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FENCKEL'S DUALITY THEOREM IN
GENERALIZED GEOMETRIC PROGRAMMING

by
Elmor L. Peterson

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Elmor L. Peterson*

Abstract. Fenchel’s duality theorem is extended to generalized geometric programming with explicit constraints -- an extension that also generalizes and strengthens Slater's version of the Kuhn-Tucker theorem.

Key words: Fenchel’s duality theorem, generalized geometric programming, convex programming, ordinary programming, Slater’s constraint qualification, Kuhn-Tucker theorem.

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1. Introduction. Although many implications of this extension have already been discussed in the author's recent survey paper [1], a proof of it is given here for the first time.

This proof utilizes the unconstrained version that has already been established by independent and somewhat different arguments in [2] and [3]. In doing so, it exploits the main result from [4] and also requires some of the convexity theory in [3]—especially the theory having to do with the "relative interior" (cf. S) of an arbitrary convex set $S \subseteq \mathbb{R}^n$ (N-dimensional Euclidean space).

2. The unconstrained case. We begin with the following notation and hypotheses:

- $\mathcal{X}$ is a nonempty closed convex cone in $\mathbb{R}^n$.
- $\varphi$ is a (proper) closed convex function with a nonempty (effective) domain $\mathcal{C} \subseteq \mathbb{R}^n$.

Now, given $\mathcal{X}$ and $\varphi$, consider the resulting "geometric programming problem" $\mathcal{P}$.

**Problem $\mathcal{P}$. Using the feasible solution set**

$$\mathcal{X} \cap \mathcal{C},$$

**calculate both the problem infimum**

$$\varphi^* = \inf_{x \in \mathcal{X}} \varphi(x)$$

and the optimal solution set

$$\mathcal{X}^* = \{ x \in \mathcal{X} | \varphi(x) = \varphi^* \}.$$
Geometric duality is defined in terms of both the "dual cone"

$$ y \hat{\in} \{ y \in E^* \mid 0 \leq \langle y, x \rangle \} \text{ for each } x \in \mathbb{R} $$

and the "conjugate transform function" $h$ whose (effective) domain

$$ h^* \{ y \in E^* \mid \sup_{x \in C} \{ \langle y, x \rangle - \varphi(x) \} \text{ is finite} \} $$

and whose functional value

$$ h(y) = \sup_{x \in C} \{ \langle y, x \rangle - \varphi(x) \}. $$

In particular, given the geometric programming problem $f$, consider the resulting "geometric dual problem" $\mathcal{S}$.

**PROBLEM $\mathcal{S}$**: Using the feasible solution set

$$ \hat{\mathcal{S}} = \mathcal{Q} \cap \mathcal{S}, $$

calculate both the problem infimum

$$ \hat{\mathcal{Q}} = \inf_{y \in \mathcal{S}} h(y) $$

and the optimal solution set

$$ \hat{\mathcal{S}} = \{ y \in \mathcal{S} \mid h(y) = \hat{\mathcal{Q}} \}. $$

Fenchel's duality theorem in the context of dual problems $f$ and $\mathcal{S}$ is one of the most important theorems in geometric programming. It can be stated in the following way.
Theorem 1. If problem \( \mathcal{D} \) has both a feasible solution \( y^* \in (r(\mathcal{D})) \cap (r(\mathcal{J})) \) and a finite infimum \( \psi \), then

(1) problem \( \mathcal{J} \) has both a nonempty feasible solution set \( \mathcal{J} \) and a finite infimum \( \varphi \), and

\[ 0 = \varphi \leq \psi, \]

(2) problem \( \mathcal{D} \) has a nonempty optimal solution set \( \mathcal{D}^* \).

This theorem is established as Theorem 31.4 on page 335 of [3].

The implications of Theorem 1 are given on page 26 of [1]. An important extension of it is established in the next section.

3. The constrained case. To incorporate explicit constraints into generalized geometric programming, we introduce the following notation and hypotheses:

I and J are two nonintersecting (possibly empty) positive-integer index sets with finite cardinality \( o(I) \) and \( o(J) \) respectively;

\( x^k \) and \( y^k \) are independent vector variables in \( E_n^k \) for \( k \in [0] \cup I \cup J \),

and \( x^I \) and \( y^I \) denote the respective Cartesian products of the vector variables \( x^i \), \( i \in I \), and \( y^i \), \( i \in I \) while \( x^J \) and \( y^J \) denote the respective Cartesian products of the vector variables \( x^j \), \( j \in J \), and \( y^j \), \( j \in J \); so the Cartesian products \( (x^I, x^J) \) and \( (y^I, y^J) \) are independent vector variables in \( E_n^I \times I \), where

\[ \alpha = \sum_{i \in I} n_i \gamma_i, \]

(1) \( \alpha \) and \( \gamma \) are independent vector variables with respective components \( \alpha_i \)

and \( \gamma_i \) for \( i \in I \), and \( \gamma \) and \( K \) are independent vector variables with
respective components $\mathscr{E}_j$ and $\mathcal{K}_j$ for $j \in J$;

$X$ and $Y$ are nonempty closed convex dual cones in $E_n$, and $\mathcal{E}_k$ and $\mathcal{H}_k$ are (proper) closed convex conjugate functions with respective (effective) domains $C_k = \mathcal{E}_k^*$ and $D_k = \mathcal{H}_k^*$ for $k \in \{0\} \cup \{J\}$.

Now, let

$$\mathcal{Z} = \{(x^0, x^1, ..., x^J, \epsilon) \in E_n \mid (x^0, x^1, x^J) \in X; \alpha = 0; \epsilon \in F_0(J), \}$$

where $\mathcal{O}(I) \circ \mathcal{O}(J) = T_i$. In addition, let

$$\mathcal{C} = \{(x^0, x^1, ..., x^J, \epsilon, \kappa) \in E_n \mid x^0 \in C_0; x^1 \in C_1, \alpha \in \kappa; \text{ and } \mathcal{E}_j^*(x^j) + \alpha \leq 0, \quad \forall j \in J; \quad (x^j, \kappa_j) \in C_j^*, \quad j \in J, \}$$

and let

$$\varphi(x^0, x^1, ..., x^J, \epsilon, \kappa) = \mathcal{O}(x^0) + \sum_{j \in J} \mathcal{E}_j^*(x^j, \kappa_j),$$

where the (closed convex) function $\mathcal{E}_j^*$ has a domain

$$\mathcal{C}_j^* = \{(x^j, \kappa_j) \mid \text{either } \kappa_j = 0 \text{ and } \sup_{d^j \in D_j} \langle x^j, d^j \rangle = \infty, \text{ or } \kappa_j > 0 \text{ and } x^j \in \kappa_j C_j \}$$

and functional values

$$\mathcal{E}_j^*(x^j, \kappa_j) = \begin{cases} 
\sup_{d^j \in D_j} \langle x^j, d^j \rangle & \text{if } \kappa_j = 0 \text{ and } \sup_{d^j \in D_j} \langle x^j, d^j \rangle < \infty \\
0 & \text{if } \kappa_j > 0 \text{ and } x^j \in \kappa_j C_j.
\end{cases}$$

The resulting problem $\mathcal{C}$ can clearly be stated in the following way.
PROBLEM A. Consider the objective function \( \tilde{g} \) whose domain

\[
\tilde{C} = \{(x, \xi) \mid x \in C_k, k \in [0] \cup I, \text{ and } (x^I, \xi^J) \in C^I_j, \ j \in J\}
\]

and whose functional value

\[
\tilde{g}(x, \xi) = \sum_{k=0}^m x^k + \sum_{j \in J} \tilde{g}^+(x^I_j, \xi^J_j).
\]

Using the feasible solution set

\[
\bar{S}^* = \{(x, \xi) \in \bar{C} \mid x \in X, \text{ and } \xi^l(x^I) \leq 0, \ l \in I\},
\]

calculate both the problem infimum

\[
\psi^* = \inf_{(x, \xi) \in \bar{S}} \tilde{g}(x, \xi)
\]
and the optimal solution set

\[
\bar{G}^* = \{(x, \xi) \in \bar{S} \mid \tilde{g}(x, \xi) = \psi^*\}.
\]

Now, section 3 of [4] shows that

\[
\gamma = \{(y^0, y^I, y^J, \beta) \in \bar{E}_I \mid (y^I, y^J) \in \gamma; \beta = 0, \lambda \in \bar{E}_{o(I)}\}.
\]

Section 3 of [4] also shows that

\[
\delta = \{(y^0, y^I, y^J, \beta) \in \bar{E}_I \mid y^0 \in D_0; (y^I, y^J) \in D^+_I, \ i \in I; y^J \in D_J, \beta_j \in E_{oI}, \text{ and } h_j(y^J) + \beta_j \leq 0, \ j \in J\},
\]

and that

\[
h(y^0, y^I, y^J, \beta) = h_0(y^0) + \sum_{l \in I} h^+_l(y^I_l, \lambda^I_l),
\]
where the (closed convex) function $h_4^+$ has a domain

$$
\eta_4^+ = \{(y^4, \lambda_4) \, | \, \lambda_4 \geq 0 \text{ and } \sup_{c^4 \in C_4} \langle y^4, c^4 \rangle < +\infty, \text{ or } \lambda_4 > 0 \text{ and } y^4 \in \lambda_4 D_4 \} 
$$

and functional values

$$
h_4^+(y^4, \lambda_4) = \begin{cases} 
\sup_{c^4 \in C_4} \langle y^4, c^4 \rangle & \text{if } \lambda_4 = 0 \text{ and } \sup_{c^4 \in C_4} \langle y^4, c^4 \rangle < +\infty \\
\lambda_4 y^4 / \lambda_4 & \text{if } \lambda_4 > 0 \text{ and } y^4 \in \lambda_4 D_4. 
\end{cases} 
$$

The resulting problem $\mathcal{S}$ can clearly be stated in the following way.

**PROBLEM B.** Consider the objective function $H$ whose domain

$$
D = \{(y, \lambda) \, | \, y^k \in D_k, \, k \in \{0\} \cup \mathbb{J}, \, \text{and} \, y^i, \lambda_i \in \mathcal{D}_i, \, i \in \mathbb{I}\} 
$$

and whose functional value

$$
H(y, \lambda) = h_0(y^0) + \sum_{i \in \mathbb{I}} h_4^+(y^4_i, \lambda_i). 
$$

Using the feasible solution set

$$
T = \{ (y, \lambda) \in D \, | \, y \in \mathcal{Y}, \, \text{and} \, h_j(y^j) < 0, \, j \in \mathbb{J} \}, 
$$

calculate both the problem infimum

$$
\hat{\varphi} = \inf_{(y, \lambda) \in T} H(y, \lambda) 
$$

and the optimal solution set

$$
T^\ast = \{ (y, \lambda) \in T \, | \, H(y, \lambda) = \hat{\varphi} \}.$$
It is worth noting that dual problems A and B provide the only completely symmetric duality that is presently known for general (closed) convex programming with explicit constraints. Moreover, [1] and some of the references cited therein show that all other duality in convex programming can be viewed as a special case. For the fundamental relations between geometric duality and ordinary Lagrangian duality see [5].

Fenchel's duality theorem in the context of dual problems A and B is one of the most important theorems, as well as one of the deepest theorems, in geometric programming. It can be stated in the following way.

**Theorem 7.** If

(i) **Problem B has a feasible solution** $(y', \lambda')$ **such that**

$$h_j(y'_j) < 0$$

$j \in J$,

(ii) **Problem B has a finite infimum** $\check{y}$,

(iii) **there exists a vector** $(y^+, \lambda^+)$ **such that**

$$y^+ \in (ri Y),$$

$$y^+ = (ri D_k)$$

$k \in \{0\} \cup J$,

$$y^+_k \in (ri D_k)$$

$k \in \{0\} \cup J$,

$$\lambda^+_k \in (ri A^+_k)$$

$k \in \{0\} \cup J$,

then

(i) **problem A has both a nonempty feasible solution set** $S$ **and a finite infimum** $\check{v}$, **and**

$$0 = v + \check{y},$$
(II) problem A has a nonempty optimal solution set $S^*$. 

Proof. We obviously need only show that the Fenchel hypothesis in Theorem 1 (i.e., the hypothesis that there exists a vector $y^* \in (ri)\gamma \cap (ri)\beta$) is equivalent to hypotheses (i) and (iii) in Theorem 2.

Toward that end, we first use the formulas for $\gamma$ and $\beta$ to derive comparable formulas for $(ri)\gamma$ and $(ri)\beta$--two derivations that make crucial use of the following basic facts:

(A) $(ri U) = U$ when $U$ is a vector space,

(B) $(ri V) = \bigcap_{k=1}^{n} V_k$ when $V = \bigcap_{k=1}^{n} V_k$ and the sets $V_k$ are convex,

and

(C) $(ri W) = (\text{int } W)$, the "interior" of $W$, when $W$ is a convex set with the same "dimension" as the space in which it is embedded.

Fact (A) is established on page 44 of [3]; fact (B) can be obtained inductively from the formula at the top of page 49 of [3], and fact (C) is explained on page 44 of [3].

Now, the formula for $\gamma$ along with facts (A) and (B) implies that

$$(ri)\gamma = \{ (y^0, y^0, \lambda, y^0, \beta) \in E \mid (y^0, y^0, y^0) \in (ri)\gamma; \lambda \in E^0(1) ; \beta = 0 \}.$$

Moreover, the formula for $\beta$ along with facts (A) and (B) implies that

$$(ri)\beta = \{ (y^0, y^0, \lambda, y^0, \beta) \in E \mid y^0 \in (ri)D_0 ; \lambda > 0 \text{ and } y^0 \in E_i (ri)D_i, \text{ and } h_j(y^0) + \beta_j < 0, \lambda \in E \},$$
by virtue of both the equation
\[ \text{ri } D^+_k = \{ (y^*, \lambda_k) \mid \lambda_k > 0 \text{ and } y^* \in \lambda_k (\text{ri } D^+_k) \} \]
and the equation
\[ \text{ri } \{ (y^j, \beta_j) \mid y^j \in D_j \text{ and } h_j (y^j) + \beta_j < 0 \} = \{ (y^j, \beta_j) \mid \beta_j \in \mathbb{R}_+, \text{ } y^j \in \text{ri } D_j \text{, and } h_j (y^j) + \beta_j < 0 \}. \]

To derive the latter equation, simply use Theorem 6.8 on page 49 of [3] along with fact (C). To derive the former equation, first consider the point-to-set mapping \( V^{+}_k \lambda_k \) where
\[ V^{+}_k (\lambda_k) \triangleq \{ y^* \mid (y^*, \lambda_k) \in D^+_k \} \]
and
\[ \lambda_k^{+} \triangleq [\lambda_k \mid V^{+}_k (\lambda_k) \text{ is not empty}] \]

Now, Corollary 6.8.1 on page 50 of [3] implies that
\[ \text{ri } D^+_k = \{ (y^*, \lambda_k) \mid \lambda_k \in \text{ri } \lambda_k^{+} \text{ and } y^* \in \text{ri } V^{+}_k (\lambda_k) \}. \]

Moreover, the definition of \( D^+_k \) clearly shows that \( \lambda_k^{+} = [\lambda_k > 0] \), which means of course that
\[ \text{ri } \lambda_k^{+} = [\lambda_k > 0]. \]

Furthermore, for \( \lambda_k > 0 \) the definition of \( D^+_k \) clearly shows that
\[ V^{+}_k (\lambda_k) = \lambda_k D^+_k, \]
which means that
\[ \text{ri } V^{+}_k (\lambda_k) = \lambda_k \text{ri } D^+_k \text{ for } \lambda_k \in \text{ri } \lambda_k^{+}, \]}
by virtue of Corollary 6.6.1 on page 48 of [3]. Consequently, our
derivation of the preceding formula for $(r,2)$ is complete.

In particular then, the Fenchel hypothesis in Theorem 1 simply
asserts that

there exists a vector $(y^0, y^1, \lambda, y^0, 0) = y^*$
such that $(y^0, y^1, y^0) \in (r,2); y^0 \in (r, D_0);$

$\lambda_i > 0$ and $y^j \in \lambda_i (r, D_j), i \in I; y^j \in (r, D_j)$

and $h_j(y^j) < 0, j \in J.$

To complete our proof, we now show that this hypothesis is in fact equivalent
to the hypothesis

there exists a vector $(y^0, y^1, \lambda, y^0)$
such that $(y^0, y^1, y^0) \in Y; y^0 \in D_0;$

$(y^i, \lambda^i) \in D^i, i \in I; y^j \in D^j$ and $h_j(y^j) < 0, j \in J$

and there exists a vector

$(y^0, y^1, \lambda, y^0)$ such that

$(y^0, y^1, y^0) \in (r,2); y^0 \in (r, D_0); \lambda_i > 0$

and $y^j \in \lambda_i (r, D_j), i \in I; y^j \in (r, D_j), j \in J.$

Obviously, a vector $(y^0, y^1, \lambda, y^0)$ that satisfies the former hypothesis
satisfies both parts of the latter hypothesis. On the other hand,
Theorem 6.1 on page 45 of [3] and Theorem 7.1 on page 51 of [3] imply that a convex combination \( \alpha (y^0, y^1, \ldots, y^j) + \beta (y^0, y^1, \ldots, y^k) \) of vectors \( y^0, y^1, \ldots, y^j \) and \( y^0, y^1, \ldots, y^k \) that satisfy the latter hypothesis will satisfy the former hypothesis for sufficiently small \( \beta > 0 \). \quad \text{q.e.d.}

Although the condition \( h_j (y^j)^T < 0 \), \( j \in \mathcal{J} \) in hypothesis (1) of Theorem 2 resembles the well-known "Slater constraint qualification", it is of course to be deleted when \( \mathcal{J} \) is empty -- which is the situation in most applications. However, the analogous condition \( g_\ell (x^\ell)^T < 0 \), \( \ell \in \mathcal{I} \) in hypothesis (1) of the (unstated) dual of Theorem 2 obtained from Theorem 2 by interchanging the symbols \( A \) and \( B \), the symbols \( x \) and \( y \), the symbols \( \mathcal{K} \) and \( \mathcal{L} \), the symbols \( i \) and \( j \), the symbols \( \mathcal{I} \) and \( \mathcal{J} \), the symbols \( \omega \) and \( i \), the symbols \( X \) and \( T \), the symbols \( C \) and \( D \), the symbols \( S \) and \( T \), and the symbols \( S^* \) and \( T^* \) is essentially the Slater constraint qualification. In fact, we shall now see that the "ordinary programming" case of the dual of Theorem 2 actually strengthens Slater's version of the "Kuhn-Tucker theorem".

The ordinary programming case occurs when

\[ J = \emptyset, \]

\[ n_k = m \text{ and } C_k = C_0 \text{ for some set } C_0 \subseteq \mathbb{R}^m \quad \text{for } k \in \{0\} \cup \mathcal{I}, \]

and

\[ \Delta \text{ column space of } \begin{bmatrix} y \\ u \\ \vdots \\ u \end{bmatrix} \text{ where there is a total of } 1 + o(1) \]

\[ \text{identity matrices } U \text{ that are } m \times m. \]
In particular, an explicit elimination of the vector space condition \( x \in \mathcal{X} \) by the linear transformation

\[
\begin{pmatrix}
  x_0^*
  \\
  u
  \\
  \vdots
  \\
  u
\end{pmatrix} = \begin{bmatrix}
  U && \\
  U && \\
  \vdots && \\
  U
\end{bmatrix} z
\]

shows that the resulting problem \( A \) is equivalent to the very general ordinary programming problem

\[
\begin{align*}
\text{Minimize} & \quad g_0(z) \\
\text{subject to} & \quad g_i(z) \leq 0 \quad i \in I \\
& \quad z \in \mathcal{C}_0.
\end{align*}
\]

Now, the Slater constraint qualification for the preceding problem simply requires the existence of a feasible solution \( z' \) such that \( g_i(z') < 0 \), \( i \in I \). Moreover, Slater's version of the Kuhn-Tucker theorem asserts that the existence of such a "Slater solution" \( z' \) and the existence of a finite infimum \( \varphi \) are sufficient to guarantee the existence of a Kuhn-Tucker (Lagrange) multiplier vector \( \lambda^* \).

To strengthen the preceding theorem with the aid of the dual of Theorem 2, first note that the image \( x' = (x', z', \ldots, z') \) of a Slater solution \( z' \) under the given linear transformation satisfies hypothesis (i) of the dual of Theorem 2. Then, note that the existence of a finite infimum \( \varphi \) is simply hypothesis (ii) of the dual of Theorem 2. Now, the convexity of \( \mathcal{C}_0 \) implies the existence of a vector \( z^* \in (\text{ri} \mathcal{C}_0) \), by virtue of Theorem 6.2 on page 45 of [3]. Moreover, its image \( x^* = (x^+, z^+, \ldots, z^+) \) under the given linear transformation clearly satisfies hypothesis (iii)
of the dual of Theorem 2 -- because \((r(\lambda)) = X\) by virtue of fact \((\lambda)\), and because \(J = \emptyset\). Consequently, the dual of Theorem 2 implies that both \(T^*\) and \(T^\theta\) are nonempty and that \(\theta = \emptyset + \emptyset\). In view of Corollary 7A of [5], we conclude from the nonemptiness of \(T^\theta\) that a Kuhn-Tucker (Lagrange) vector \(\lambda^*\) exists. Finally, note that we have also shown the existence of another vector \(\gamma^*\); so the Slater version of the Kuhn-Tucker theorem has actually been strengthened.

More significant implications of Theorem 2 are given on page 47 of [1].

References