenchel's duality theorem is extended to generalized geometric programming with explicit constraints -- an extension that also generalizes and strengthens Slater's version of the Kuhn-Tucker theorem.

Key words: Fenchel's duality theorem, generalized geometric programming, convex programming, ordinary programming, Slater's constraint qualification, Kuhn-Tucker theorem.

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1. Introduction. Although many implications of this extension have already been discussed in the author's recent survey paper [1], a proof of it is given here for the first time.

This proof utilizes the unconstrained version that has already been established by independent and somewhat different arguments in [2] and [3]. In doing so, it exploits the main result from [4] and also requires some of the convexity theory in [3]--especially the theory having to do with the "relative interior" (ri S) of an arbitrary convex set $S \subseteq E_N$ (N-dimensional Euclidean space).

2. The unconstrained case. We begin with the following notation and hypotheses:

> $\mathcal{X}$ is a nonempty closed convex cone in E_n,
>
> g is a (proper) closed convex function with a nonempty (effective) domain $\mathcal{C} \subseteq E_n$.

Now, given $\mathcal{X}$ and g, consider the resulting "geometric programming problem" $\mathcal{A}$.

PROBLEM $\mathcal{A}$. Using the feasible solution set

$$\mathcal{S} \triangleq \mathcal{X} \cap \mathcal{C},$$

calculate both the problem infimum

$$\varphi \triangleq \inf_{x \in \mathcal{S}} g(x)$$

and the optimal solution set

$$\mathcal{S}^* \triangleq \{x \in \mathcal{S} \mid g(x) = \varphi\}.$$

Geometric duality is defined in terms of both the "dual cone"

$$\mathcal{Y} \stackrel{\Delta}{=} \{y \in E_n \mid 0 \le \langle x,y \rangle \text{ for each } x \in \mathcal{X}\}$$

and the "conjugate transform function" h whose (effective) domain

$$\mathcal{D} \stackrel{\Delta}{=} \{y \in E_n \mid \sup_{x \in \mathcal{C}} [\langle y,x \rangle - g(x)] \text{ is finite}\}$$

and whose functional value

$$h(y) \stackrel{\Delta}{=} \sup_{x \in \mathcal{C}} [\langle y,x \rangle - g(x)].$$

In particular, given the geometric programming problem $\mathcal{A}$, consider the resulting "geometric dual problem" $\mathcal{B}$.

PROBLEM $\mathcal{B}$. Using the feasible solution set

$$\mathcal{T} \stackrel{\Delta}{=} \mathcal{Y} \cap \mathcal{D},$$

calculate both the problem infimum

$$\psi \stackrel{\Delta}{=} \inf_{y \in \mathcal{T}} h(y)$$

and the optimal solution set

$$\mathcal{T}^* \stackrel{\Delta}{=} \{y \in \mathcal{T} \mid h(y) = \psi\}.$$

Fenchel's duality theorem in the context of dual problems $\mathcal{A}$ and $\mathcal{B}$ is one of the most important theorems in geometric programming. It can be stated in the following way.

Theorem 1. *If problem $\mathcal{B}$ has both a feasible solution $y^\circ \in (\mathrm{ri}\,\mathcal{Y}) \cap (\mathrm{ri}\,\mathcal{D})$ and a finite infimum ψ, then*

(I) *problem $\mathcal{A}$ has both a nonempty feasible solution set $\mathcal{S}$ and a finite infimum φ, and*

$$0 = \varphi + \psi,$$

(II) *problem $\mathcal{A}$ has a nonempty optimal solution set $\mathcal{S}^*$.*

This theorem is established as Theorem 31.4 on page 335 of [3].

The implications of Theorem 1 are given on page 26 of [1]. An important extension of it is established in the next section.

3. The constrained case. To incorporate explicit constraints into generalized geometric programming, we introduce the following notation and hypotheses:

I and J are two nonintersecting (possibly empty) positive-integer index sets with finite cardinality $o(I)$ and $o(J)$ respectively;

x^k and y^k are independent vector variables in E_{n_k} for $k \in \{0\} \cup I \cup J$, and x^I and y^I denote the respective Cartesian products of the vector variables x^i, $i \in I$, and y^i, $i \in I$ while x^J and y^J denote the respective Cartesian products of the vector variables x^j, $j \in J$, and y^j, $j \in J$; so the Cartesian products $(x^0, x^I, x^J) \stackrel{\Delta}{=} x$ and $(y^0, y^I, y^J) \stackrel{\Delta}{=} y$ are independent vector variables in E_n, where

$$n \stackrel{\Delta}{=} n_0 + \sum_I n_i + \sum_J n_j;$$

α and λ are independent vector variables with respective components α_i and λ_i for $i \in I$, and β and κ are independent vector variables with respective components β_j and κ_j for $j \in J$;

X and Y are nonempty closed convex dual cones in E_n, and g_k and h_k are (proper) closed convex conjugate functions with respective (effective) domains $C_k \subseteq E_{n_k}$ and $D_k \subseteq E_{n_k}$ for $k \in \{0\} \cup I \cup J$.

Now, let

$$\mathcal{X} \stackrel{\Delta}{=} \{(x^0, x^I, \alpha, x^J, \kappa) \in E_{\eta} \mid (x^0, x^I, x^J) \in X;\ \alpha = 0;\ \kappa \in E_{o(J)}\},$$

where $n + o(I) + o(J) = \eta$. In addition, let

$$\mathcal{C} \stackrel{\Delta}{=} \{(x^0, x^I, \alpha, x^J, \kappa) \in E_{\eta} \mid x^0 \in C_0;\ x^i \in C_i,\ \alpha_i \in E_1, \text{ and } g_i(x^i) + \alpha_i \le 0,\ i \in I;\ (x^j, \kappa_j) \in C_j^+,\ j \in J\},$$

and let

$$\mathcal{G}(x^0, x^I, \alpha, x^J, \kappa) \stackrel{\Delta}{=} g_0(x^0) + \sum_J g_j^+(x^j, \kappa_j),$$

where the (closed convex) function g_j^+ has a domain

$$C_j^+ \stackrel{\Delta}{=} \{(x^j, \kappa_j) \mid \text{either } \kappa_j = 0 \text{ and } \sup_{d^j \in D_j} \langle x^j, d^j \rangle < +\infty, \text{ or } \kappa_j > 0 \text{ and } x^j \in \kappa_j C_j\}$$

and functional values

$$g_j^+(x^j, \kappa_j) \stackrel{\Delta}{=} \begin{cases} \sup_{d^j \in D_j} \langle x^j, d^j \rangle & \text{if } \kappa_j = 0 \text{ and } \sup_{d^j \in D_j} \langle x^j, d^j \rangle < +\infty \\ \kappa_j g_j(x^j/\kappa_j) & \text{if } \kappa_j > 0 \text{ and } x^j \in \kappa_j C_j. \end{cases}$$

The resulting problem $\mathcal{A}$ can clearly be stated in the following way.

PROBLEM A. Consider the objective function G whose domain

$$C \overset{\Delta}{=} \{(x,\kappa) \mid x^k \in C_k,\ k \in \{0\} \cup I,\ \text{and}\ (x^j,\kappa_j) \in C_j^+,\ j \in J\}$$

and whose functional value

$$G(x,\kappa) \overset{\Delta}{=} g_0(x^0) + \sum_J g_j^+(x^j,\kappa_j).$$

Using the feasible solution set

$$S \overset{\Delta}{=} \{(x,\kappa) \in C \mid x \in X,\ \text{and}\ g_i(x^i) \le 0,\ i \in I\},$$

calculate both the problem infimum

$$\varphi \overset{\Delta}{=} \inf_{(x,\kappa) \in S} G(x,\kappa)$$

and the optimal solution set

$$S^* \overset{\Delta}{=} \{(x,\kappa) \in S \mid G(x,\kappa) = \varphi\}.$$

Now, section 3 of [4] shows that

$$\mathcal{Y} = \{(y^0,y^I,\lambda,y^J,\beta) \in E_n \mid (y^0,y^I,y^J) \in Y;\ \beta = 0,\ \lambda \in E_{o(I)}\}.$$

Section 3 of [4] also shows that

$$\mathcal{D} = \{(y^0,y^I,\lambda,y^J,\beta) \in E_n \mid y^0 \in D_0;\ (y^i,\lambda_i) \in D_i^+,\ i \in I;\ y^j \in D_j,$$

$$\beta_j \in E_1,\ \text{and}\ h_j(y^j) + \beta_j \le 0,\ j \in J\},$$

and that

$$h(y^0,y^I,\lambda,y^J,\beta) = h_0(y^0) + \sum_I h_i^+(y^i,\lambda_i),$$

where the (closed convex) function h_i^+ has a domain

$$D_i^+ \overset{\Delta}{=} \{(y^i, \lambda_i) \mid \text{either } \lambda_i = 0 \text{ and } \sup_{c^i \in C_i} \langle y^i, c^i \rangle < +\infty, \text{ or } \lambda_i > 0 \text{ and } y^i \in \lambda_i D_i\}$$

and functional values

$$h_i^+(y^i, \lambda_i) \overset{\Delta}{=} \begin{cases} \sup_{c^i \in C_i} \langle y^i, c^i \rangle & \text{if } \lambda_i = 0 \text{ and } \sup_{c^i \in C_i} \langle y^i, c^i \rangle < +\infty \\ \\ \lambda_i h_i (y^i/\lambda_i) & \text{if } \lambda_i > 0 \text{ and } y^i \in \lambda_i D_i . \end{cases}$$

The resulting problem $\mathcal{B}$ can clearly be stated in the following way.

PROBLEM B. <u>Consider the objective function</u> H <u>whose domain</u>

$$D \overset{\Delta}{=} \{(y, \lambda) \mid y^k \in D_k, \ k \in \{0\} \cup J, \ \underline{\text{and}} \ (y^i, \lambda_i) \in D_i^+, \ i \in I\}$$

<u>and whose functional value</u>

$$H(y, \lambda) \overset{\Delta}{=} h_0(y^0) + \sum_I h_i^+(y^i, \lambda_i) .$$

<u>Using the feasible solution set</u>

$$T \overset{\Delta}{=} \{(y, \lambda) \in D \mid y \in Y, \ \underline{\text{and}} \ h_j(y^j) \le 0, \ j \in J\},$$

<u>calculate both the problem infimum</u>

$$\psi \overset{\Delta}{=} \inf_{(y, \lambda) \in T} H(y, \lambda)$$

<u>and the optimal solution set</u>

$$T^* \overset{\Delta}{=} \{(y, \lambda) \in T \mid H(y, \lambda) = \psi\} .$$

It is worth noting that dual problems A and B provide the only completely symmetric duality that is presently known for general (closed) convex programming with explicit constraints. Moreover, [1] and some of the references cited therein show that all other duality in convex programming can be viewed as a special case. For the fundamental relations between geometric duality and ordinary Lagrangian duality see [5].

Fenchel's duality theorem in the context of dual problems A and B is one of the most important theorems, as well as one of the deepest theorems, in geometric programming. It can be stated in the following way.

Theorem 2. If

(i) problem B has a feasible solution (y', λ') such that

$$h_j(y'^j) < 0 \qquad j \in J,$$

(ii) problem B has a finite infimum ψ,

(iii) there exists a vector (y^+, λ^+) such that

$$y^+ \in (\text{ri } Y),$$

$$y^{+k} \in (\text{ri } D_k) \qquad k \in \{0\} \cup J,$$

$$(y^{+i}, \lambda_i^+) \in (\text{ri } D_i^+) \qquad i \in I,$$

then

(I) problem A has both a nonempty feasible solution set S and a finite infimum φ, and

$$0 = \varphi + \psi,$$

(II) <u>problem</u> A <u>has a nonempty optimal solution set</u> S*.

Proof. We obviously need only show that the Fenchel hypothesis in Theorem 1 (i.e. the hypothesis that there exists a vector $\mathcal{Y}^{\circ} \in (\text{ri}\,\mathcal{Y}) \cap (\text{ri}\,\mathcal{D})$) is equivalent to hypotheses (i) and (iii) in Theorem 2.

Toward that end, we first use the formulas for $\mathcal{Y}$ and $\mathcal{D}$ to derive comparable formulas for $(\text{ri}\,\mathcal{Y})$ and $(\text{ri}\,\mathcal{D})$ -- two derivations that make crucial use of the following basic facts:

(A) $(\text{ri } U) = U$ when U is a vector space,

(B) $(\text{ri } V) = \times_1^{\eta} (\text{ri } V_k)$ when $V = \times_1^{\eta} V_k$ and the sets V_k are convex,

and

(C) $(\text{ri } W) = (\text{int } W)$, the "interior" of W, when W is a convex set with the same "dimension" as the space in which it is embedded.

Fact (A) is established on page 44 of [3]; fact (B) can be obtained inductively from the formula at the top of page 49 of [3]; and fact (C) is explained on page 44 of [3].

Now, the formula for $\mathcal{Y}$ along with facts (A) and (B) implies that

$$(\text{ri}\,\mathcal{Y}) = \{(y^0, y^I, \lambda, y^J, \beta) \in E_{\eta} \mid (y^0, y^I, y^J) \in (\text{ri } Y);\ \lambda \in E_{o(I)};\ \beta = 0\}.$$

Moreover, the formula for $\mathcal{D}$ along with facts (A) and (B) implies that

$$(\text{ri}\,\mathcal{D}) = \{(y^0, y^I, \lambda, y^J, \beta) \in E_{\eta} \mid y^0 \in (\text{ri } D_0);\ \lambda_i > 0 \text{ and } y^i \in \lambda_i (\text{ri } D_i),$$

$$i \in I;\ y^j \in (\text{ri } D_j),\ \beta_j \in E_1, \text{ and } h_j(y^j) + \beta_j < 0,\ j \in J\},$$

by virtue of both the equation

$$(\text{ri } D_i^+) = \{(y^i, \lambda_i) \mid \lambda_i > 0 \text{ and } y^i \in \lambda_i (\text{ri } D_i)\}$$

and the equation

$$(\text{ri } \{(y^j, \beta_j) \mid y^j \in D_j \text{ and } h_j(y^j) + \beta_j \le 0\}) =$$

$$\{(y^j, \beta_j) \mid \beta_j \in E_1,\ y^j \in (\text{ri } D_j),\ \text{ and } h_j(y^j) + \beta_j < 0\}.$$

To derive the latter equation, simply use Theorem 6.8 on page 49 of [3] along with fact (C). To derive the former equation, first consider the point-to-set mapping $Y_i^+ : \Lambda_i^+$ where

$$Y_i^+[\lambda_i] \overset{\Delta}{=} \{y^i \mid (y^i, \lambda_i) \in D_i^+\}$$

and

$$\Lambda_i^+ \overset{\Delta}{=} \{\lambda_i \mid Y_i^+[\lambda_i] \text{ is not empty}\}.$$

Now, Corollary 6.8.1 on page 50 of [3] implies that

$$(\text{ri } D_i^+) = \{(y^i, \lambda_i) \mid \lambda_i \in (\text{ri } \Lambda_i^+) \text{ and } y^i \in (\text{ri } Y_i^+[\lambda_i])\}.$$

Moreover, the definition of D_i^+ clearly shows that $\Lambda_i^+ = \{\lambda_i \ge 0\}$, which means of course that

$$(\text{ri } \Lambda_i^+) = \{\lambda_i > 0\}.$$

Furthermore, for $\lambda_i > 0$ the definition of D_i^+ clearly shows that $Y_i^+[\lambda_i] = \lambda_i D_i$, which means that

$$(\text{ri } Y_i^+[\lambda_i]) \equiv \lambda_i (\text{ri } D_i) \text{ for } \lambda_i \in (\text{ri } \Lambda_i^+),$$

by virtue of Corollary 6.6.1 on page 48 of [3]. Consequently, our derivation of the preceding formula for $(\mathrm{ri}\,\mathcal{D})$ is complete.

In particular then, the Fenchel hypothesis in Theorem 1 simply asserts that

> there exists a vector $(y^0, y^I, \lambda, y^J, 0) = y^\circ$
>
> such that $(y^0, y^I, y^J) \in (\mathrm{ri}\ Y)$; $y^0 \in (\mathrm{ri}\ D_0)$;
>
> $\lambda_i > 0$ and $y^i \in \lambda_i (\mathrm{ri}\ D_i)$, $i \in I$; $y^j \in (\mathrm{ri}\ D_j)$
>
> and $h_j(y^j) < 0$, $j \in J$.

To complete our proof, we now show that this hypothesis is in fact equivalent to the hypothesis

> there exists a vector $(y'^0, y'^I, \lambda', y'^J)$
>
> such that $(y'^0, y'^I, y'^J) \in Y$; $y'^0 \in D_0$;
>
> $(y'^i, \lambda'_i) \in D_i^+$, $i \in I$; $y'^j \in D_j$ and $h_j(y'^j) < 0$, $j \in J$
>
> --- and there exists a vector
>
> $(y^{+0}, y^{+I}, \lambda^+, y^{+J})$ such that
>
> $(y^{+0}, y^{+I}, y^{+J}) \in (\mathrm{ri}\ Y)$; $y^{+0} \in (\mathrm{ri}\ D_0)$; $\lambda_i^+ > 0$
>
> and $y^{+i} \in \lambda_i (\mathrm{ri}\ D_i)$, $i \in I$; $y^{+j} \in (\mathrm{ri}\ D_j)$, $j \in J$.

Obviously, a vector (y^0, y^I, λ, y^J) that satisfies the former hypothesis satisfies both parts of the latter hypothesis. On the other hand, Theorem 6.1 on page 45 of [3] and Theorem 7.1 on page 51 of [3] imply that a convex combination $\alpha(y'^{0},y'^{I},\lambda',y'^{J})+\beta(y^{+0},y^{+I},\lambda^{+},y^{+J})$ of vectors $(y'^{0},y'^{I},\lambda',y'^{J})$ and $(y^{+0},y^{+I},\lambda^{+},y^{+J})$ that satisfy the latter hypothesis will satisfy the former hypothesis for sufficiently small $\beta>0$. q.e.d.

Although the condition $h_j(y'^{j})<0$, $j\in J$ in hypothesis (i) of Theorem 2 resembles the well-known "Slater constraint qualification", it is of course to be deleted when J is empty -- which is the situation in most applications. However, the analogous condition $g_i(x'^{i})<0$, $i\in I$ in hypothesis (i) of the (unstated) dual of Theorem 2 (obtained from Theorem 2 by interchanging the symbols A and B, the symbols x and y, the symbols κ and λ, the symbols g and h, the symbols i and j, the symbols I and J, the symbols φ and ψ, the symbols X and Y, the symbols C and D, the symbols S and T, and the symbols S* and T*) is essentially the Slater constraint qualification. In fact, we shall now see that the "ordinary programming" case of the dual of Theorem 2 actually strengthens Slater's version of the "Kuhn-Tucker theorem".

The ordinary programming case occurs when

$$J=\emptyset,$$

$$n_k=m \text{ and } C_k \overset{\Delta}{=} C_o \text{ for some set } C_o\subseteq E_m \qquad k\in\{0\}\cup I,$$

and

$$X\overset{\Delta}{=}\text{column space of }\begin{bmatrix}U\\U\\ \cdot\\ \cdot\\ \cdot\\U\end{bmatrix}\quad\text{where there is a total of } 1+o(I) \text{ identity matrices } U \text{ that are } m\times m.$$

In particular, an explicit elimination of the vector space condition $x \in X$ by the linear transformation

$$\begin{pmatrix} x^0 \\ x^I \end{pmatrix} = \begin{bmatrix} U \\ U \\ \cdot \\ \cdot \\ \cdot \\ U \end{bmatrix} z$$

shows that the resulting problem A is equivalent to the very general ordinary programming problem

$$\underline{\text{Minimize}}\ g_0(z)\ \underline{\text{subject to}}$$

$$g_i(z) \le 0 \qquad i \in I$$

$$z \in C_o .$$

Now, the Slater constraint qualification for the preceding problem simply requires the existence of a feasible solution z' such that $g_i(z') < 0$, $i \in I$. Moreover, Slater's version of the Kuhn-Tucker theorem asserts that the existence of such a "Slater solution" z' and the existence of a finite infimum φ are sufficient to guarantee the existence of a Kuhn-Tucker (Lagrange) multiplier vector λ^*.

To strengthen the preceding theorem with the aid of the dual of Theorem 2, first note that the image $x' = (z', z', \ldots, z')$ of a Slater solution z' under the given linear transformation satisfies hypothesis (i) of the dual of Theorem 2. Then, note that the existence of a finite infimum φ is simply hypothesis (ii) of the dual of Theorem 2. Now, the convexity of C_o implies the existence of a vector $z^+ \in (\text{ri}\ C_o)$, by virtue of Theorem 6.2 on page 45 of [3]. Moreover, its image $x^+ = (z^+, z^+, \ldots, z^+)$ under the given linear transformation clearly satisfies hypothesis (iii) of the dual of Theorem 2 -- because (ri X) = X by virtue of fact (A), and because $J = \phi$. Consequently, the dual of Theorem 2 implies that both T and T* are nonempty and that $0 = \varphi + \psi$. In view of Corollary 7A of [6], we conclude from the nonemptyness of T* that a Kuhn-Tucker (Lagrange) vector λ^* exists. Finally, note that we have also shown the existence of another vector y*; so the Slater version of the Kuhn-Tucker theorem has actually been strengthened.

More significant implications of Theorem 2 are given on page 47 of [1].

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