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EXTERNALITIES, PUBLIC GOODS, AND THE GENERIC STRUCTURE OF PARETO SETS*/

by

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Abstract

The generic structure of the local pareto set is studied for a mathematical formulation of exchange economies which includes a form of taxation, public goods, externalities, production, and location type problems. Necessary conditions for the existence of a pareto point are derived. Using transversality theory, the generic structure of the set of points satisfying these conditions is established. As a by-product, the type of externalities which can effect the structure of the pareto set for a pure exchange economy without production is isolated. Catastrophe theory is used to show how perturbations of non-generic economies change the problem into the generic setting.

§1. Introduction

A local pareto optima (LPO) for a vector valued function is a point in the domain about which neighboring points cannot increase the value of some component without decreasing the value of some other component. Such points are of interest since they are a local solution for certain types of games. In recent years several authors have characterized these points for smooth pay-off or utility functions (A partial list includes [10,11,12,13].) In these papers the domain is usually some restriction of a hyperplane in Euclidean space, and restrictions are imposed upon the arguments of the functions - restrictions which arise in a most natural fashion since the modeling problem usually is a standard exchange economy without production or externalities.

The purpose of this paper is to remove some of these restrictions so that we can characterize the set of local pareto points for location type problems and for economies modeled to include externalities, public goods, production, etc. Thus, one of the reasons for this study is to understand the structure of the set of solution concepts for these more general types of problems. A second reason , and this is the one which led to this research, is that an approach we shall use in a subsequent paper to investigate information theory (as defined in [5,9]) requires knowledge of the local pareto set for a generic choice of utility functions.

In Section 2 we formulate our problem and provide some simple examples. These examples are meant to motivate our terminology, and later they will be used to illustrate our results. The examples we will consider are listed in the above paragraph.

In Section 3 first order necessary conditions for a LPO are given. These conditions are in terms of an interaction between the structure of the feasible set (we will define this term in the next section) and the gradients of the utility functions. This interaction leads to a division of the LPOs into two groups - dictatorial and non-dictatorial. The former type behave much like their title suggests - a subset of agents dictates the location of a LPO for all agents. Most of our results concern the latter type. For non-dictatorial LPOs this interaction between the feasible set and the utility functions is further illustrated by a naive taxation-subsidy rule. The section ends with a sufficiency condition for a point to be a LPO.

The above first order analysis does not hold for satiation points - that is, a point which is a local maximum for some agent's pay-off or utility function. These and similar points are studied with a second order analysis in Section 4.

In Section 5 we use the Thom transversality theorem to obtain the generic structure of the set of LPOs in feasibility space. In the first part of the section a plausibility argument is provided to describe Thom's theorem. In the second part of this section, our result is stated and proved for one type of LPOs.

It turns out that even in the non-dictatorial case agents can form a hierarchy of coalitions. This is discussed in Theorem 2, Section 3. The hierarchy is studied further in Section 6 where the generic behavior of such LPOs is obtained. Several examples are given to illustrate the results and the clustering of agents.

In Section 7 these results are tied together to provide a picture of the generic structure of the set of LPOs. However, there are many examples where the set of LPOs do not conform to this structure. Consequently a small perturbation of the utility functions may drastically change the structure of this LPO set. In this section we provide some examples and describe how the old set changes. The analysis used is, essentially, Thom's catastrophe theory [3,14,15].

Some of these results were presented at the National Bureau of Economic Research Conference on Decentralization at Northwestern, April, 1976. Part of Section 7 is included as a response to some questions posed to me at this meeting by J. Green and H. Sonnenschein.

§2. Mathematical Representation and Examples.

Assume that M is a smooth m-dimensional manifold, Ω is a codimension n, smooth, imbedded submanifold of M, $(n < m)$, and $u = (u_1, \ldots, u_a)$ is a smooth mapping from M into $\mathbb{E}^a$ where "a" is a positive integer. M will be called the commodity bundle, Ω the feasible set, and u is the pay-off or utility function. Furthermore, assume that M, and consequently Ω, has a given coordinate representation.

Definition 2.1. Point $p \in \Omega$ is said to be a local pareto optimal point (LPO) if there exists some open neighborhood V of p in Ω such that if $q \subset V - \{p\}$ and if for some i we have $u_i(q) > u_i(p)$, then there is some j such that $u_j(q) < u_j(p)$.

Our main goal is to obtain necessary conditions for p to be a LPO, and to obtain the generic representation of the set of points in $\mathring{\Omega}$ (the interior of Ω) which satisfy these conditions. It turns out that the answers will be in terms of the normal bundle of Ω at point p, which we shall denote by N_p.

This representation of the problem seems to be reasonably natural as we hope to show with the following three elementary examples from economics. In later sections these examples will be used to illustrate our results.

1. Standard exchange economy: Assume that an economy has a fixed quantity of each of c commodities and that there are a agents. Each agent's holdings of these commodities can be represented by a vector in the positive orthant of $\mathbb{E}^c$, i.e. $\mathbb{E}^c_+$. Thus the commodity bundle for all agents is $M = (\mathbb{E}^c_+)^a = \mathbb{E}^c_+ \times \cdots \times \mathbb{E}^c_+$. The dimension of M is ca. The constraint fixing the total amount

of goods in this productionless economy restricts the feasibility set to be $\Omega = M \cap \{(\underline{x}_1,\ldots,\underline{x}_a) | \underline{x}_i \epsilon \mathbb{E}^c, \sum_{i=1}^{a} \underline{x}_i = \underline{b}\}$ where $\underline{b} \epsilon \mathring{\mathbb{E}}^c$ represents the total amount of goods. This is a submanifold (with boundary) of dimension $ca - c = c(a - 1)$.

Since Ω is defined implicitly by c equations, the normal bundle N_p for any $p \epsilon \mathring{\Omega}$ is given by the gradient of the defining equations. A simple computation show that

$$N_p = \{(\underline{q},\underline{q},\ldots,\underline{q}) | \underline{q} \epsilon \mathbb{E}^c\}.$$

The ith agent has a pay-off or utility function u_i which is a smooth function from Ω to $\mathbb{E}$. Assume that this function depends only upon the ith agent's holdings. That is, assume the existence of functions $\overline{u}_i\colon \mathbb{E}_+^c \to \mathbb{E}$ such that $u_i(\underline{x}_1,\ldots,\underline{x}_i,\ldots,\underline{x}_a) = \overline{u}_i(\underline{x}_i)$ for all $\underline{x} \epsilon M$ and $i = 1,2,\ldots,a$. Thus, $\nabla u_i = (\underline{0},\underline{0},\ldots,\nabla\overline{u}_i,\underline{0})$.

2. Externalities:

a. Classical. Example 1 can be extended to a externality problem by dropping the requirement that $u_i(\underline{x}_1,\ldots,\underline{x}_i,\ldots,\underline{x}_a) = \overline{u}_i(\underline{x}_i)$. In other words, the ith agent's utility function depends not only upon his holdings, but also upon selective holdings of other agents. Traditionally, these have been interpreted as having either positive or negative effects upon the agent. For example, the heat in a neighbor's apartment may keep mine at a more comfortable temperture, but another neighbor's pollutant is not welcome. The only notational change from example one is the form of ∇u_i.

b. Location. This example, which typlifies location type problems, is due to W. Khilstrom and H. Sonnenschein, and it was communicated to me by H. Sonnenschein. Assume there is a lake with a public beach at one end. The beach is so attractive that two different families would like to camp on the lake shore as close as possible to the public beach. However each family dislikes the other - enough so that it will influence where they will camp. Representing the lake as S^1, and each family's preference function as $u_i(\theta_1, \theta_2)$ where θ_i is the location of the ith family, we see that Ω is $S^1 \times S^1$, or a torus, and $M = \Omega$. However, if we admit that possible (but not necessarily feasible) locations for one, or both families are any where on $\mathbb{R}^2$, then M becomes either $\mathbb{R}^3$ or $\mathbb{R}^4$. If $M = \Omega$, then the normal bundle consists of $\{0\}$. If $M = \mathbb{R}^3$, then the normal bundle N_p is the linear space generated by the normal vector to Ω at p. A similar representation holds if $M = \mathbb{R}^4$, except that the normal bundle is now two dimensional.

3. Public Goods. Assume an economy with a agents has a fixed amount of c private goods and d public goods. Let $\mathscr{U}_{\underline{b}} = \{\underline{x} \epsilon \mathbb{R}^c_+)$ all components of $\underline{x}$ are less than or equal to the corresponding component of $\underline{b}\}$. Let f be a smooth regular function from $\mathbb{R}^c_+$ into $\mathbb{R}^d_+$. The purpose of f is to describe a conversion rule whereby private goods are converted into public goods. Let $\underline{d}^*$ represent the initial level of public goods.

Assuming that $\underline{b}$ gives the total amount of private goods, then all possible levels of public goods are given by $\mathscr{D} = \underline{d}^* + f(\mathscr{U}_{\underline{b}})$, which is a manifold with boundary in $\mathbb{R}^d_+$. If $\underline{d}' \epsilon \mathscr{D}$, then all possible representations of private goods is given by $\Omega_{\underline{d}'} = \{(\underline{x}_1, \ldots, \underline{x}_a) \epsilon (\mathbb{R}^c_+)^a | \Sigma \underline{x}_i \epsilon f^{-1}(\underline{d}' - \underline{d}^*)\}$ (Since f is regular, $f^{-1}(\underline{d}' - \underline{d}^*)$ is a smooth manifold.)

So, our total allocation space, or feasibility space Ω, is represented by the disjoint union $\bigcup_{\underline{d}'\in\mathscr{D}} \Omega_{\underline{d}'}$. This can be given the structure of a manifold with boundary, and it is a ca dimensional submanifold of $\mathbb{R}_+^{ca} \times \mathbb{R}_+^d = M$. This dimension statement follows from the fact that $\mathscr{D}$ is d dimensional, $f^{-1}(\underline{d}' - \underline{d}^*)$ is a c - d dimensional submanifold (by the implicit function theorem), and $\Sigma\underline{x}_i = \underline{e}'\epsilon f^{-1}(\underline{d}' - \underline{d}^*)$ is a c(a - 1) dimensional submanifold. Thus $\mathring{\Omega}$ is a (ca - c) + (c - d) + d = ca dimensional submanifold of M.

Submanifold Ω can be represented implicitly as $\underline{d}' - f(\Sigma\underline{x}_i) = 0$ for $\underline{d}'\in\mathscr{D}$. Thus, for $p\in\mathring{\Omega}$, the normal bundle is generated by the d equations $(-\nabla f_i(\Sigma x_j),\ldots,-\nabla f_i(\Sigma\underline{x}_j),\underline{e}_i)$, $i = 1,2,\ldots,d$, where $\underline{e}_i = (0,\ldots,1,\ldots,0)$ and $f = (f_1,\ldots,f_d)$.

The ith agent's utility function u_i is a smooth mapping from M to $\mathbb{R}$. If the problem is without externalities, then there are mappings $\overline{u}_i\colon \mathbb{R}_+^c \times \mathbb{R}_+^d \to \mathbb{R}$ such that $u_i(\underline{x}_1,\ldots,\underline{x}_i,\ldots,\underline{x}_a;\ \underline{d}') = \overline{u}_i(\underline{x}_i,\underline{d}')$ for all $(x,d')\epsilon M$ and $i = 1,2,\ldots,a$. Otherwise, such a representation does not hold.

Other examples, including a Debreu model for production [2] could be included, but the above suffices for our purposes of illustrating the results.

§3. Necessary conditions for a LPO.

A LPO can be interpreted as a maximum point for u_i subject to the fixed constraints defining Ω and the variable constraints of u_j, $j \neq i$. This interpretation suggests that at a LPO some combination of the utility functions behaves like a maximum point on Ω; namely, some combination of the gradients of the components is either zero or normal to Ω. Our first theorem states this is precisely what occurs.

Theorem 1. *If* $p \in \mathring{\Omega}$ *is a* LPO, *then the following hold*.

a) *If* $m - n \geq a$, *then* p *is a critical point for* $u\colon \Omega \to \mathbb{E}^a$,

b) *If* $\nabla u_i(p) \neq 0$ *for* $i = 1,2,\ldots,a$, *then there exists non-zero* $\mu = (\mu_1,\ldots,\mu_a) \in \mathbb{E}^a_+$ *such that*

3.1) $$\Sigma \mu_i \nabla u_i(p) \in N_p$$

It turns out that in order to analyze the setting when p is a LPO but $\nabla u_i(p) = 0$ for some i, we must divide the problem into two cases. The first is when $\nabla u_i(p)$ *must* be zero. Here the characterization is the same as part b of the above statement. The second is when $\nabla u_i(p)$ *may* be equal to zero. In this case the analysis is somewhat different. Both of these cases will be considered in the next section.

Proof. a) Let p be a LPO in the interior of $\mathring{\Omega}$, and let $\alpha(t)$ be any smooth curve from $(-\epsilon,\epsilon)$ to Ω where $\alpha(0) = p$. Then $u_i(\alpha(t)) - u_i(\alpha(0))$ cannot be positive for all choices of index i.

Consequently, $u_i(\alpha(t)) - u_i(\alpha(0)) + O(t^2) = (\nabla u_i(\alpha(0)) \cdot \alpha'(0))t$ cannot be positive for all choices of i and small values of t. Since $\alpha(t)$ can be selected so that $\alpha'(0)$ agrees with any vector in $T\Omega_p$, the tangent space of Ω at p, it follows that $D\,u(p)(T\Omega_p) = \{D\,u(p)(\underline{v}) | \underline{v} \in T\Omega_p\}$ does not meet $\mathring{\mathbb{E}}^a_+$. That is, $D\,u\colon T\Omega_p \to \mathbb{E}^a$ is not surjective. Since $m - n = \dim T\,\Omega_p \geq a$, it follows that $D\,u(p)|T\,\Omega_p$ is not of maximal rank, or that p is a critical point for $u|\Omega$.

b) We have that $u(\alpha(t)) - u(p)$ does not meet $\mathring{\mathbb{E}}^a_+$ for sufficiently small t. Since $\nabla u_i(p) \neq 0$ for all t, this behavior can be approximated by $D\,u_p(T\Omega_p)$. Since $D\,u(p)(T\Omega_p)$ is a linear space not meeting $\mathring{\mathbb{E}}^a_+$, it has a non-zero normal vector $\underline{\mu} \in \mathbb{E}^a_+$. By use of standard arguments in linear algebra, this means that $\sum_{i=1}^{a} \mu_i \nabla u_i(p) \in N_p$. (Either act upon $\underline{\mu}$ with the adjoint of matrix $D\,u(p)$, or take the dot product of $\underline{\mu}$ and $D\,u(p)(\underline{h})$ for arbitrary $\underline{h} \in T\Omega_p$.) This completes the proof.

An immediate corollary is the well-known representation of a pareto point for a standard exchange economy where satiation points for the utility functions are not admitted.

Corollary 1.1. <u>In a pure exchange economy</u> (Example 1) <u>assume that</u> $\nabla u_i \neq \underline{0}$ <u>on</u> $\mathring{\Omega}$ for all $i = 1,2,\ldots,a$. <u>If</u> $p \in \mathring{\Omega}$ <u>is a</u> LPO, <u>then there exists</u> $\underline{\lambda} = (\lambda_1,\ldots,\lambda_a) \in \mathring{\mathbb{E}}^a_+$ <u>such that</u> $\lambda_i \nabla \bar{u}_i(p) = \lambda_j \nabla \bar{u}_j(p)$ <u>for all</u> i,j.

Proof. The assumption of no externalities implies that $\nabla u_i(p) = (\underline{0},\underline{0},\ldots,\nabla\bar{u}_i(p),\underline{0},\ldots,\underline{0})$. According to the theorem and the definition of N_p (Section 2), it follows that there exists $\underline{q}\epsilon\mathbb{R}^c$ such that

$$(\lambda_1 \nabla\bar{u}_1(p), \lambda_2 \nabla\bar{u}_2(p), \ldots, \lambda_a \nabla\bar{u}_a(p)) = (\underline{q}, \ldots, \underline{q}).$$

Since some $\lambda_i > 0$, all $\lambda_i \neq 0$, or $\underline{\lambda}\epsilon\mathring{\mathbb{R}}^a_+$. The conclusion now follows.

<u>Example 2a</u>. The characterization of LPOs can change with different types of utility functions when M and Ω remain the same. To see this, consider a simple externality problem whereby the first agent's utility function depends only upon his holdings, while all other agents' utility functions depend upon their own holdings <u>and</u> the first commodity of the first agent.

Let x^i_j denote the jth agent's holdings of the ith commodity, and let $\nabla_i \bar{u}_i$ denote the gradient of the ith agent's utility function with respect to the agent's <u>own</u> holdings. With this notation it follows from Theorem 1 and the form of N_p for this problem that there exists $\underline{\lambda}\epsilon\mathring{\mathbb{R}}^a_+$ such that $\lambda_i \nabla_i \bar{u}_i(p) = \lambda_j \nabla_j \bar{u}_j(p)$ for $i,j \geq 2$;

$$\lambda_1 \frac{\partial u_i(p)}{\partial x^\alpha_1} = \lambda_j \frac{\partial u_j(p)}{\partial x^\alpha_j} \text{ for } \alpha \geq 2 \text{ and } j \geq 2; \quad \text{and}$$

$$\sum_{i=1}^{a} \lambda_i \frac{\partial u_i(p)}{\partial x^1_i} = \lambda_j \frac{\partial u_j(p)}{\partial x^1_j} \text{ for } j \geq 2.$$

A similar representation holds for more complicated externality relationships. Notice, if some agent's utility function is such that $\nabla u_k \epsilon N_p$ where ∇ is the gradient with respect to all coordinates, then some of the components of $\underline{\lambda}$ may be zero.

<u>Example 2b</u>. If $M = \Omega$ for Example 2b, then $N_p = \{0\}$. This

means that at a LPO p there exist λ_1, λ_2 non-negative, $\lambda_1^2 + \lambda_2^2 > 0$, such that $\lambda_1 \nabla u_1(p) = -\lambda_2 \nabla u_2(p)$. It may be that one of the λ's equals zero. This corresponds to the possibility that one or both of the gradients may be zero.

If $M = \mathbb{E}^3$, and u_1 depends upon r, θ_1, θ_2, while u_2 depends only upon θ_1 and θ_2, then the necessary conditions for a LPO are the existence of $(\lambda_1, \lambda_2) \in \mathbb{E}_+^2$ such that the projection of $\lambda_1 \nabla u_1(p)$ onto $T\,\Omega_p$ equals $-\lambda_2 \nabla u_2(p)$. The possibility that $\nabla u_1(p) \in N_p$ is admitted since λ_2 may be zero.

This last example illustrates the fact that several of the components of $\underline{\mu}$ may be equal to zero. In the above example $\lambda_2 = 0$ corresponds to agent 1 determining the location for <u>both</u> families. Analytically it corresponds to the case where $D\,u(p)(T\Omega_p)$ meets $\mathbb{E}_+^a - \{0\}$ but not $\mathring{\mathbb{E}}_+^a$. This phenomena is further illustrated in the following example.

Example. For Example 2a, assume that $c = 1$, $a = 3$, $u_2(\underline{x}) = x_2$, $u_3(\underline{x}) = x_3$, and $u_1(x_1, x_2, x_3) = -((q_1 - x_1)^2 + (x_2 - q_2)^2 + (q_3 - x_3)^2)$ where $q = (q_1, q_2, q_3) \notin \Omega$. If p is the orthogonal projection of q onto Ω, then p is a LPO. Indeed, it is the maximum point for $u_1|\Omega$, and $\nabla u_1(p) \in N_p$.

$$D\,u(q)(h) = \begin{pmatrix} -2(p_1-q_1) & -2(p_2-q_2) & -2(p_3-q_3) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0 \\ h_2 \\ h_3 \end{pmatrix}$$

where $h \in T\Omega_p$. (That is, $h_1 + h_2 + h_3 = 0$). Clearly, $D\,u(p)(T\Omega_p)$ meets the x_2, x_3 coordinate plane, and $\underline{\mu} = (1,0,0)$.

A LPO of this type need not always corresponds to a maximum of the utility function for some agent (e.g., change the exponents from 2 to 3 for the $(x_2 - q_2)$ term), but it does suggest that an agent, or subgroup of agents, are dictating the remainder of the agents' moves or choices. We will call this type of LPO a dictatorial LPO. Notice that a dictatorial LPO can only occur in situations which admit externalities. For the most part we shall not examine this type of point in detail; primarily because it doesn't correspond to the game theoretic notion whereby <u>all</u> players have some free will . Nevertheless, we will state how the following theorems can be modified to apply to the dictatorial LPOs.

Let A be a set of vectors. The cone defined by A is the set of vectors of the form $\sum_{i=1}^{c} \lambda_i \underline{v}_i$ where $\underline{v}_i \in A$ and the λ_i's are either all non-positive or positive.

Definition 3.1. <u>Local pareto point</u> $p \in \mathring{\Omega}$ <u>is said to be strongly dictatorial if the following hold for some non empty set of</u> indices B

a) $\sum_{i \in B} \lambda_i \nabla u_i(p)$ <u>is not in the cone spanned by</u> N_p <u>and</u> $\{\nabla u_j(p)\}_{j \in B'}$ <u>where</u> $\lambda_i \geq 0$ <u>and not all of them are equal to</u> <u>zero</u>. <u>Set</u> $B' = \{1,2,\ldots,a\} - B$.

b) <u>Let</u> $f_i(x)$ <u>be an arbitrary smooth function depending upon the same variables as</u> $u_i(x)$, $i \in B$. <u>Then</u> p <u>is a</u> LPO <u>for</u> the utility functions $\{u_j\}_{j \in B'}$, $\{f_i\}_{i \in B}$.

<u>Agent</u> $i \in B$ <u>is called ineffectual at</u> p.

It is clear that agent $i \epsilon B$ plays no role in determining p as a LPO. In the preceding example, agents 2 and 3 are ineffectual. When the exponent on the $(x_2 - q_2)$ term is changed to 3, then only agent 3 is ineffectual.

It turns out that a strongly dictatorial point is too strong for our first order analysis. Thus we give a weaker definition.

Definition 3.2. Local pareto point $p \epsilon \mathring{\Omega}$ is said to be first order dictatorial if the following hold for some non-empty set of indices B

a) $\sum_{i \epsilon B} \lambda_i \nabla u_i(p)$ is not in the cone spanned by N_p and $\{\nabla u_j(p)\}_{j \epsilon B'}$ where $\lambda_i \geq 0$ and not all of them are equal to zero.

b) Let $A_i = (A_1, \ldots, A_n)$ be an arbitrary vector where the index of any non-zero component agrees with the index of some non-zero component of $\nabla u_i(p)$, $i \epsilon B$. Then there exists some $\mu \epsilon \mathbb{E}^a_+$ such that the system $\nabla u_1(p), \nabla u_2(p), \ldots, \underline{A_i}, \ldots, \nabla u_a(p)$ satisfies Equation 3.1 where the A_i's replace the $\nabla u_i(p)$'s.

Proposition 3.1a. A strongly dictatorial LPO $p \epsilon \mathring{\Omega}$ is a first order dictatorial point.

b. LPO $p \epsilon \mathring{\Omega}$ is a first order dictatorial point if and only if there does not exist $\mu \epsilon \mathring{\mathbb{E}}^a_+$ which satisfies Equation 3.1.

c. If $T\Omega_p \cap (\bigcap_{i \epsilon B'} \mathrm{Ker} Du_i(p)) = \{\underline{0}\}$, where B' is the complement of the set of indices of the first order ineffectual agents, then

<u>the first order dictatorial point is a strong dictatorial point</u>.

For most problems the third statement is useless. This is because it requires $m - n \leq \text{card } B' < a$, which cannot be satisfied for any of the examples in Section 2 unless $c = 1$. To see that this inequality holds, notice that for any line on $T\Omega_p$ passing through $\underline{0}$, there is $\nabla u_i(p)$, for some $i \in B'$, which is <u>not</u> in the orthogonal complement of the line. This means that $\{\nabla u_i(p)\}_{i \in B'}$ spans a $m - n$ dimensional space, which in turn implies that $m - n \leq \text{card } B'$. There are some elementary second order conditions which greatly relax this inequality.

The purpose of condition a is to ensure that the agents in B cannot be incorporated into the decision process. The first order dictatorial point should be viewed as those LPOs in $\mathring{\Omega}$ for which the <u>first order</u> preferences of some set of agents are ignored in the processes of selection. Indeed, these first order preferences may blocked by higher order desires of other agents. This is illustrated by the following example which, incidently, shows that a first order dictatorial LPO need not be a strong dictatorial LPO.

For Example 2a, let $c = a = 2$,

$$u_1(x) = -(x_1^1 - q_1^1)^2 - (x_2^1 - q_2^1)^2 - (x_1^2 - p_1^2)^2 + (x_2^2 - p_2^2)^2,$$

$$u_2(x) = x_2^1 + x_1^2 - (x_2^2 - p_2^2)^2$$

where $p = (p_1^1, p_1^2, p_2^1, p_2^2)$ is the projection of $(q_1^1, p_1^2, q_2^1, p_2^2)$ onto $\mathring{\Omega}$, and $0 \neq (q_1^1 - p_1^1) = (q_2^1 - p_2^1)$.

$B = \{2\}$ and p is a first order dictatorial LPO. However

p is not a strong dictatorial LPO as we see by choosing $f_2(x) = (x_2^2 - p_2^2)^2$. Notice that f_2 was selected to correspond to a direction in Ker D u_p where D^2u_1 is positive definite.

Since we are primarily interested in a first order theory, we will drop the modifier "first order".

Proof. a. Immediate.

b. Any $\underline{\mu}\epsilon\mathbb{E}_+^a$ which satisfies Equation 3.1 is a normal vector to $Du_p\ (T\Omega_p)$. Thus if there does not exist $\underline{\mu}\epsilon\mathring{\mathbb{E}}_+^a$ satisfying Equation 3.1, then the normal space is in $\mathbb{E}_+^a - \mathring{\mathbb{E}}_+^a$. This means for some non-empty set of indices B if index $i\epsilon B$, then the ith component of any $\underline{\mu}$ which satisfies Equation 3.1 must be zero. Without loss of generality assume that $B = \{b, b + 1, \ldots, a\}$. This means that if we replace $\nabla u_i(p)$ with $\underline{A}_i$, $i\epsilon B$, then $\sum^{k-1} \mu_i \nabla u_i(p) + \sum_b^a 0 \cdot \underline{A}_i = \sum^{b-1} \mu_i \nabla u_i(p) = \sum^a \mu_i \nabla u_i(p) \epsilon N_p$. Also no linear combination $\sum_{i=b}^{a} \lambda_i \nabla u_i(p)$ where $\lambda_i \geq 0$, but not all equal zero, is in the cone spanned by N_p and $\{\nabla u_i(p)\}_{i\epsilon B'}$ for otherwise there would exist some normal vector to $D\ u_p(T\Omega_p)$ with non-zero components with indices in B. This would be a contradiction. This is so since by construction there exists $\underline{\mu}\epsilon\mathring{\mathbb{E}}_+^{b-1}$ such that $\sum^{b-1} \mu_i \nabla u_i(p) \epsilon N_p$.

If $p\epsilon\mathring{\Omega}$ is a dictatorial LPO, then we can assume without loss of generality that set $B = \{b, \ldots, a\}$ identifies the ineffectual agents. Assume the conclusion is false, that is there is $\underline{\lambda}\epsilon\mathring{\mathbb{E}}_+^a$ which is in the normal space to $Du_p\ (T\Omega_p)$. This means that $\sum_b^a \lambda_i \nabla u_i(p)$ is an element of the cone spanned by N_p and $\{\nabla\ u_i(p)\}_{i\epsilon B'}$. This contradiction completes the proof.

c. Due to the assumption, for any non-zero $\underline{h}\epsilon T\Omega_p$ there is some $i \in B'$ such that $\nabla u_i(p) \cdot \underline{h} < 0$. This means that for any point q sufficiently close to p, there is some $i \in B'$ such that $u_i(q) < u_i(p)$. That is, the pareto optimality of p is determined by u_i, $i \in B'$. Consequently, p remains a LPO independent of the choice of $\{u_i(x)\}$, $i \in B$.

Although we are considering non-dictatorial LPOs, the agents can be subdivided in a fashion which resembles coalitions.

Theorem 2. Let $p \in \mathring{\Omega}$ be a non-dictatorial LPO such that $\nabla u_i(p) \neq \underline{0}$ for all i. If the codimension of $Du_p\ (T\Omega_p)$ is k, then there exist

i) k non-empty subsets $S_1, S_2, \ldots, S_k$ such that
$$\bigcup_{i=1}^{k} S_i = \{1,2,\ldots,a\},$$ but S_i is not a subset of
$$\bigcup_{\substack{j=1 \\ j \neq i}}^{k} S_j, \quad i = 1,2,\ldots,k.$$

and

ii) k unique linearly independent unit vectors $\underline{\mu}^1, \ldots, \underline{\mu}^k \in \mathbb{E}_+^a$ where the subscripts of the non-zero components of $\underline{\mu}^i$ correspond to the elements of S_i;

such that

$$\sum_{j \in S_i} \mu_j^i \nabla u_j(p) \in N_p, \quad i = 1,2,\ldots,k. \tag{3.2}$$

Conversely, let $p \in \mathring{\Omega}$ and $\nabla u_i(p) \neq 0$ for all i. Assume there exist k vectors $\underline{\mu}^1, \ldots, \underline{\mu}^k \in \mathbb{E}_+^a$ such that both Equation 3.2 and condition i are satisfied if and only if $\mu_j^i \neq 0$, where $j \in S_i$. Then the codimension of $Du_p\ (T\Omega_p)$ is at least k. If, in addition, these conditions cannot be satisfied for any choice of k + 1 <u>vectors with their corresponding sets</u> S_i, <u>then the codimension of</u> $Du_p\ (T\Omega_p)$ <u>is equal to</u> k. <u>In either case, should</u> p <u>be a</u> LPO, <u>it is non-dictatorial</u>.

Let $\Theta_u = \{p\epsilon\mathring{\Omega} \mid \exists\ \underline{\mu}\epsilon\mathring{\mathbb{R}}^a_+$ with the property that $\Sigma\ \mu_i \nabla u_i(p)\epsilon N_p\}$. According to this theorem and Proposition 3.11 set Θ_u is the set of points in $\mathring{\Omega}$ which satisfy the first order necessary conditions for a non-dictatorial LPO. In a later section we shall characterize the generic structure of this set.

If the subsets S_i are pair wise disjoint, then we call the partition of agents "coalition-like". We shall show by means of an example, which follows the proof of this theorem, that not all divisions of the agents need to be coalition-like. Indeed, as we shall show in Section 6, the non coalition-like case is more general.

A similar partition theorem holds for dictatorial LPOs except that some of the vectors $\underline{\mu}^j$ admit zero entries which correspond to the "ineffectual" agents in S_j.

Proof. The codimension of $Du_p\ (T\Omega_p)$ is k if and only if the normal space at the origin is k dimensional. Since p is non-dictatorial, this normal space meets the interior of $\mathbb{R}^a_+$. If k = 1, the unit vector $\underline{\mu}'$ is uniquely defined and $S_1 = \{1,2,\ldots,a\}$.

If k > 1, then the normal space has a basis of k linearly independent unit vectors. Since p is non-dictatorial, $Du_p\ (T\Omega_p)$ does not meet $\mathbb{R}^a_+ - \{\underline{0}\}$, so all the basis vectors, $\underline{\alpha}^1, \underline{\alpha}^2, \ldots, \underline{\alpha}^k$, can be selected in $\mathring{\mathbb{R}}^a_+$. Let $\underline{\alpha}^i$ be the ith row of k × n matrix A. Use the Gaussian elimination to determine A_k, the row echelon form of A.

Let $\underline{\mu}^1, \ldots, \underline{\mu}^k$ be the unit vectors obtained by normalizing the row vectors of A_k, and let S_i be the set of indices corresponding to the non-zero entries of vector $\underline{\mu}^i$, $i = 1,2,\ldots,k$. According to the properties of the row echelon form of a matrix, the S_i's satisfy the set theoretic property asserted in the theorem. Geometrically the S_i's identify coordinate planes of $\mathbb{E}^a_+$ which have a one-dimensional intersection with the normal plane, and the $\underline{\mu}^i$'s define the directions of these lines. Since this normal plane passes through the interior of $\mathbb{E}^a_+$, all the components of $\underline{\mu}^i$ are non-negative. The uniqueness assertion follows from the uniqueness of a row echelon form. It follows immediately that $\sum_{j\in S_i} \mu^i_j \nabla u_j(p) \in N_p$.

The converse is almost immediate. Since the vectors $\underline{\mu}^i$ satisfy Equation 3.2, they are in the normal space of $Du_p\ (T\Omega_p)$. It only remains to show that they are linearly independent. This follows from the set theoretic condition on the sets S_i, which states that each set S_i has some element, say $\ell(i)$, which is not contained in S_j, $j \neq i$. Therefore, the only way the $\ell(i)$th component of $\Sigma\, \gamma_j \underline{\mu}^j = \underline{0}$ can be zero is if $\gamma_i = 0$. This proves the assertion.

Since $\underline{\mu}^j \in \mathbb{E}^a_+$, and since $\cup S_i = \{1,2,\ldots,a\}$, it follows that $\Sigma\underline{\mu}^j \in \mathring{\mathbb{E}}^a_+$. This shows that p is non-dictatorial.

The set theoretic condition implies that constant multiples of the $\underline{\mu}^i$'s define the rows of a matrix which is in row echelon form for some relabeling of the axes of $\mathbb{E}^a$. More precisely, relabel the coordinate axes in the following way. Relabel the $\varkappa(1)$ unique elements of S_1 as $1,2,\ldots,\varkappa(1)$, the $\varkappa(2)$ unique elements of S_2 as $\varkappa(1) + 1,\ldots,\varkappa(1) + \varkappa(2)$, etc., and the remaining elements as $\sum_{i=1}^{k} \varkappa(i) + 1,\ldots,n$. In this relabelled coordinate system, multiply

$\underline{\mu}^i$ by the multiplicative inverse of its first non-zero component. It is clear that the new vectors form the row vectors of a matrix in a row echelon form.

In light of the preceding paragraph, the second part of the converse asserts that any matrix describing the normal space to $D\, u_p(T\Omega_p)$ has less than $k + 1$ non-zero rows in its row echelon form. Thus, the normal space is of dimension k. This completes the proof.

As we stated, non coalition-like divisions of the agents exist. To see this, consider Example 2a, where $c = 3$, $a = 2$, $\underline{b} = (4,4)$, and

$$u_1(x) = x_1^1 - (x_1^2 - 2)^2 + x_2^1 - x_3^2,$$

$$u_2(x) = x_1^2 + x_2^2 - x_3^1 - (x_2^1 - 1)^2,$$

$$u_3(x) = x^1 + x_3^2.$$

Then $p = (1,2;\ 1,1;\ 2,1)$ is a LPO, and $\nabla u_1(p) = (1,0,1,0,0,-1)$, $\nabla u_2(p) = (0,1,0,1,-1,0)$, $\nabla u_3(p) = (0,0,0,0,1,1)$. Also, $\nabla u_1 + \nabla u_3$, $\nabla u_2 + \nabla u_3 \epsilon N_p$. Therefore, $S_1 = (1,3)$, $\underline{\mu}' = (1,0,1)$; $S_2 = (2,3)$, $\underline{\mu}^2 = (0,1,1)$, and a simple computation shows that $Du_p\ (T\Omega_p)$ is the line $t(1,1,-1)$ where $t \epsilon \mathbb{R}$.

We have seen how the structure of Ω can effect the characterization of a LPO. Another aspect of this interaction between Ω and u is that a change in the structure of the feasible set Ω can sharply alter the location of possible LPOs. This is, of course, due to the concommitant change in the structure of form of N. Such a change in the structure of Ω may be the result of a taxation, or a subsidy, to discourage of encourage certain behavior. Geometrically, this is a change in the form of the normal space N which imposes a restriction on the location of set Θ_u. In a private goods, pure exchange economy (Example 1 or 2a), this could be represented by replacing the fixed quantity $\underline{b}$ with a smooth mapping $\underline{b}$: $\mathbb{R}_+^c \to \mathbb{R}_+^c$, or by the mapping $\underline{b}$: $(\mathbb{R}_+^c)^a \to \mathbb{R}_+^c$. The former reflects a commodity centered approach which is independent of the agents, while the latter allows for adjustments depending upon the agent. Other representations are, of course, possible.

This geometric approach has the advantage that once the restrictions on the locations of Θ_u have been stated, the tax can be determined by choosing an appropriate N. This choice leads to some differential equations and the solution of these equations is the tax. An extreme but simple example of how this works follows.

Assume for Example 1 that c = 2, a = 2, and that both agents desire the first commodity, i.e. $\frac{\partial \bar{u}_1}{\partial x_1^1} > 0$, $\frac{\partial \bar{u}_2}{\partial x_2^1} > 0$ for all $p \in \Omega$.

Furthermore, assume it is desired that both agents have at least some of the second commodity, which is, say, spinach. That is, we wish to keep Θ_u away from the boundary of Ω given by $x_1^2 = 0$, or $x_2^2 = 0$. One way of doing this is to penalize the holding of the first commodity in the region of this boundary. For example, should the normal space be the space spanned by $(\gamma_1, \gamma_2; 0, \gamma_4)$ and $(0,1; 0,1)$, then there cannot be a LPO in this region. This is because $\frac{\partial \bar{u}_2}{\partial x_2^1} > 0$.

The basis for the normal space for this new Ω is given by $\nabla(\Sigma x_i^j - b_j(x))$; or for j = 1, by

$(1,0;\ 1,0) - (\frac{\partial b_1}{\partial x_1^1}, \frac{\partial b_1}{\partial x_1^2}; \frac{\partial b_1}{\partial x_2^1}, \frac{\partial b_1}{\partial x_2^2})$. The stated goal can be achieved should for some $\varepsilon > 0$, we have

$$\frac{\partial b_1}{\partial x_j^1} = \begin{cases} 1 & \text{if} \quad 0 \le x_j^2 \le \epsilon/5 \\ 0 & \text{if} \quad x_j^2 \ge \epsilon \end{cases} \qquad j = 1,2.$$

Let $c(y)$ be any smooth function such that

$$c(y) = \begin{cases} 1 & 0 \le y \le \epsilon/4 \\ 0 & y \ge 3\epsilon/4. \end{cases}$$

A solution would be

$$b_2 = \text{constant}, \ b_1 = \text{constant} + x_2^1\, c(x_2^2) + x_1^1\, c(x_1^2).$$

For this solution, in an $\epsilon/4$ region of the stated boundaries, the normal space is given by basis vectors $(1,0;\ 0,0)$ and $(0,1;\ 0,1)$, or $(0,0;\ 1,0)$ and $(0,1;\ 0,1)$. The tax, actually in this case it is a subsidy, is separable in the sense that b_1 is a constant plus a sum of functions each of which depends upon only the holdings of one agent.

A third type of "taxation" which leads to interesting geometric properties of Ω, would be where goods are either added or subtracted from an agent's holdings in certain regions $(\mathbb{E}_+^c)^q$. In a non separable setting this could lead to a tax law which cannot be represented as in function in $\mathbb{E}^c$.

Theorem 2 gives first order necessary conditions for a point $p \in \mathring{\Omega}$ to be a LPO. These conditions are by no means sufficient. While it is not the purpose of the current paper to discuss sufficient conditions for $p \in \mathring{\Omega}$ to be a LPO, we will conclude this section by giving an elementary one. We do so to further illustrate the role of the μ_i's and to motivate some examples we will discuss in a later section of this paper.

Theorem 3. <u>For some utility function</u> u, <u>let</u> $p \epsilon \Theta_u$. <u>Furthermore, let</u> $\underline{\lambda} \epsilon \mathring{\mathbb{E}}^a_+$ <u>be such that</u> $\Sigma \lambda_i \nabla u_i(p) \epsilon N_p$. <u>If</u> $W = \Sigma \lambda_i u_i$ <u>has a local weak maximum at</u> p, <u>then</u> p <u>is a</u> LPO.

If p is dictatorial, then $\underline{\lambda} \epsilon \mathbb{E}^a_+$ and p must be a strict local maximum point for W.

Proof. If p is a local maximum point for W, then for all q in a sufficiently small neighborhood of p we have $0 \geq W(q) - W(p) = \underline{\lambda} \cdot (\underline{u}(q) - \underline{u}(p))$, where $\underline{u} = (u_1, \ldots, u_a)$. That is, unless $\underline{u}(q) = \underline{u}(p)$, the angle between vectors $\underline{\lambda}$ and $(\underline{u}(q) - \underline{u}(p))$ is greater than $\pi/2$. Since $\underline{\lambda}$ is in $\mathring{\mathbb{E}}^a_+$, $(\underline{u}(q) - \underline{u}(p))$ is not in $\mathbb{E}^a_+$, which in turn implies that p is a LPO. This completes the proof.

These sufficient conditions can be improved. All we need for p to be a LPO is that the angle formed by $\underline{\lambda}$ and $(\underline{u}(q) - \underline{u}(p))$ is greater than ω where ω is the maximum of the angles formed by $\underline{\lambda}$ and the coordinate axis of $\mathbb{E}^a_+$. Clearly $\omega \leq \pi/2$.

An important point is that this theorem holds for any choice of $\underline{\lambda}$ with the stated property. In cases where $D\,u(T\Omega_p)$ has co-dimension greater than 1, this provides some flexibility. Indeed, it is a relatively simple matter to construct an example from Example 2a, for which the conditions are satisfied for some choices of an admissible $\underline{\lambda}$, but not for others.

This theorem has several obvious corollaries. For example, the usual second order conditions for a maximum of a function on a manifold will translate into a second order condition for u. The one we will use later is the second order condition for the standard exchange economy and for Example 2a; namely, if $p \epsilon \Theta_u$, $\underline{\lambda}$ the corresponding vector in $\mathring{\mathbb{R}}^a_+$, and if $\Sigma\, \lambda_i D^2 u_i(p)$ is negative definite, then p is a LPO. For more general choices of Ω, Morse Theory [8] provides other second order conditions.

4. Vanishing gradients

In our first order analysis of a LPO, we imposed the condition that $\nabla u_i(p) \neq 0$ for $i = 1,2,\ldots,a$. This restriction excludes some interesting problems such as satiation points, and it will influence our characterization of the generic structure of Θ_u. Furthermore, as we shall see, this condition may introduce a bias concerning the possible locations of LPOs. Namely, there are choices of Ω where $\nabla u_i(p)$ must be zero for certain p to be non-dictatorial LPOs. In this section we shall characterize these two types of LPOs.

The characterization of the first type of LPO, where $\nabla u_i(p) = 0$, is an extension of the basic idea used in the preceding section. In the preceding section we showed that if $\underline{h} \in T\Omega_p$ was such that $\nabla u_i(p)\cdot\underline{h} > 0$, then there exists j such that $\nabla u_j(p)\cdot\underline{h} < 0$. But the normal hyperplane of each gradient $\nabla u_i(p)$ divides TM_p into an open half space which includes the gradient, and a closed half space which includes the negative of the gradient. Of course, this closed half-space consists of the vectors $\underline{h}$ such that $\nabla u_i(p)\cdot\underline{h} \leq 0$. Therefore, the above condition implies that each $\underline{h}$ is in the closed half space of $\nabla u_i(p)$ for some choice of i. In other words, $T\Omega_p$ is contained in the union of these half spaces. If this union equals TM_p, then there exist $\lambda_i \geq 0$, not all equal to zero, such that $\Sigma\, \lambda_i \nabla u_i(p) = \underline{0}$. Otherwise this summation is in $N_p - \{\underline{0}\}$.

Suppose $\nabla u_i(p) = 0$ for $i \in C$ where $p \in \mathring{\Omega}$ is a LPO. Let C' be the complement of set C. Assume the union of the closed half spaces of $\nabla u_j(p)$, $j \in C'$, contains $T\Omega_p$. Then a second order analysis of $u_i(p)$, $i \in C$, needs to be performed only for $\underline{h} \in \left(\bigcap_{j \in C'} \operatorname{Ker} Du_p\right) \cap T\Omega_p$; that is, for $\underline{h}$ in $T\Omega_p$ and all the hyperplanes. On the other hand, should non-zero $\underline{h} \epsilon T\Omega_p$ be in the intersection of the open half spaces defined by ∇u_j, $j \epsilon C'$, then $D^2 u_i(p)(\underline{h},\underline{h}) < 0$ for some $i \epsilon C$. This approach emphasizes the second order behavior of u_i, $i \epsilon C$, in terms of the cone defined by ∇u_j, $j \epsilon C'$. Such an analysis is possible and preferred for such problems as finding sufficient conditions for a point p to be strongly dictatorial, but this approach runs counter to our theme of a first order analysis. The approach we wish to take is to emphasize the first order theory and find the minimal requirements for $\nabla u_j(p)$, $j \epsilon C'$. We do this by reversing the above analysis. That is, we first determine all $\underline{h}$ such that $Du_i(p)(\underline{h},\underline{h})$ is negative definite for some $i \epsilon C$. All other vectors in $T\Omega_p$ must be in the union of the closed half planes of ∇u_j, $j \epsilon C'$.

Assume that $\nabla u_i(p) = 0$. Recall that $D^2 u_i(p)$ is a symmetric bilinear functional on TM_p. Let $W_i = \{\underline{h} \epsilon TM_p | D^2 u_i(p)(\underline{h},\underline{h}) < 0\}$, and let $K_i = \{\underline{v} \epsilon TM_p | D^2 u_i(p)(\underline{v},\underline{h}) = 0 \text{ for all } \underline{h} \epsilon TM_p\}$, the null space of $D^2 u_i(p)$. For set of indices B, let $L_{B,p} = \{\underline{h} \epsilon TM_p | \underline{h}$ is in the space spanned by the tangent vectors for those coordinate axis which are represented as variables in $u_i(x)$ for some $i \epsilon B\}$.

Theorem 4. <u>Assume that</u> $p \epsilon \mathring{\Omega}$ <u>is a</u> LPO, <u>and that</u> $\nabla u_i(p) = 0$ <u>if and only if</u> $i \epsilon C$. <u>Let</u> $K = (\bigcup_{i \epsilon C} K_i) \cap L_{C,p}$, <u>and let</u> W <u>be any subspace contained in</u> $TM_p - (\overline{\cup W_i} \cup K)$. <u>Then there exist non-negative</u> μ_i, <u>not all zero, such that</u> $\sum_{i \epsilon C'} \mu_i \nabla u_i(p)$ <u>is in the orthogonal complement of</u> $T\Omega_p \cap W$ <u>in</u> TM_p.

Sets K and $\overline{\cup W_i} - W_i$ determine the directions where a higher order analysis may apply. To avoid a higher order analysis, we can require $K = \{\underline{0}\}$. Since we are only interested in necessary conditions, the second set can be ignored. Notice that according to this theorem, if $C' = \emptyset$, then $W = \{\underline{0}\}$.

Proof. Following the proof of Theorem 1, let $\alpha(t)$ be an arbitrary smooth curve on Ω such that $\alpha(0) = p$. Then for each index i, $u_i(\alpha(t)) - u_i(p) = \nabla u_i(p)\ \alpha'(0) + \{D^2u_i(p)(\alpha'(0), \alpha'(0)) + \nabla u_i(p)\cdot\alpha''(0)\}t^2/2! + O(t^3)$. If $i\in C$, then the right hand side reduces to $D^2u_i(p)(\alpha'(0),\alpha'(0))$. Should $\alpha'(0)\in UW_i$, then there is $i\in C$ such that $u_i(\alpha\ (t)) < u_i(p)$. Consequently, for small values of t, $\alpha(t)$ is not at a preferred point to p. On the other hand, if for sufficient small values of t we have $u_i(\alpha(t)) \geq u_i(p)$ for all $i\in C$, then $D^2u_i(p)(\alpha'(0),\alpha'(0)) \geq 0$ for $i\in C$. This implies $h = \alpha'(0)\notin UW_i$.

Since p is a LPO, there exists $j\in C'$ such that $u_j(\alpha(t)) < u_j(p)$. Using an analysis similar to that of Theorem 1, it follows that Du_p maps this subset of $T\Omega_p$ into $\mathbb{R}^d \times \underline{0}$ which misses $\mathring{\mathbb{R}}^d_+ \times \underline{0}$, where d is the cardinality of set C'. It remains to identify the vectors $\underline{h}$ in this subset.

As shown above, $\underline{h}$ need not be in UW_i. Furthermore, $\underline{h}$ need not be in K since the pareto optimality of points in this direction may be determined by higher order terms in an expansion of $u_i(x)$ for some $i\in C$. For the same reason, $\underline{h}$ need not be in $\overline{UW_i} - W_i$. This exhausts the directions for which the pareto optimality of p may be determined by higher order terms. Thus, this subset must contain $T\Omega_p - \{\overline{UW_i} \cup K\}$. But $T\Omega_p - \{\overline{UW_i} \cup K\}$ contains $T\Omega_p \cap W$. Because $T\Omega_p \cap W$ is a linear space, $D\ u_p(T\Omega_p \cap W)$ is a linear space which misses $\mathring{\mathbb{R}}^d_+ \times \underline{0}$. Thus there exists a non-zero vector $\underline{\mu}\in\mathbb{R}^d$ such that $\sum_{i\in C'} \mu_i \nabla u_i(p)$ is orthogonal to $T\Omega_p \cap W$.

This completes the proof.

As indicated in the above proof, $\{(\overline{UW_i} - W_i) \cup K\}$ identifies those variables where a higher order analysis is required to obtain a sharper characterization of LPO p. For example, if $u_1(x) = -(x_1^1 - p_1^1)^2 + (x_1^2 - p_1^2)^2 + \varepsilon(x_1^3 - p_1^3)^2$ where $p\in\mathring{\Omega}$ is a LPO, then $K_1 = \{\underline{h} | h_1^1 = h_1^2 = 0\}$ and K_1 is the space spanned by $(0,0,1; 0,0,\ldots)$. Should $\varepsilon < 0$, then no further analysis need be made in the h_1^3 direction. This is so because its pareto optimality is determined by the fourth order term. On the other hand, if $\varepsilon > 0$, then the pareto optimality of p, at least for the p_1^3, component, is determined by other agents. Both possibilities are admitted by the above corollary.

A similar example exhibiting the role of $\overline{UW_i} - W_i$ can be given with two agents where $a > 2$. Let

$$u_1(x) = (x_1^1 - p_1^1)^2 - (x_2^1 - p_2^1)^2 + \varepsilon\{(x_1^1 - p_1^1)^4 + (x_2^1 - p_2^1)^4\} \text{ and}$$

$$u_2(x) = -(x_1^1 - p_1^1)^2 + (x_2^1 - p_2^1)^2. \text{ Set } \overline{\bigcup_{i=1}^{2} W_i} - \bigcup_{i=1}^{2} W_i =$$

$\{\underline{h}\in TM_p | (h_1^1)^2 = (h_2^1)^2\}$. If $\varepsilon < 0$, then the pareto optimality of p in the direction of these two lines is given by agent 1. If $\varepsilon > 0$, and this set intersects $T\Omega_p$, then agents other than 1 and 2 must determine the pareto optimality of the component of this point.

It is clear that an analysis similar to that given in the proof of Theorem 2 will lead to a subdivision of the indices in C'. Also, a second order dictatorial LPO could be defined where for an ineffectual agent i, $\nabla u_i(p)$ can be replaced by any vector for which the index of a non zero component agrees with the index of some non-zero bilinear symmetric functional on TM_p which has its null space contained in the null space of $D^2u_i(p)$. We would expect that after such changes, the conclusion of Corollary 1.2 would still hold.

To sharpen the statement of Theorem 4, we would like to restrict the dimension of the orthogonal complement, which in turn means increasing the dimension of W. For example, is it always possible to choose W so that it is the space spanned by $TM_p - (\overline{\cup W_i} \cup K)$, which we will call $\mathscr{W}$? The answer is no, even if the higher order terms are eliminated, i.e., $K = \{\underline{0}\}$. To see this, consider $c = 1$, $a = 4$, $p \in \mathring{\Omega}$, $u_1(x) = 4(x_1 - p_1)^2 - (x_2 - p_2)^2 - [(x_1 - p_1)^4 + (x_2 - p_2)^4]$, $u_2(x) = 4(x_2 - p_2)^2 - (x_1 - p_1)^2$, $u_3(x) = -3x_2 - x_1 + 2x_3 - (x_3 - p_3)^2 + 2x_4$, and $u_4(x) = x_2 - x_3 - x_4 - (x_4 - p_4)^2$. A simple computation shows that p is a pareto optimum, and that the space spanned by $TM_p - (\overline{\cup W_i})$ is TM_p. Vector $\underline{h} = (h_1,h_2,h_3,h_4) \in T\Omega_p$ can be expressed as $(h_1,h_2,h_3, -(h_1 + h_2 + h_3))$. Because $\begin{pmatrix} \nabla u_3(p) \\ \nabla u_4(p) \end{pmatrix} : T\Omega_p \to \mathbb{E}^2$ is onto, we see that the theorem cannot be sharpened in this fashion.

This example may seem non-typical since $(\frac{\partial u_3}{\partial x_3}, \frac{\partial u_3}{\partial x_4})$ is a negative multiple of $(\frac{\partial u_4}{\partial x_3}, \frac{\partial u_4}{\partial x_4})$. However, by computing the normals to the linear spaces in $T\Omega_p - \overline{\bigcup_{i=1}^{2} W_i}$, it can be shown that if there does not exist nonnegative λ_1,λ_2 such that $\lambda_1 \nabla u_3(p) + \lambda_2 \nabla u_4(p) \in N_p$, then the condition given in the preceding sentence must hold. For this problem $\underline{h} \in T\Omega_p \cap W$ if $h_1 + h_2 + h_3 + h_4 = 0$ and $h_1 = \beta h_2$ where $\beta \in (-2,-1/2) \cup (1/2,2)$. The corresponding normal space is given by $(a, \beta a + (\beta - 1)c, c, c)$.

To establish that the negative multiple statement holds, we use the fact that

$$\begin{vmatrix} \frac{\partial u_3}{\partial x_1} & \frac{\partial u_4}{\partial x_1} \\ \frac{\partial u_3}{\partial x_i} & \frac{\partial u_3}{\partial x_i} \end{vmatrix} \neq 0 \quad \text{for} \quad i = 2,3,4.$$

A similar statement holds in general. Let β be a parameter in some set which parameterizes the linear subspaces of maximal dimension, W_β, which are contained in $T\Omega_p - \overline{\cup W_i}$. That is, $\cup W_\beta = T\Omega_p - \overline{\bigcup_{i \in C} W_i}$, W_β is of maximal dimension, and $\beta_1 \neq \beta_2$ implies $W_{\beta_1} \neq W_{\beta_2}$. Since W_β is a linear subspace, it can be represented as the intersection of hyperplanes. Recall $\mathscr{W}$ is the linear span of $T\Omega_p - \overline{\cup W_i}$. Assume that any W_β can be expressed in terms of the intersection of $\mathscr{W}$ and ℓ variable hyperplanes. Furthermore, let ℓ be the minimum number required.

Notice that any subset containing $T\Omega_p - \overline{\cup W_i}$ which can be expressed as the union of subspaces obtained by the intersection of $\mathscr{W}$ and $\gamma < \ell$ variable hyperplanes must properly contain $T\Omega_p - \overline{\cup W_i}$. Indeed, it contains the linear span of a subset of $T\Omega_p - \overline{\cup W_i}$.

Corollary 4.1. Assume that $p \in \mathring{\Omega}$ is a LPO such that $\nabla u_i(p) = o$ if and only if $i \in C$. Furthermore, assume $K_i \cap L_{i,p} = \{\underline{0}\}$ for $i \in C$. Let $\ell \geq 0$ be the minimum number of variable hyperplanes required to describe arbitrary W_β. Then one of the following must occur.

1. The rank of Du_p and the cardinality of C' is at least $\ell + 1$.

2. $T\Omega_p - \overline{\cup W_i}$ is properly contained in some set which can be expressed as the union of linear subspaces, each of which is the intersection of $\mathscr{W}$ and γ variable hyperplanes, $\gamma < \ell$. Furthermore, for each of these linear subspaces there exist non negative λ_i, not all zero, such that $\Sigma\, \lambda_i \nabla u_i(p)$ is in the orthogonal complement.

In other words, either we need at least ℓ agents in C', or

$\sum_{i \in C'} \lambda_i \nabla u_i(p)$ must satisfy some more stringent conditions. It follows from this corollary that the above example is the simplest non-trivial example exhibiting the combination between second and first order conditions.

Proof. Without loss of generality, let $C' = \{1,2,\ldots,d\}$ and $C = \{d+1,\ldots,a\}$. According to Theorem 4, for each $\underline{\beta}$ there is a non-zero $(\lambda_1,\ldots,\lambda_d) \in \mathbb{R}_+^d$ such that $(D\, u_p)^* \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix}$ is in the normal space corresponding to $\underline{\beta}$. Here $(D\, u_p)^*$ is the adjoint of $D\, u_p$. Trivially, there exist subspaces of $\mathbb{R}^d$ from which a solution for the above relationship can be found for each choice of $\underline{\beta}$; for example, just take the span of all solutions. Let g be the minimal dimension for subspaces with this property. Clearly $d \geq \text{rank } D\, u^* = \text{rank } D\, u \geq q$. Let $\mathscr{A}$ be the set of solutions of unit length from such a g dimensional subspace. They can be parameterized by $g - 1$ parameters $\underline{\alpha} = (\alpha_1,\ldots,\alpha_{g-1})$. That is, for each $\underline{\beta}$, there is some parameter $\underline{\alpha}$ which corresponds to a $\underline{\lambda}_\alpha$ such that $W_{\underline{\beta}}$ is in the hyperplane defined by $\sum_{i=1}^{d} \lambda_{i,\alpha} \nabla u_i(p)$. Consequently $W_{\underline{\beta}} = T\Omega_p - \overline{UW_i}$ is contained in the part of the union of the hyperplanes defined by $\underline{\lambda} \in \mathscr{A}$ which meet $\mathscr{W}$. Since the hyperplanes are defined in terms of $g - 1$ parameters, either $g = \ell + 1$, or statement two is satisfied. This completes the proof.

For some problems, this determination of the subspace is simple. This is because we are only concerned with that part of the normal space to W_β which is contained in the span of $\{\nabla u_i(p)\}_{i \in C'}$. This is true in the pure exchange model without externalities (Example 1).

Corollary 4.2. <u>For the pure exchange economy without externalities, let</u> $p \in \mathring{\Omega}$ <u>be a</u> LPO <u>such that</u> $\nabla u_i(p) = \underline{0}$ <u>if and only if</u> $i \in C$. <u>Then there exist positive</u> λ_i, $i \in C'$, <u>such that</u> $\lambda_i \nabla u_i(p) = \lambda_k \nabla u_k(p)$ <u>for</u> $i,k \in C'$.

We also have that if $K_i \cap L_{i,p} = \{0\}$, then $\overline{\bigcup_{i \in C} W_i} \cap L_{C,p} \supset L_{C,p} \cap T\Omega_p$. This describes the second order behavior.

A more complete analysis of this type of LPO must consider the second order behavior of u_i, $i \in C$, particularly in the linear subspace defined by Ker Du_p. This will appear elsewhere where it will be incorporated as a part of detailed study of higher order terms.

We now turn to the second type of LPO where $\nabla u_i(p) = 0$ for some choice of i. This second type of LPO is due to an interesting, delicate interaction between the structure of Ω, the type of utility function admitted, and the characteristics of a non-dictatorial LPO versus a dictatorial LPO. It may turn out that due to the structure of Ω and the type of utility functions admitted, the only way some $p \in \Omega$ can be a non-dictatorial LPO is for $\nabla u_j(p) = 0$ for some choice of j - a condition which suggests some u_j may be at a maximum point. To see how this may occur, let $M = \mathbb{E}^6$ with coordinates $x = (x_1^1, x_1^2;\ x_2^1, x_2^2;\ x_3^1, x_3^2)$, $\Omega = S^5 = \{x \in M \mid \sum_{j=1}^{3} \sum_{i=1}^{2} (x_j^i)^2 = 1\}$, and $u_j(x) = u_j(x_j^1, x_j^2)$, $j = 1,2,3$. For this choice of Ω, the direction given by $p \in \Omega$ defines the normal bundle for Ω. Thus, should the jth block of p be zero, for example $p = (\frac{1}{2}, \frac{1}{2};\ 0,0;\ \frac{1}{2}, \frac{1}{2})$, then a necessary condition for p to be a non-dictatorial LPO is that $\nabla u_j(p) = 0$. On the other hand, such a p may be a dictatorial LPO without ∇u_j satisfying this condition. Of course, should u_j depend upon some other coordinate, say x_1^1, then the situation changes. This interaction between Ω and the type of utility function admitted will play a role in our analysis of the structure of θ_u.

In preparation for this analysis, we introduce the following notation. Let Z be a proper subset of $\{1,2,\ldots,a\}$, possibly empty, and let $\mathscr{U}$ be some collection of utility functions. An example of $\mathscr{U}$ would be the set of utility functions defined in any of our examples in Section 2. Let $\tilde{M}_Z = \{p\in\mathring{\Omega} \mid$ for all $u\in\mathscr{U}$, if p is a non-dictatorial LPO, then $\nabla u_j(p)$ must equal zero for $j\in Z\}$. Notice that if $Z_1 \supset Z_2$, then $\tilde{M}_{Z_2} \supset \tilde{M}_{Z_1}$. We eliminate this inclusion property by defining $M_Z = \tilde{M}_Z - (\bigcup_{Z_i \supsetneq Z} \tilde{M}_{Z_i})$.

For the above example, $\tilde{M}_{\{i\}}$ is the 3 dimension submanifold obtained by the intersection of S^5 with the 4 dimensional linear subspace of $\mathbb{E}^6$ defined by the condition that the ith block of p is zero. $M_{\{i,j\}} = \tilde{M}_{\{i,j\}} = \tilde{M}_{\{i\}} \cap \tilde{M}_{\{j\}}$ is a one dimensional submanifold of S^5, thus $M_{\{i\}}$ is a 3 dimensional submanifold of S^5.

Since we do not admit a bias property concerning the location of pareto points, the sets $\tilde{M}_Z = \{p\in\mathring{\Omega} \mid$ if $v\in N_p$, then all components of v corresponding to variables of $u_i(x)$, $i\in Z$, are zero.$\}$ From this characterization, it follows immediately that for the standard exchange economy and Example 2a, if $Z \neq \emptyset$, then $M_Z = \emptyset$. This shows that for a large number of interesting examples $M_\emptyset = \mathring{\Omega}$. Indeed, it is not until we consider location type problems, introduce production or complicated conversion rules in public goods problems, or admit certain types of taxation that $M_Z \neq \emptyset$ for $Z \neq \emptyset$.

Notice that $M_\emptyset$ is an open set, but that M_Z; $Z \neq \emptyset$, is not. The first order characterization of non-dictatorial LPOs follows immediately, and it is the same of Theorem 1. Thus, we list it as a corollary.

Corollary 1.2. Let $p \in \mathring{\Omega} \cap M_Z$ and assume that p is a LPO. Then $\nabla u_i(p) = \underline{0}$ for $i \in Z$. Furthermore, if $\nabla u_j(p) \neq \underline{0}$ for $j \in Z'$, then there exists positive non negative μ_i, not all zero, such that $\sum_{i \in Z'} \mu_i \nabla u_i(p) \in N_p$.

Should $\nabla u_j(p) = \underline{0}$ for some $j \in Z'$, then a statement of the type given in Corollary 1.2 holds.

Although we do not do so, we could impose the condition that M_Z is a lower dimensional submanifold. For those choices of Ω and $\mathscr{U}$ which could plausibly be of interest to economics and do not satisfy this condition, the structure of Ω could be subdivided into a finite number of separate problems. A simple example where M_Z is not a lower dimensional subspace is where $M = \mathbb{R}^3$, Ω is a unit cylinder of finite height parallel to some coordinate axis which is capped on top and bottom by a hemisphere, and $u_i(x) = u_i(x_i)$, $i = 1,2,3$. An example closer to economics can be constructed using the (taxation) mechanism discussed at the end of the previous section.

5. Generic Behavior of LPOs. Preliminaries

Recall that set Θ_u is the set of points in $\mathring{\Omega}$ which satisfy the first order necessary conditions for a LPO. A natural problem is to characterize the structure of Θ_u in $\mathring{\Omega}$, at least for "most" utility functions. This we shall do here. Our main tool to realize this objective is the Thom transversality theorem. While we shall refer the reader to references [4,10] for a proof of this powerful theorem, we will begin this section with a plausibility argument to illustrate some of the structure of this result.

The basic idea is the following. The necessary conditions for LPOs are given in terms of relationships between first order derivatives of the utility functions at different points in $\mathring{\Omega}$. The goal is to use these relationships to determine the structure of Θ_u back in $\mathring{\Omega}$. Notice the similarity between this statement and the classical problem where n equations in m unknowns, $m > n$, are set equal to zero. The implicit function theorem is used to show that under some regularity conditions the set of points satisfying this zero restriction forms a $m - n$ dimensional submanifold in the domain.

The basic idea for our problem is the same. A target space is created which includes the domain M, range $\mathbb{R}^a$, and all admissible first derivatives. To do this, for $f,g \in C^\infty(M,\mathbb{R}^a)$ we say that $f \underset{r}{\sim} g$ at x_0 if the Taylor series of order r for f at x_0 agree with that of g at x_0. (Recall, M has a given coordinate system, so, at least locally, Taylor series are well defined.) This forms an equivalence relationship, and the disjoint union of the resulting equivalence classes forms $J^r(M,\mathbb{R}^a)$. Thus, $J^0(M,\mathbb{R}^a)$ can be identified with $M \times \mathbb{R}^a$, and $J^1(M,\mathbb{R}^a)$ can be identified with $M \times \mathbb{R}^a \times L(T_pM,\mathbb{R}^a)$, where for each $p \in M$, $L(T_pM,\mathbb{R}^a)$ is the space of linear maps from T_pM to $\mathbb{E}^a$. In our problem, these linear maps can be identified with $(\nabla u_1,\ldots,\nabla u_a)$. $J^r(M,\mathbb{E}^a)$ is endowed with a natural manifold structure. This completes the target space.

We now need a mapping between the domain M and the target space $J^r(M,\mathbb{E}^a)$. For a utility mapping u, let $j^r u\colon M \to J^r(M,\mathbb{E}^a)$ be the map which at each $x\in M$ identifies u with the appropriate equivalence class, that is, the rth order Taylor series evaluated at x. It will be shown later in this section that the necessary conditions for a LPO define a submanifold Σ in J^1. So our problem is transferred to determining the structure of $(j^1u)^{-1}(\Sigma)$ in $\mathring{\Omega}$.

In general, let Σ be some submanifold in J^r. The object is to understand the structure of $(j^r u)^{-1}(\Sigma)$ in M. Judging from the classical implicit function theorem, we might expect this to be a submanifold of the appropriate codimension. It turns out that this is true under the assumption that a generalized version of the regularity conditions hold (see [2]).

Definition 5.1. Let $\mathscr{M},\mathscr{N}$ be manifolds and $\mathscr{A}$ a submanifold of $\mathscr{N}$. Let $f\colon \mathscr{M}\to\mathscr{N}$ be a smooth mapping. f is said to have a transverse intersection with $\mathscr{A}$ ($f \pitchfork \mathscr{A}$) if for each $p\in\mathscr{M}$ either $f(p)$ is disjoint from $\mathscr{A}$, or $Df(p)(T_p\mathscr{M}) + T_{f(p)}\mathscr{A} = T_{f(p)}\mathscr{N}$.

In other words, when $f(\mathscr{M})$ intersects $\mathscr{A}$, it crosses it in such a fashion that small perturbations of f cannot separate the sets - the two sets do not meet tangentially. This transversality condition plays the role of the "regularity condition" in the implicit function theorem.

Theorem 5 [4]. Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds, $\mathcal{A}$ a submanifold of $\mathcal{N}$, and f a smooth mapping from $\mathcal{M}$ to $\mathcal{N}$. If $f \pitchfork \mathcal{A}$, then either $f^{-1}(\mathcal{A})$ is empty, or $f^{-1}(\mathcal{A})$ defines a smooth submanifold of $\mathcal{M}$ with codimension $f^{-1}(\mathcal{A})$ = codimension $\mathcal{A}$.

Thus, our problem of characterizing the structure of $\Theta = (j^1 u)^{-1}(\Sigma)$ reduces to showing that $j^1 u \pitchfork \Sigma$. For a given map this may be a difficult task to verify. Fortunately, for our purposes, this is not necessary, thanks to the Thom jet transversality theorem [4,10]. This theorem asserts that "most" mappings satisfy this regularity condition.

Theorem 6. Let $C^{r+1}(M,\mathbb{R}^a)$ have the Whitney C^{r+1} topology, and let Σ be a smooth submanifold of $J^r(M,\mathbb{R}^a)$. Then there exists a residual set $\mathcal{B} \subset C^\infty(M,\mathbb{R}^a)$ such that if $u \in \mathcal{B}$, then $j^r u \pitchfork \Sigma$. Indeed, should M be precompact and Σ closed, then $\mathcal{B}$ is open dense. If the closure of Σ is a finite union of submanifolds, the dimension of each bounded by that of Σ, then $\mathcal{B}$ can be chosen to be open-dense.

An important fact, and one we shall use here, is that this theorem holds even should we impose a restriction upon which components function u_i can depend, $i = 1,2,\ldots,a$. This follows immediately from the proof [4,10].

We are now prepared to use these results to determine the structure of Θ_u for various classes of utility functions where a "class of utility functions" implies a restriction on the type of admitted externalities. By this we mean that class $\mathcal{U}$ is determined by the restrictions imposed upon which coordinates each agent's utility function can depend. Furthermore, we require this coordinate restriction to be global; that is, we will not admit classes which, for example, prohibit the dependency of u_1 on x_3^2 over part of M, but permit such dependencies over other regions. Such extensions are interesting from the viewpoint of economics and possible to analyze by using the same methods, but they will not be investigated here. Of course, *individual* utility functions may have this property but the class does not.

In order to keep the statements and proofs of the theorems from becoming overly technical, we shall initially analyze a restricted type of externality. These restrictions will be relaxed later in this section.

Definition 5.2. *A class* $\mathcal{U}$ *of smooth utility functions is said to be* M_Z *admissible if it satisfies the following:*

1. $u \in \mathcal{U}$, $p \in \theta_u \cap M_Z$, *implies that* $\sum_{i \in S} \lambda_i \nabla u_i(p) \notin N_p$ *for any proper subset* S *of indices contained in* Z' *and* $\lambda_i > 0$.

2. *Let* $p \in M_Z \cap \mathring{\Omega}$ *and* $v \in T_p\Omega$. *There exists* $u \in \mathcal{U}$ *such that* $Du_p(\underline{v}) \neq \underline{0}$.

3. M_Z *is non-empty*.

Condition 1 prohibits "coalition" LPOs of the type described in Theorem 2 for $k > 1$, and it requires $\nabla u_i(p) \neq \underline{0}$ for all i. Condition 2 ensures that Ω has been reduced to the appropriate dimension; namely, it guarantees that the definition of class $\mathcal{U}$ does not contain constraints which permit a reduction in the size of Ω. For example, if class $\mathcal{U}$ does not permit the ith agent's utility function from depending on one of his holdings - a holding included in the structure of M_Z, then the class does permit some other agent's utility function to depend on this holding. Condition 3 was discussed in Section 4. Examples 1 and 3, and the type of externalities for Example 2a which follows Theorem 1 satisfy this definition. Assume that M_Z is either a submanifold which does not include any of its boundary points, or (M_Z-interior M_Z) is a lower dimensional submanifold without boundary. Even though the latter set is not the boundary of M_Z, we will denote it as ∂M_Z.

Theorem 7. Let $\mathscr{U}$ be an M_Z admissible class of utility functions, and let $C^2_{\mathscr{U}}(M,\mathbb{E}^a)$ be all C^2 utility functions of class $\mathscr{U}$ where $C^2_{\mathscr{U}}$ has the Whitney C^2 topology. There exists a residual set $\mathscr{B}$ in $C^2(M,\mathbb{E}^a)$ such that $u \in \mathscr{B}$ implies that $\theta_u \cap M_Z$ is either empty, or it is a smooth submanifold in the interior of M_Z with dimension $\dim M_Z - (m - n) + \text{card } Z' - 1$, and in ∂M_Z with dimension $\dim \partial M_Z - (m - n) + \text{card } Z' - 1$. In particular, $\theta_u \cap M_\emptyset$ is a submanifold of dimension $a - 1$.

If the dimension number is negative, then the submanifold is empty. If it is zero, then the submanifold is the union of isolated points.

If M_Z is pre-compact, then $\mathscr{B}$ can be chosen to be open-dense.

By use of Theorems 5 and 6, it is sufficient to establish that the first order conditions which define θ_u form a manifold in J^1. The way we do this is first to find a representation for the normal bundle in J^1, which we would denote as N_1, and then to express the defining conditions for θ_u in terms of N_1. The only minor complication in the second task is that the defining conditions for θ_u

are in terms of the directions given by ∇u_i, not the magnitudes. In anticipation of this fact, the gradient vectors will be expressed in terms of scalar multiples of directions. In doing so, we use that fact that no $\nabla u_i(p)$ is equal to $\underline{0}$ for $p \in \theta_u \cap M_Z$ and $i \in Z'$.

Proof. By definition, M is either an open subset of $\mathbb{R}^m$, or it is a manifold which can be locally identified with $\mathbb{R}^m$; in fact, in the latter case this identification is readily established by the coordinate system which is given on M. For example, in the location problem where M is a torus, M can be locally identified with an open subset of $\mathbb{R}^2$ through the coordinates θ and φ. Thus, locally ∇u_i can be identified with a vector in $\mathbb{R}^m$. Moreover the specification of class $\mathscr{U}$ requires that $m - k_i$ of these entries must be zero. So, locally ∇u_i can be identified with some vector in a k_i dimensional subspace of $\mathbb{R}^m$. Let $\underline{A}_i = (A_i^1, \ldots, A_i^m)$ denote any non-zero vector in this space, and let $k = k_1 + \cdots + k_a$.

We shall prove the theorem for $M_\emptyset$, and at the end of the proof we shall indicate what changes are necessary to handle the general case.

Locally $J_u^1(M, \mathbb{R}^a)$ can be smoothly identified with $\mathcal{V} = \nu \times \mathbb{R}^a \times L$ where ν is an open subset of M and L is the k dimensional linear subspace of $(\mathbb{R}^m)^a$ determined by the entries $(\underline{A}_1, \ldots, \underline{A}_a)$. By choosing ν sufficiently small, the normal bundle for $\mathring{\Omega}$ can be locally represented as $(\mathring{\Omega} \cap \nu) \times (\underline{0} \times \mathbb{R}^n) \subset \nu \times \mathbb{R}^m$ where n is the codimension of $\mathring{\Omega}$ in M. Denote this representation as N_ν. These identifications will be used to define a manifold in $J_u^1(M, \mathbb{R}^a)$ which corresponds to the LPOs.

Let $f\colon \tilde{\nu} \to \nu \times \mathbb{E}^m$ be defined as

$$f(p,q;\ \underline{A}_1,\ldots,\underline{A}_a) = (p;\ \sum_{i=1}^{a} A_i^1, \sum_{i=1}^{a} A_i^2, \ldots, \sum_{i=1}^{a} A_i^m).$$

Since f is linear in the second argument, it follows from condition 2 of Definition 5.2 (this condition defines which components of $\underline{A}_i$ can be non-zero) that $f \pitchfork N_\nu$. According to Theorem 5, $f^{-1}(N_\nu)$ is a smooth submanifold in $\tilde{\nu}$ with codimension $n + (m-n) = m$.

Cover $\mathring{\Omega}$ with a countible number of open sets ν_i which possess the properties of ν as expressed above. Since f is linear, actually a bundle mapping, and M has a given coordinate representation, standard arguments concerning local vector bundle representations handle the overlap conditions between $\nu_i \cap \nu_j$. Therefore, the above construction defined locally leads to a submanifold, N_1, of codimension m in $J^1\,(M,\mathbb{E}^a)$. Actually, this gives rise to a subvector bundle over $\mathring{\Omega}$.

As a subvector bundle, N_1 admits vectors where $\underline{A}_i = \underline{0}$. In our problem, this is not admitted. So, let N be those elements of N_1 such that no $\underline{A}_i$ is equal to zero. This is an open condition since it is the intersection of N_1 with the finite number of open sets given by the complement Λ, where $\Lambda = \cup \Lambda_i$ and $\Lambda_i = \{(p,q;\ \underline{A}_1,\ldots,\underline{A}_a) \mid \underline{A}_i = 0\}$. Consequently N is a manifold with the same codimension. Notice, this last construction may eliminate some points from $\mathring{\Omega}$. For example, let $\mathring{\Omega} = S^5 = \{\sum_{i=1}^{3} \sum_{j=1}^{2} (x_i^j)^2 = 1\} \subset \mathbb{E}^6$, and let $u_i(x_i^1, x_i^2)$, $i = 1,2,3$. At $p = (\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; 0,0) \in S^5$ there is no $\underline{A}_3 \neq \underline{0}$ which is in N_1. Actually, in this problem the above construction eliminates any point of $\mathring{\Omega}$ which has a double zero in one of the three pairing of coordinates. Indeed, by definition the part of $\mathring{\Omega}$ which remains is $M_{\emptyset}$. This is a submanifold of $\mathring{\Omega}$ by condition 3 and the fact that $\bigcup_{Z\neq\emptyset} M_Z$ is closed in $\mathring{\Omega}$.

Let $\underline{\mathscr{A}}_i = (\mathscr{A}_i^1,\ldots,\mathscr{A}_i^m) = \lambda_i\underline{A}_i$ where $\lambda_i = |\underline{A}_i|^{-1}$ and $\underline{A}_i \neq \underline{0}$. With this notation, the entries in N can be (locally) represented as entries in $S^{k_1-1} \times \cdots \times S^{k_a-1} \times (\mathring{\mathbb{R}}_+)^a$.

Let π: $J^1(M,\mathbb{R}^a) - \Delta \to M \times S^{k_1-1} \times \cdots \times S^{k_a-1}$ be given (locally) by $\pi(p,q;\ \lambda_1\underline{\mathscr{A}}_1,\ldots,\lambda_a\underline{\mathscr{A}}_a) = (p,\underline{\mathscr{A}}_1,\ldots,\underline{\mathscr{A}}_a)$. We claim that $\pi(N)$ is a submanifold of dimension $(m-n) + [k - (m-n) - 1] = k - 1$, which is codimension $(m + k - a) - (k - 1) = m + 1 - a$. This will follow from the condition that should $(p,q;\ \lambda_1\underline{\mathscr{A}}_1,\ldots,\lambda_n\underline{\mathscr{A}}_n)\in N$, then $(p,q;\ \underline{\mathscr{A}}_1,\ldots,\lambda_n\lambda_1^{-1}\underline{\mathscr{A}}_n)\in N$. ($\lambda_1 \neq 0$, since $\underline{A}_1 \neq \underline{0}$.) Furthermore, it will turn out that the $\lambda_i\lambda_1^{-1}$ are uniquely determined for given p and $(\underline{\mathscr{A}}_1,\ldots,\underline{\mathscr{A}}_n)$. Consequently, π restricted to this submanifold of N is one to one, yielding that $\pi(N)$ is a manifold of dimension $\{\dim\mathring{\Omega} + k - (m-n)\} - 1$.

Condition 1 of Definition 5.2, implies that should $(p,\lambda_1\underline{\mathscr{A}}_1,\ldots,\lambda_a\underline{\mathscr{A}}_a)\in N$, then any collection of (a-1) of the vectors forms a linearly independent set. Therefore the matrix

$$A = \begin{pmatrix} \underline{\mathscr{A}}_1 \\ \underline{\mathscr{A}}_2 \\ \vdots \\ \underline{\mathscr{A}}_a \end{pmatrix}$$

has rank at least $a - 1$. If A has rank $a - 1$, then A: $\mathbb{R}^m \to \mathbb{R}^a$ defines a hyperplane in $\mathbb{R}^a$. Let $\underline{\mu} = (\mu_1,\ldots,\mu_a)$ be a normal unit vector to this hyperplane. There are only two choices for $\underline{\mu}$, and we assert that one of them requires $\mu_i > 0$, $i = 1,2,\ldots,a$. This follows from the fact that $\Sigma\, \lambda_i \underline{\mathscr{A}}_i \epsilon N_p$. So $(\Sigma\, \lambda_i \underline{\mathscr{A}}_i)\cdot \underline{h} = \underline{\lambda}\cdot A\underline{h} = 0$ for $\underline{h} \epsilon \mathbb{E}^m$. That is, $\underline{\lambda}$ is a normal vector to the hyperplane given by the image of A, which means that $\underline{\lambda}$ is a constant multiple of $\underline{\mu}$. Thus the $\mu_i > 0$, and $(1,\lambda_2/\lambda_1,\ldots,\lambda\ /\lambda_1)$ is uniquely determined. Notice that in this case $\Sigma\, \lambda_i \underline{\mathscr{A}}_i = \underline{0}$ and the $\underline{A}_i$'s are in $T\mathring{\Omega}_p$.

A similar argument holds if A has rank a, except here $\underline{h}$ is restricted to vectors in $\mathbb{E}^{m-n} \times \underline{0}$ which can be identified with $T\Omega_p$. Then, according the definition of N, $A|\mathbb{E}^{m-n} \times \underline{0}$ defines a codimension one hyperplane in $\mathbb{E}^a$. Let $\underline{\lambda}$ be the normal vector. This means that $(1,\lambda_2/\lambda_1,\ldots,\lambda_a/\lambda_1)$ is uniquely determined. Since this vector is uniquely determined, $\pi(N)$ is a submanifold.

Let $\tilde{N} = \pi^{-1}(\pi(N))$. By representing the vectors in $L - \Lambda$ in "polar coordinates", it can be viewed as a projection. From this it follows that $d\pi_p$ is surjective, so the transversality condition $\pi \pitchfork \pi(N_1)$ is trivially satisfied. Consequently it follows from Theorem 4 that $\tilde{N}$ is a submanifold in $J^1\ (M,\mathbb{E}^a)$ with codimension $m + 1 - a$. By construction, $\tilde{N}$ is a submanifold of $M_\emptyset \times \mathbb{E}^a \times L \subset \mathring{\Omega} \times \mathbb{E}^a \times L$ with codimension $m - n + 1 - a$.

Notice that $(p,q;\ \underline{A}_1,\ldots,\underline{A}_a)\epsilon\tilde{N}$ implies the existence of $\underline{\mu} = (\mu_1,\ldots,\mu_a)\epsilon\mathring{\mathbb{E}}^a_+$ such that $(p,q;\ \mu_1\underline{A}_1,\ldots,\mu_a\underline{A}_a)\epsilon N$. In other words, should $j^1u(x)\epsilon\tilde{N}$ for $u\epsilon C^2_{\mathcal{U}}$, then $x\epsilon\Theta_u \cap M_\emptyset$; or $\Theta_u = (j^1u)^{-1}(\tilde{N})$ According to Theorem 5, there exists a residual set of functions $\mathscr{B}$ in $C^2_{\mathcal{U}}$ such that if $u\epsilon\mathscr{B}$, then either $\Theta_u \cap M_\emptyset$ is empty, or it is a submanifold of dimension $(m-n) - (m - n + 1 - a) = a - 1$.

If the closure of Ω is compact in M and if only a finite number of coordinate charts are needed to cover $\mathring{\Omega}$ and to provide the local trivial representation for the normal bundle, then $\mathscr{B}$ can be selected so it is open-dense. This follows by the structure of $\tilde{N}$ in $J^1_{\mathcal{U}}$ and Theorem 6.

Now assume $Z \neq \emptyset$. We shall indicate what changes in the above argument are needed to complete the proof. The normal bundle, N_1, is again a subfiber bundle over $M_Z \times \mathbb{R}^a$ with fiber codimension $m - n$. This has fiber dimension $\sum_i^a k_i - (m-n)$. Notice that the components $\underline{A}_i$, $i \in Z$, are all zero, so this fiber lies in the space defined by $\underline{A}_i$, $i \in Z$. Thus the space can be divided into the components defined by indices in Z', which we call $L_{Z'}$, and those in Z, labeled $L_{Z'}$, and N_1 lies in the former. From the fiber in N_1, we remove all $\underline{A}_i = \underline{0}$, and project into $M_Z \times \prod_{i \in Z'} S^{k_i - 1}$. This is a manifold of dimension $\dim M_Z + (\sum_1^a k_i - (m-n) - 1)$, or codimension $\dim M_Z + \sum_{i \in Z} k_i - \text{card } Z' - \{\dim M_Z + (\sum_1^a k_i - (m-n) - 1)\} = -\sum_{i \in Z} k_i + (m-n) + 1 - \text{card } Z'$. The inverse image of this projection is a submanifold of $M_Z \times \mathbb{R}^a \times L_{Z'}$ with codimension $-\sum_{i \in Z} k_i + (m-n) + 1 - \text{card } Z'$. Thus when we attach L_Z, we have that $\tilde{N}$ is a submanifold of $M_Z \times \mathbb{R}^a \times L$ with codimension $(m-n) + 1 - \text{card } Z'$. The conclusion now follows.

Corollary 7.2 [10]. <u>For the standard exchange economy, there exists an open dense set of utility mappings $\mathscr{B} \subset C^2_{\mathcal{U}}(M, \mathbb{R}^a)$ such that if $u \in \mathscr{B}$, then either θ_u is empty in $\mathring{\Omega}$, or it forms an (a-1) dimensional submanifold of $\mathring{\Omega}$.</u>

For the strongly dictatorial case, the situation is different. The term a is replaced with the number of agents which play a role in determing the location of the LPO. For example, if $a = 3$

and $c > 1$, and if u_1 determines the location of LPO p, then $\nabla u_1(p) \epsilon N_p$ but the values of $\nabla u_2(p)$ and $\nabla u_3(p)$ are arbitrary. The codimension of Σ in J^1 is then $ac - c = 2c$. This follows from the fact that once c appropriate components of $\nabla u_1(p)$ are given, the remaining 2c components are determined. Thus $(j^1 u)^{-1}(\Sigma)$ is, generically, a $c(a-1) - 2c = 0$ dimensional manifold of $\mathring{\Omega}$. The basic idea is that the dimension of the submanifold is determined by the number of participating agents. For practical purposes, ineffectual agents are projected out of the problem. The result should not be surprising because the pareto points correspond to local maxima for u_1, and it states that they should be isolated in the generic case.

§6. Generic Behavior of LPOs: General Case

Condition 1 of Definition 5.2 does not allow for the division of agents into subgroups in the fashion described in Theorem 2. In order to permit a greater flexibility in the choice of externalities for the utility mappings, this first restriction of Definition 5.2 will be relaxed.

Definition 6.1. Let $S_1,\ldots,S_\alpha$ be non-empty subsets of $\{1,2,\ldots,a\}$ such that i) $S_i \not\subset \bigcup_{\substack{j=i \\ j\neq i}}^{\alpha} S_j$, $i = 1,2,\ldots,\alpha$, but ii) $\{1,2,\ldots,a\} \subset \bigcup_{i=1}^{\alpha} S_j$. Let $N_1,\ldots,N_\alpha$ be subvector bundles of N. **A utility function** u is said to be $(S_1,N_1;\ S_2,N_2;\ldots;S_\alpha,N_\alpha)$ admissible at $p\in\mathring{\Omega}$ if

1. there exist positive constants $\mu_{i,k}$ such that $\sum_{i\in S_k} \mu_{i,k}\nabla u_i(p)\in(N_k)_p$, $k = 1,2,\ldots,\alpha$, and

2. it is impossible to select positive μ_i's such that this condition holds for any non-empty subset of any S_k.

The conditions on the subsets S_i and on the gradient functions are motivated by the conclusion of Theorem 2. In the example following the proof of Theorem 2, the sum of the gradients of functions with indices in S_1 can be in N only if it is in the subspace spanned by (1,0; 1,0; 1,0). Restrictions of this type are natural for problems with externalities. This is the reason the subvector bundles N_i are introduced.

Of course, in this example, some LPO's may be determined by (S,N). Indeed, this is why the above definition is local in nature. This reflects the fact that a given function u may be $(S_1,N_1;\ldots;S_\alpha,N_\alpha)$ admissible at some point, but $(S_1^*,N_1^*;\ldots,;S_{\alpha_1}^*,N_{\alpha_1}^*)$ admissible at some other point. We combine these elements in a straightforward fashion. Let P be a collection of elements $\{S_1,N_1;\ S_2,N_2;\ldots,;S_n,N_m\}$. We shall require admissible utility functions to satisfy conditions similar to those listed in Definition 5.2 for some element of P.

Unfortunately, by admitting a more general class of utility functions, an adjustment in the definition of M_Z (Section 2) is needed. We give such a modification here. For this definition we will be using an element $\{S_1,N_1;\ldots,;S_\alpha,N_\alpha\}$ from set P. We require all of Definition 6.1 to hold <u>except</u> Sentence 2. This sentence precludes zero gradients, which must occur for M_Z, $Z \neq \emptyset$.

Let $\delta = (S_1,N_1;\ldots,;S_\alpha,N_\alpha)\epsilon P$. For subsets of indices Z, let $\tilde{M}_{Z,\delta} = \{p\epsilon\mathring{\Omega}\,|\,\text{for all } u\epsilon\mathscr{U}, \text{ if } p\epsilon\theta_u \text{ and } j\epsilon Z, \text{ then } \nabla u_j(p) = 0\}$. Because the classes $\mathscr{U}$ admit any smooth function depending upon the required variables, there is no bias introduced. Thus these sets can be characterized in terms of the structure of $\mathring{\Omega}$. In this case we have $\tilde{M}_{Z,\delta} = \{p\epsilon\mathring{\Omega}\,|$ if $i\epsilon Z$ and in S_j, $j\epsilon B$, then $\bigcap_{j\epsilon B} N_{j,p}$ has zero coordinates in all the entries corresponding to the admissible variables of $u_i(\underline{x})\}$. Let $M_{Z,\delta} = \tilde{M}_{Z,\delta} - \left(\bigcup_{Z_i \supsetneq Z} M_{Z_i,\delta}\right)$.

Furthermore, let $M_{Z,P} = \bigcup_{\delta\epsilon P} M_{Z,\delta}$.

It follows from the above construction that $M_{\emptyset,\delta}$, and consequently $M_{\emptyset,P}$, is an open set. Notice that they are non-empty sets equal to $\mathring{\Omega}$ for pure exchange economies (Example 1 and 2a).

Definition 6.2. <u>Let</u> P <u>be a collection of elements</u>

$\{S_1,N_1;\ldots,S_k,N_k\}$. A class $\mathscr{U}$ of utility functions is said to be P-admissible if the following are satisfied

1) Let $u \in \mathscr{U}$. If there is $p \in M_P$, P and $\lambda_i > 0$ such that $\Sigma\, \lambda_i \nabla u_i(p) \in N_p$, then for some element of P, u is $\{S_1,N_1;\ldots,;S_\alpha,N_\alpha\}$ admissible at p.

2. For any $\underline{v} \in T\Omega_p$, $p \in \mathring{\Omega}$, there exists $u \in \mathscr{U}$ such that $Du_p(\underline{v}) \neq 0$.

3. $M_{\emptyset,P} \neq \emptyset$.

This definition is stated in terms of $M_{\emptyset,P}$. Because the generic characterization of P admissible points is more complex, we shall initially restrict our attention to $M_{\emptyset,P}$. At the end of this section we shall describe $M_{Z,P}$.

Throughout this section we continue our assumption that class $\mathscr{U}$ restricts only the coordinates on which utility function u_i, $i = 1,2,\ldots,a$, may depend. We extend this assumption by assuming these permitted dependencies define the sets N_i, at least over $M_{\emptyset,P}$.

The goal is to characterize θ_u for $u \in \mathscr{U}$, where $\mathscr{U}$ is P-admissible. The next theorem asserts that generically we can expect θ_u to be the finite union of submanifolds of varying dimensions where the dimensions depend upon the elements in P. In the non-coalition like case, the actual dimensions become somewhat unwieldly. For expository reasons we defer the actual statement of the dimensions for the non-coalition like case to later in this section. At this time we obtain an upper bound for the dimension of these submanifolds.

A corollary of this analysis is that even in the non-coalition-like case, certain classes of indices result which have some of the flavor of the coalition-like case. That is, we obtain a higher order clustering of the indices. We try to emphasize this similarity and clustering effect by the choice of parameters used to state the theorem. The definition of these relevant parameters follows.

Let $\{S_1,N_1; S_2,N_2;\ldots;S_\alpha,N_\alpha\}\epsilon P$. For these subsets we say that S_i is chain related to S_j if there exists subsets $S_{k_1},\ldots,S_{k_\ell}$ such that $S_i \cap S_{k_1} \neq \emptyset$, $S_{k_1} \cap S_{k_2} \neq \emptyset,\ldots,S_{k_\ell} \cap S_j \neq \emptyset$. Clearly this is an equivalence relationship. If there are β equivalence classes, let $\mathbb{C}_\gamma$, $\gamma = 1,2,\ldots,\beta$, be the set of indices of elements in the γth equivalence class.

For set $\mathbb{C}_\gamma$, let s_γ equal the cardinality of $\bigcup_{i\epsilon\mathbb{C}_\gamma} S_i$, and let $\underline{n}_\gamma$ denote the dimension of the N_γ, the subspace spanned by vector spaces N_i, $i\epsilon\mathbb{C}_\gamma$ where n_i is the dimension of N_i. Notice that $\underline{n}_\gamma \leq \sum_{i\epsilon\mathbb{C}_\gamma} n_i$. Finally, let g_i be number of coordinates permitted by utility functions in S_i, and $\underline{g}_\gamma$ the number of coordinates permitted by utility functions in $\bigcup_{i\epsilon\mathbb{C}_\gamma} S_i$. Integer g_i is then the dimension of the subspace spanned by $\sum_{i\epsilon S_i} \nabla u_i(p)$ for all possible choices of $u\epsilon\mathcal{U}$. Thus $g_i \leq \sum_{j\epsilon S_i} k_j$ and $\underline{g}_\gamma \leq \sum_{i\epsilon\mathbb{C}_\gamma} g_i$.

For any element of P which is coalition-like, the sets $\mathbb{C}_\gamma$ consist of a single element, $\beta = \alpha$, $\underline{n}_i = n_i$ and $\underline{g}_i = g_i$.

Theorem 8. <u>Let</u> $\mathcal{U}$ <u>be a</u> P-admissible <u>class of utility functions</u>. <u>Let</u> $C^2_{\mathcal{U}}(\mathring{\Omega},\mathbb{R}^a)$ <u>be all</u> C^2 <u>utility functions in</u> $\mathcal{U}$, <u>where</u> $C^2_{\mathcal{U}}(\mathring{\Omega},\mathbb{R}^a)$ <u>has the Whitney</u> C^2 <u>topology</u>. <u>There exists a residual</u>

set $\mathscr{B}$ in $C^2(\mathring{\Omega},\mathbb{E}^a)$ such that $u\in\mathscr{B}$ implies that either $\theta_u \cap M_{\emptyset,P}$ is empty, or it is the finite union of smooth submanifolds in $\mathring{\Omega}$ corresponding to the different entries of P. Indeed, the submanifold corresponding to $\{S_1,N_1;\ldots;S_\alpha,N_\alpha\}\in P$, where this element has $\beta \leq \alpha$ equivalence classes, has dimension less than or equal to $m - n - \{\sum_{j=1}^{\beta}(\underline{g}_\gamma - \underline{n}_\gamma)\} + \alpha - \beta$. If the element of P defines a coalition-like structure, then this is an equality. If this integer is zero and the submanifold is non-empty, then it corresponds to a union of isolated points. If this integer is negative, then the submanifold is empty. If $\mathring{\Omega}$ is precompact then $\mathscr{B}$ can be chosen to be open-dense.

The hypothesis of Theorem 7 corresponds to the special case where P consists of the single element $(\{1,2,\ldots,a\},N)$, $g = m$, and $n_1 = n$.

It is remarkable that for the pure exchange economy the coalition-like elements of P add nothing to the structure of θ_u, at least in the generic case. Indeed, this is true even for the non coalition-like setting if $\beta \geq 2$.

Corollary 8.1. Let $\mathscr{U}$ be a P-admissible set of utility functions for a pure exchange economy with externalities. Assume all elements of P, other than $\{(1,2,\ldots,a),N\}$, form at least two equivalence classes. Then there exists an open dense set $\mathscr{B} \subset C^2_{\mathscr{U}}(M,\mathbb{E}^a)$ such that $u\in\mathscr{B}$ implies that either θ_u is empty, or it is an $(a-1)$ dimensional submanifold of $\mathring{\Omega}$: If P does contain an element with one equivalence class, then the part of θ_u corresponding to this element is either empty, or a submanifold with dimension not exceeding $\underline{n}_1 - c + a - 1$.

In other words, the set of C^2 class $\mathcal{U}$ utility functions for which θ_u is non-empty and differs from the above specified structure is contained in a nowhere dense set of $C^2_{\mathcal{U}}(M,E^a)$.

Following the proof of Theorem 9, we shall show by means of an example that the part of θ_u due to a one equivalence class element of P need not be empty in the generic setting.

Proof of Corollary. In the exchange economy model, $m = ca$, $n = c$. Generically, element $(\{1,2,\ldots,a\},N)\in P$ gives rise to an $(a-1)$ dimensional submanifold in $\mathring{\Omega}$ for θ_u. All other elements of P define, generically, submanifolds of dimension less than or equal to $m - n - \{\sum_{\gamma=1}^{\beta}(\underline{g}_\gamma - \underline{n}_\gamma)\} + a - \beta$, where $\beta \geq 2$. By the nature of the N_γ's and the normal bundle, at least $(\underline{n}_\gamma a)$ components are needed to specify N_γ, so $\underline{g}_i \geq \underline{n}_i a$. In fact, should each agent in an equivalence class depend upon his own holding, or at least some other agent in this class depends upon these holdings, then $\underline{g}_\gamma \geq \underline{n}_\gamma a + \underline{s}_\gamma(c - \underline{n}_\gamma)$. Thus in this setting the dimension of the submanifold is bounded above by

$$ca - c - \{\sum_{\gamma=1}^{\beta}(\underline{n}_\gamma a + \underline{s}_\gamma(c - \underline{n}_\gamma) - \underline{n}_\gamma\} + a - \beta =$$

$$- c + \Sigma\, \underline{n}_\gamma(\underline{s}_\gamma - a + 1) + a - \beta \text{ because } \Sigma\, \underline{s}_\gamma = a.$$

If $\beta \geq 2$, $\underline{s}_\gamma + 1 \leq a$; so the dimension of the submanifold is bounded above by $-c + \sum_{\gamma=1}(\underline{s}_\gamma - a + 1) + a - \beta = -c - \beta a + a + \beta + a - \beta =$ $-c + (2 - \beta)a < 0$. If $\beta = 1$, $\underline{s}_\gamma = a$, so the dimension of the submanifold is bounded above by $-c + \underline{n}_\gamma + a - 1$.

Now, should the definition of $\mathcal{U}$ specify that some equivalence class exists where none of the utility functions of agents in this class depend upon p_i of their combined holdings, then $\underline{g}_i \geq \underline{n}_i a + \underline{s}_i(c - \underline{n}_i) - p_i$. Condition 2 of Definition 6.2 implies that the utility functions for some collection of agents must depend upon these p_i holdings. This means some $\underline{g}_j$ is augmented by part of the sum p_i. In the final summation $\Sigma\, \underline{g}_\gamma$, these adjustment either cancel, or decrease the established upper bound on the dimension of the submanifold. This completes the proof.

The proof of Theorem 8 is similar to that of Theorem 7. The basic idea is to find a representation for the N_i's in $J^1_{\mathcal{U}}$. Using the "polar coordinate" representation $\underline{A}_i = \lambda_i \underline{\mathscr{A}}_i$, the set in J^1 corresponding to N_i is projected onto the unit vectors to determine the directions leading to an element in N_i. The inverse image of this projection then gives the set of vectors which can be modified in length to end up in the appropriate normal space. All of this goes through with essentially the same proof as for Theorem 7. The major complication is keeping track of the dimensions. Thus we emphasize this counting argument and slide over some of the points considered in the proof of Theorem 7. In particular, we shall no longer be careful about the need to have a local representation to obtain a trivial representation for some of the vector bundles.

Proof of Theorem 8. Let $\{S_1,N_1;\ldots,S_\alpha,N_\alpha\}\epsilon P$. We shall determine the structure of that part of $\theta_u \cap M_{\emptyset,P}$ which corresponds to this element of P.

Since $\mathbb{R}^a$ is an Euclidean space, the submanifold of J^1 corresponding to a restriction from M to $\mathring{\Omega}$ can be viewed as a vector bundle over $M_{\emptyset,P}$. By standard local arguments, let π_{S_i} be the natural projection bundle mapping from this space to $M_\emptyset \times \mathbb{R}^{S_i}$ where $\mathbb{R}^{S_i} = \prod_{j\epsilon S_i} \mathbb{R}^{k_j}$. Namely, at each $p\epsilon\mathring{\Omega}$, π_{S_i} is the projection from $\mathbb{E}^a \times \mathbb{E}^k$ to $\mathbb{E}^{S_i}$. Let f_{S_i} be a bundle mapping from $M_{\emptyset,P} \times \mathbb{E}^{S_i}$ onto $M_{\emptyset,P} \times \mathbb{E}^{g_i}$ given by $f_{S_i}(p;\underline{A}_{j_1},\ldots,\underline{A}_{j_p}) = (p;\ \sum_{j\in S_i} \underline{A}_j)$. It follows from the definition of g_i that this map is surjective.

Locally N_i can be represented as a subbundle of $\mathring{\Omega} \times \mathbb{E}^{g_i}$ with fiber dimension n_i. Since df_{S_i} is surjective, transversality conditions are trivially satisfied. This means that by piecing together local arguments in the standard fashion, we have that $f_{S_i}^{-1}(N_i)$ is a submanifold of $M_{\emptyset,P} \times \mathbb{E}^{S_i} \subset \mathring{\Omega} \times \mathbb{E}^{S_i}$. Actually it is a sub vector bundle with fiber codimension $g_i - n_i$.

Not all of the vectors in a fiber of $f_{S_i}^{-1}(N_i)$ need to satisfy the same independence property enjoyed by the elements of $f^{-1}(N_i)$ in the proof of Theorem 6. The problem here stems from the possibility that P may admit other elements which subdivide set S_i into two or more subsets. Thus, in order to keep only those vectors which are defined by the stated element of P, we retain those vectors in $f_{S_i}^{-1}(N_i)$ for which any combination of $s_i - 1$ of the s_i vectors forms a linearly independent set. Since this is a construction similar to that used to define Δ, the resulting set is a submanifold (indeed, an open subset) of $f_{S_i}^{-1}(N_i)$ with the same codimension. Denote this new set as $\tilde{f}_{S_i}^{-1}(N_i)$. It is not a vector bundle since it does not admit the $\underline{0}$ vector in any fiber.

Since π_{S_i} is a projection, $\tilde{N}_i = \pi_{S_i}^{-1}(\tilde{f}_{S_i}^{-1}(N_i))$ is a submanifold with codimension $g_i - n_i - n$. The term n is a result of our restriction from M to $M_{\emptyset,P}$. Let $N' = \bigcap_{i=1}^{\alpha} \tilde{N}_i$. It follows from this construction that N' is an open subset of $N'' = \bigcap_{i=1}^{\alpha} \pi_{S_i}^{-1}(f_{S_i}^{-1}(N_i))$. We shall show that N'' is a subvector bundle over $M_\emptyset$. This will show that N' is a submanifold in $J^1_{\mathcal{U}}$ with the same codimension as N''.

Because $\pi_{S_i}^{-1}(f_{S_i}^{-1}(N_i))$ is a subvector bundle over $\mathring{\Omega}$, $i = 1,2,\ldots,\alpha$, N'' is also a subvector bundle if the intersection of the fibers has constant dimension for arbitrary $p \in M_{\emptyset,P}$. However, this follows immediately from our restriction that for $p \in M_{\emptyset,P}$ the non-zero coordinates of the N_i's and those of the A_i's are defined in terms of class $\mathscr{U}$.

If the S_i's are pairwise disjoint, an alternate construction for N' is available. In this setting it is easy to see that N'' can be represented as the Whitney sum (see, for example, [6]) $M_\emptyset \times \mathbb{E}^a \times f_{S_i}^{-1}(N_1) \oplus \cdots \oplus f_{S_\alpha}^{-1}(N_\alpha)$, where this is interpreted as the product of $M_\emptyset \times \mathbb{E}^a$ with the fiber given by the direct sum of the fibers of $f_{S_1}^{-1}(N_1),\ldots,f_{S_\alpha}^{-1}(N_\alpha)$. Thus N' has dimension $a + m - n + k - \sum_{i=1}^{\alpha}(g_i - n_i)$.

We assert in the non coalition-like setting that N'' is a subvector bundle of the Whitney sum $M_{\emptyset,P} \times \mathbb{E}^a \times N_{\mathbb{C}_1} \oplus \cdots \oplus N_{\mathbb{C}_\beta}$, where $N_{\mathbb{C}_\gamma}$ is a fiber of codimension $\underline{g}_\gamma - \underline{n}_\gamma$ corresponding to the γth equivalence class of this particular element of P. To see that this is so, let π_γ be the natural projection bundle mapping from $J^1_{\mathscr{U}}$ to $M_{\emptyset,P} \times \mathbb{E}^{\underline{s}_\gamma}$, and f_γ the bundle mapping from $M_{\emptyset,P} \times \mathbb{E}^{\underline{s}_\gamma}$ onto $M_{\emptyset,P} \times \mathbb{E}^{\underline{g}_\gamma}$ given by $f_\gamma(p; \underline{A}_{j_1},\ldots,\underline{A}_{j_\ell}) = (p; \sum_{\alpha \in \mathbb{C}_\gamma} \sum_{j \in S_\alpha} \underline{A}_j)$. Then, following the above argument, we have that $\pi_\gamma^{-1}(f_\gamma^{-1}(N_\gamma))$ is a subvector bundle over $M \times \mathbb{E}^a$ with fiber codimension $\underline{g}_\gamma - \underline{n}_\gamma$. By the disjointness of the equivalence classes or by using a Whitney sum argument, $N''' = \bigcap_{\gamma=1}^{\beta} \pi_\gamma^{-1}(f_\gamma^{-1}(N_\gamma))$ is a vector subbundle over $M \times \mathbb{E}^a$ with fiber codimension $\Sigma\, \underline{g}_\gamma - \underline{n}_\gamma$.

We now show that if $\underline{v}$ is a vector in a fiber of N'', then $\underline{v}$ is in N'''. To see this, decompose $\underline{v} = \sum_{\gamma=1}^{\beta} \underline{v}_\gamma$ and $\underline{v}_\gamma = \sum_{i \in \mathbb{C}_\gamma} v_i$.

Redefining π_{S_i} as a mapping from $\mathbb{R}^a \times \mathbb{R}^k$ to $\mathbb{R}^{S_i} \subset \mathbb{R}^{\underline{s}_\gamma}$, it follows that $\pi_\gamma \underline{v}_\gamma = \sum_{i \in \mathbf{C}_\gamma} \pi_{S_i} v_i$. With this identification of $\mathbb{R}^{S_i}$ as a subspace of $\mathbb{R}^{\underline{s}_\gamma}$, $f_{S_i} = f_\gamma | \mathbb{R}^{S_i}$, so $f_\gamma(\pi_\gamma \underline{v}_\gamma) = \sum_{i \in \mathbf{C}_\gamma} f_{S_i}(\pi_{S_i} v_i) \in N_\gamma$. Thus $\underline{v}_\gamma \in \pi_\gamma^{-1}(f_\gamma^{-1}(N_\gamma))$, and $\underline{v} \in N'''$, which proves our assertion.

The above construction ensures that any vector in N' is non zero. So, define $\underline{\mathscr{A}}_i = \lambda_i^{-1} \underline{A}_i$ where $\lambda_i = |\underline{A}_i|$. By an argument similar to that used in the proof of Theorem 7, h(N') is a submanifold of $M_{\emptyset,P} \times S^{k_1} \times \cdots \times S^{k_a}$ where $h\colon J^1 - \Delta \to M_{\emptyset,P} \times S^{k_1} \times \cdots \times S^{k_a}$ is defined as $h(p,q; \lambda_1 \underline{\mathscr{A}}_1, \ldots, \lambda_a \underline{\mathscr{A}}_a) = (p, \underline{\mathscr{A}}_1, \ldots, \underline{\mathscr{A}}_a)$.

In the coalition-like setting, h(N') has dimension $(m-n) + k - \Sigma(g_i - n_i) - \alpha$, or codimension $(m-n) + k - a - [(m-n) + k - \Sigma(g_i - n_i) - \alpha] = \Sigma(g_i - n_i) + \alpha - a$. To see that this is so, let $\underline{v} \in N'$. There is a unique decomposition $\underline{v} = \sum_{i=1}^{\alpha} \underline{v}_i$ where $\underline{v}_i = \sum_{j \in S_i} \underline{A}_j$. Thus $\underline{v} = \sum_{i=1}^{\alpha} \sum_{j \in S_i} \lambda_j \underline{\mathscr{A}}_j$. Now, using the independence property of vectors in S_i, an argument similar to that used in the proof of Theorem 7 shows that $h(f^{-1}(N_i))$, a mapping from $\mathbb{R}^{S_i} \to \prod_{j \in S_i} S^{k_j}$, is of one dimension less than the fiber $f^{-1}(N_i)$. Because there are α sets S_i, this equation follows. In other words, any $\underline{v} \in N'$ is given uniquely by the directions $\underline{\mathscr{A}}_j$ and α choices of λ_j, one from each set S_j.

In the general case, the dimension is $(n-n) + k -$ fiber codim of $N' - \beta$, which is codimension $\geq$ fiber codim of $N' + \beta - a \geq \Sigma(\underline{g}_\alpha - \underline{n}_\alpha) + \beta - a$. To see this, note that any vector from the γth equivalence class can be expressed as $\sum_{i \in \mathbf{C}_\gamma} \sum_{j \in S_i} \lambda_j \underline{\mathscr{A}}_j$. The independence property of the $\underline{A}_j$'s where $j \in S_i$ and $\Sigma\, \underline{A}_j \in N_i$, implies that once the directions $\underline{\mathscr{A}}_j$ are known, then a choice of some λ_j uniquely determines the remaining scalar multiples.

So, say λ_1 of S_1 is determined. Since S_1 is one element of a chain related equivalence class, there exists another set, say S_2, such that $\ell \in S_1 \cap S_2$. The value of λ_ℓ has been determined by λ_1, and in turn, the value of λ_ℓ determines the values of the scalars with indices in S_2. Consequently λ_1 and the directions $\underline{\mathscr{A}}_i$ determine the values of λ_ℓ, $\ell \in S_1 \cup S_2$. A simple induction argument using the chain related definition of an equivalence class shows that any vector in a fiber of N' is uniquely determined by the directions $\underline{\mathscr{A}}_i$ and β scalars, one from each equivalence class.

The conclusion now follows by our standard argument applied to mapping h.

Because h can be viewed as a projection, it trivially satisfies the transversality condition, and $h^{-1}(h(N'))$ is a submanifold in $J^1|M_{\emptyset,P} \times \mathbb{E}^a \times L$ with codimension equal to fiber codim $N' + \beta - a \geq \Sigma^\beta(\underline{g}_\gamma - \underline{n}_\gamma) + \beta - a$, or equal to $\Sigma^\alpha(g_i - n_i) + \alpha - a$ in the coalition-like case. The remainder of this proof follows that of Theorem 7, yielding in the equivalence class case that the submanifolds have dimension $m - n -$ fiber codim $N' - \beta + a \leq m - n - \Sigma^\beta(\underline{g}_\gamma - \underline{n}_\gamma) + \beta - a$. In the coalition-like case it is equal to $m - n - \Sigma^\alpha(g_i - n_i) + \alpha - a$. This completes the proof.

It remains to determine the fiber codimension of N'. Before we state the result, we shall determine N' for two special cases of the pure exchange economy with externalities. Let $c = 2$, $a \geq 3$, $u_1(x) = u_1(x_1^1, x_1^2;\ x_2^1;\ x_i^2)$, $u_2(x) = u_2(x_1^2;\ x_2^1, x_2^2; x_i^1)$, $u_i(x) = u_i(x_i^1, x_i^{2i})$, $\underline{A}_1 = (A_1^1, A_1^2;\ A_1^3, 0; \ldots; 0, A_1^{2i}, \ldots)$,

$\underline{A}_2 = (0,A_2^2;\ A_2^3,\ A_2^4;\ldots;A_2^{2i-1},0;\ldots)$ and
$A_i = (0,0;\ 0,0;\ldots;\ A_i^{2i-1},A_i^{2i};0,0;\ldots)$ where $i = 3,4,\ldots,a$. Then
$P = [(\{1,2,\ldots,a\},N),\ (\{1,3,4,,\ldots,a\},\ N_1;\ \{2,3,\ldots,a\},\ N_2)]$ where
N_j is the space spanned by $(\underline{e}_j;\ \underline{e}_j;\ldots;\underline{e}_j)$ and $\underline{e}_1 = (1,0)$,
$\underline{e}_2 = (0,1)$. A straight forward computation shows for the second element of P that the fiber of $\pi_1^{-1}(f_1^{-1}(N_1))$ is
$\{(\underline{A}_1,\underline{A}_2,\ldots,\underline{A}_a)\,|\,A_1^1 = A_1^3 = A_i^{2i},\ 0 = A_1^2 = A_2^{2i} + A_i^{2i}\}$, the fiber of
$\pi_2^{-1}(f_2^{-1}(N_2))$ is $\{(\underline{A}_1,\underline{A}_2,\ldots,\underline{A}_a)\,|\,A_2^2 = A_2^4 = A_i^{2i},\ 0 = A_2^3 = A_2^{2i-1} + A_i^{2i-1}\}$, and the fiber of N' is given by the intersection of these two sets. Notice that it has dimension 2, since, for example, specifying values for A_1^1 and A_2^2 uniquely determines the remaining entries. Thus the fiber codimension of N' is
$2(a-2) + (a-2) + 3 + (a-2) + 3 - 2 = 4a - 4 = (g_1 - n_1) + (g_2 - n_2)$.
In this case for a generic choice of u, the submanifold corresponding to the second element of P is empty.

In the above example, change $u_2(x)$ to $u_2(x) = u_2(x_1^1;x_2^1,x_2^2;x_i^2)$.
Thus $\underline{A}_2 = (A_2^1,0;\ A_2^3,A_2^4;\ldots;\ 0,A_2^{2i};\ldots)$, and the second element of P becomes $(\{1,3,4,\ldots,a\},\ N_1;\ \{2,3,\ldots,a\},\ N_1)$. The fiber of $\pi_1^{-1}(N_1))$ is the same, but the fiber of $\pi_2^{-1}(f_2^{-1}(N_2))$ now becomes
$\{(\underline{A}_1,\underline{A}_2,\ldots,\underline{A}_a)\,|\,A_2^1 = A_2^3 = A_i^{2i-1},0 = A_2^4 = A_2^{2i} + A_i^{2i}\}$. In this case the fiber of N' has dimension a - 1. To see this, notice that spcifying the value of A_1^1 determines the values of A_1^3,A_i^{2i-1}, A_2^2 and A_2^4. There are a - 2 additional free variables A_i^{2i}. The fiber codimension of N' is $4a - 2 - (a-1) = 3a - 1$. While the fiber codimension differs, the submanifold in $\mathring{\Omega}$ is empty (generically).

The differences between the fiber codimension in these two examples motivates the following theorem. In the second example, the two normal spaces are the same. This plays a role in determining the codimension of N' <u>only</u> because some entry in $\pi_1^{-1}(f_1^{-1}(N_1))$ determines the entries in $\pi_2^{-1}(f_2^{-1}(N_1))$. Also, since both normal spaces are the same, certain coordinates are determined twice, namely the A_i^{2i-1} terms. There are a - 2 of these. Consequently we might expect the codimension to be given by these adjustments to the term $(g_1 - n_1) + (g_2 - n_2)$, or $(g_1 - n_1) + (g_2 - n_2) + n_2 - (a - 2) = (2a - 2) + (2a - 2) + 1 - (a - 2) = 3a - 1$. Theorem 9 is an extension of this type of computation. Before stating the theorem, terms corresponding to these adjustments will be defined.

Consider $\{S_1,N_1; S_2,N_2;\ldots;S_\alpha,N_\alpha\}$ with β equivalence classes and $\mathbb{C}_\gamma$, $\gamma = 1,2,\ldots,\beta$ the set of indices. Furthermore, assume the indices of $\mathbb{C}_\gamma$ are listed in increasing order. Let $j \in \mathbb{C}_\gamma$. Some of the coordinates of $\mathbb{R}^{g_j}$ are determined by entries with subscripts in $\bigcap_{\substack{i<j \\ i\in\mathbb{C}_\gamma}} S_i$. Consequently there are other equations relating these variables with indices $i < j$ in $\mathbb{C}_\gamma$. Let $R_{j,\delta}$ be the number of new equations, that is, those given in the definition of $\mathbb{R}^{g_j}$, which can be expressed as a linear combination of the previous equations. Let $\hat{m}_{j,\delta}$ be the number of subspaces of N_j which has at least one of these equations.

For the first example given above, $R_{2,1} = 0$; and for the second $R_{2,1} = a - 2$, namely the coordinates x_i^1, $i \geq 3$. The equation is the same in either case. Also for the second example, $\hat{m}_{2,1} = 1$.

According to our specification of the index set, if i is the smallest integer in set $\mathbb{C}_\delta$, then $R_{i,\delta} = \hat{m}_{i,\delta} = 0$.

Theorem 9. Let $\{S_1,N_1;\ S_2,N_2;\ldots;S_\alpha,N_\alpha\}$ be an element of P with β equivalence classes given by index sets $\mathbb{C}_\gamma$, $\gamma = 1,2,\ldots,\beta$. There exists a residue set $\mathscr{B} \subset C^2_{\mathcal{U}}$ such that $u \in \mathscr{B}$ implies that the part of θ_u in $M_{\emptyset,P}$ corresponding to this element of P is either empty or an

$$6.1 \qquad m - n + a - \beta - \sum_{\gamma=1}^{\beta} \sum_{i\in\mathbb{C}_\gamma} (g_i - n_i + \hat{m}_{i,\gamma} - R_{i,\gamma})$$

dimensional submanifold. If $M_{\emptyset,P}$ is precompact, then $\mathscr{B}$ can be selected to be open-dense.

By use of standard combinatoric arguments from linear algebra (arguments used to determine the dimensions of subspaces), or probability (counting measures), alternate forms of this equation can be obtained, including some which display the independence of the ordering of the subscripts. For example, for each coordinate in $\mathbb{E}^{g_\gamma}$ there may be more than one defining equation relating the components A_i^j. These equations can be expressed in matrix form, and the corank of this matrix with respect to the target space can be computed. It can be seen from the following proof that $\sum_{i\in\mathbb{C}_\gamma} R_{i,\gamma}$ is the sum of these coranks. A similar explanation in terms of how the N_i's relate to each other and the independence of these relationships holds for the term $\Sigma\, \hat{m}_{i,\gamma}$. However, the above equation seems to be the easiest to use when considering a specific example.

Proof. According to the proof of Theorem 8, all we need show is that the fiber codimension of N' is equal to the value of the double summation. This is merely a counting argument to determine the number of degrees of freedom in the equations $\sum_{\substack{i \in S_i \\ \ell \in \mathbb{C}_\gamma}} A_i^j = h_{j,\delta}$.

That is, we need to count the number of terms $h_{j,\gamma}$ which are free to be selected, and the number which are consequently determined.

Start with the equivalence class with indices in $\mathbb{C}_\gamma$, and assume for convenience that $C_\gamma = \{1,2,\ldots,r\}$. There are g_1 equations relating the components with subscripts in S_1, n_1 of which are free to be determined and $g_1 - n_1$ of which are then fixed. In S_2, there are $g_2 - n_2$ determined equations. However, $\hat{m}_{2,\gamma}$ of the n_2 equations where the right hand side was supposedly free to be determined are already fixed because they can be expressed in terms of linear combinations of the coordinates already known; that is, coordinates with subscripts in $S_1 \cap S_2$. Thus the number of determined equations is $g_2 - n_2 + \hat{m}_{2,\delta}$.

On the other hand, some of these equations consists of variables already evaluated, variables with subscripts in $S_1 \cap S_2$. Of these equations, $R_{2,\delta}$ of them add no new restrictions or information on the coordinates. Thus the number of determined equations relating to the variables is $g_2 - n_2 + \hat{m}_{2,\delta} - R_{2,\delta}$, and the total number of determined equations for variables with subscripts in $S_1 \cup S_2$ is $(g_1 - n_1) + (g_2 - n_2) + \hat{m}_{2,\delta} - R_{2,\delta}$. In other words, this is the number of linearly independent constraining equations.

Assume the equation holds for $i = m - 1$. We shall show it holds for $i = m$. To do so we need only determine the number of determined equations involving new coordinates. But this is clearly $g_m - n_m + \hat{m}_{m,\delta} - R_{m,\delta}$, and the induction argument is completed.

That this argument does not depend upon the labeling of the indices follows by a standard alteration of the above induction argument or by the fact we are computing the codimension of the fiber of N', a term which is clearly independent of the ordering. Since there may be more than one equation involving the same coordinates, and this can be true only for an equivalence class of more than one element, there is a question of consistency. However a moments reflection reveals that this is handled in the definition of an element of P. If there are inconsistent equations, then such an element can not belong to P. This corresponds to the "empty" part of our conclusion.

We now show that for a pure exchange economy Equation 6.1 is not always negative. Let $c = 2$, and let $\mathscr{U}$ be the class $u_1(x) = u_1(x_1^1, x_1^2;\ x_2^1)$ $u_2(x) = u_2(x_1^1;\ x_2^1, x_2^2)$, and $u_i(x) = u_i(x_i^1, x_i^2)$ for $i = 3,4,\ldots,a$. Then $P = [(\{1,2,\ldots,a\}, N), (\{1,3,\ldots,a\}, N_1;\ \{2,3,\ldots,a\}, N_2)]$ where $N_1 = N_2$ is the space spanned by $(\underline{e}_1;\ \underline{e}_1;\ldots;\underline{e}_1)$. Computing the parameters for the second element yields

$$g_1 - n_1 = 2(a - 2) + 3 - 1 = g_2 - n_2,$$

$$\hat{m}_{2,1} = 1,\ R_{2,1} = 2(a - 2),\ \mathfrak{s} = 1, \text{ and } \mathbb{C}_1 = \{1,2\}.$$

The dimension of the submanifold of θ_u corresponding to this element is $2a - 2 + a - 1 - [2(a-1) + 2(a-1) + 1 - 2(a-2)] = a - 4$. Thus if that part of θ_u corresponding to this element of P is non-empty then when $a = 4$, it is, generically, the union of isolated points, when $a = 5$, it is, generically, the union of lines and/or images of the circle (see Section 7), when $a = 6$, it is, generically, a two manifold, etc.

The term $R_{2,1}$ played an important role in this example. Indeed, it is a simple exercise to show for the pure exchange economy that if $\Sigma R_{i,1} = 0$, then the submanifold is empty. Actually, it is not the magnitude of this sum which is important, but rather that part of the sum corresponding to redundancies in the non-zero terms defining $N_1 \cap N_2$.

Finally we come to the question of modifying the above to include the structure of points on $M_{Z,P}$ where $Z \neq \emptyset$. The easiest way to do this is to change sentence 2 of Definition 6.1 to read that it is impossible to select positive μ_i's such that the normal condition holds for any non-empty proper subset of any $S_k \cap Z'$. With this change, the definition of all the terms and parameters extend in an obvious fashion from $M_{\emptyset,P}$ to $M_{Z,P}$. Modifications of the proofs follow the same line as those given for Theorem 7.

Corollary 9.1. Let $\delta = \{S_1,N_1;\ldots;S_\alpha,N_\alpha\}$ be an element of P with β equivalence classes given by index sets $\mathbb{C}_\gamma$, $\gamma = 1,2,\ldots,\beta$. There exists a residue set $\mathscr{B} \subset C^2$ such that $u \in \mathscr{B}$ implies that the part of θ_u in $M_{Z,\delta}$ corresponding to this element of P is either empty or an

$$\dim(M_{Z,\delta}) + \text{card } Z' - \beta - \sum_{\gamma=1}^{\beta} \sum_{i \in \mathbb{C}_\gamma} (g_i - n_i + \hat{m}_{i,\gamma} - R_{i,\gamma})$$

dimensional submanifold. If $M_{\emptyset,P}$ is precompact, then $\mathscr{B}$ can be selected to be open dense.

§7. Examples.

In the preceding two sections the generic structure of θ_u was obtained. However, set θ_u contains more than just LPOs; for example, it also includes all p which serve as a LPO for $-u$. Higher order terms are needed to distinguish between the LPOs and other entries in θ_u. This we did in Theorem 3 for second order terms where p is in $M_\emptyset$ or $M_{\emptyset,P}$.

Let $\tau_u = \{p\in\mathring{\Omega} \mid p\in\theta_u$ and $\underline{\lambda}\cdot u$ satisfies the second order sufficient conditions for a maximum$\}$. A description of the generic structure of τ_u is precisely the same as the statements given for θ_u in Theorems 7-9. To see this, consider $J^2(M,\mathbb{R}^a)$. The first order conditions are the same as derived earlier. The second order conditions are open. Consequently, the codimension of the manifold in $J^2(M,\mathbb{R}^a)$ corresponding to the definition of τ_u is the same as the codimension corresponding to the definition of θ_u. Of course, it is a submanifold, since <u>all</u> possible second order terms are permitted in the definition of θ_u. Consequently if j^2u is transverse to this manifold determined by τ, then the inverse image of this manifold under mapping j^2u is either empty or of the same dimension as θ_u. By the Thom Transversality Theorem, this is true for either a residual set $\mathscr{B}$, or $\mathscr{B}$ can be chosen to be open dense depending on the boundedness properties of $\mathring{\Omega}$, $M_\emptyset$, etc.

τ_u is <u>not</u> the set of all LPOs, but rather the set of all non-degenerate LPOs. The behavior of the total set involves an analysis of higher order terms, and it will be discussed elsewhere.

For the remainder of this paper we shall review some of the results proved earlier to point out some of the implications and how some of the results tie in with each other. This discussion will be in terms of exchange economies without externalities

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To replace material from page 63, line 5 to the bottom of page 65.

If $a = 2$, then $\theta_u \cap \mathring{\Omega}$ is a one-dimensional submanifold. But one dimensional submanifolds are the disjoint union of images of open intervals and images of the unit circle. On the other hand, θ_u can be viewed as the inverse image of a closed set. This is because $\theta_u \cap \mathring{\Omega}$ is the inverse image of $(j^2 u)^{-1}(\Sigma)$, where $\Sigma = \mathring{\Omega} \times \mathbb{R}^a \times L_u$ and $L_u = \{(\underline{A}_1, \underline{A}_2) \mid \underline{A}_i \in \mathbb{R}^c,$ $\exists\, \lambda_i > 0 \ni \lambda_1 \underline{A}_1 + \lambda_2 \underline{A}_2 = \underline{0}\}$. Since it is assumed that $\nabla u_i(p) > 0$ for $p \in \mathring{\Omega}$ and Ω is compact, for a given u we can restrict L_u to a closed subset. This means $\theta_u \cap \mathring{\Omega}$ is closed in $\mathring{\Omega}$. Consequently any images of line intervals in θ_u must be both open and closed in $\mathring{\Omega}$. This is only possible if θ_u terminates on the boundary of $\mathring{\Omega}$. Therefore, θ_u consists of the disjoint union of images of line intervals with end points on the boundaries of $\mathring{\Omega}$ and images of the unit circle. It is not difficult to construct examples where all of these occur.

This description is for θ_u. The description for τ_u is not the same. This is because τ_u is not the inverse image of a closed set. The second order sufficient conditions for a pareto optimum forms an open condition in J^2. However, it must be a subset of θ_u. This already is a statement concerning the structure of τ_u. Furthermore, should $a = c = 2$, then τ_u cannot contain the image of a circle. (For simplicity, we will call this diffeomorphic image of a circle a circle.)

Suppose it did. Then there is some value of u_1, say d_1, such that this level set supports this circle. That is, this circle meets $u_1 = d_1$ in the Edgeworth box, and if it meets any other level set of u_1, it is for values of d on only one side of d_1 (say $d > d_1$).

We next show that the circle (generically) meets the level set $u_1 = d_1$

points in the domain, and in our case we have a continuum.

Now, let p be an isolated point in the intersection of $u_1 = d_1$ and the circle. By continuity considerations, for $d > d_1$ the level set $u_1 = d$ intersects the circle in at least two points. Indeed, two points can be selected which correspond to maxima of u_2 restricted to $u_1 = d$, and two distinct families $\{p_{1,d}\}, \{p_{2,d}\}$ can be selected from the circle so that $p_{i,d} \to p$ as $d \to d_1$. Since $c = 2$, in the Edgeworth box representation of this problem the level set $u_1 = d$ is (locally) diffeomorphic to a line segment. (This follows from the inverse function theorem and the assumption that $\nabla u_1(p) \neq \underline{0}$.) Therefore, it follows from elementary analysis that when u_2 is restricted to the level set $u_1 = d$, it must have a local minimum between $p_{1,d}$ and $p_{2,d}$. Let q_d denote this point. Notice that $q_d \to p$ as $d \to d_1$.

According to the necessary conditions from Lagrange Multiplier Theory, there exist scalars $\lambda(p_{i,d})$ and $\mu(q_d)$ such that

$$\nabla u_1(p_{i,d}) = \lambda(p_{i,d}) \nabla u_2(p_{i,d})$$

and

$$\nabla u(q_d) = \mu(q_d) \nabla u_2(q_d).$$

The assumption that $p_{i,d}$ are pareto points determines the sign of $\lambda(p_{i,d})$. The condition that $\nabla u_2(p) \neq \underline{0}$, plus continuity conditions, require that for sufficiently small $d - d_1 > 0$, $\mu(q_d)$ has the same sign. Thus $p_{i,d} \in \tau_u \subset \theta_u$ and $q_d \in \theta_u$.

This gives the desired contradiction, because in any neighborhood V of p, $V \cap \theta_u$ is not a one-manifold since it always contains q_d for some d.

(Example 1, Section 2.). One advantage of Example 1 is that we can use an Edgeworth box description for $a = 2$.

For Example 1, restrict attention to those utility mappings where $\nabla u_i(p) \neq 0$ for $p \in \mathring{\Omega}$. We then have that there exists an open dense set of mappings $\mathscr{B}$ such that $u \in \mathscr{B}$ implies that for each of the sets $\Theta_u \cap \mathring{\Omega}$ and $\tau_u \cap \mathring{\Omega}$ one of the following must occur. Either the set is empty, or it is an $a - 1$ dimensional submanifold of $\mathring{\Omega}$. If the first mentioned set is empty, then so is the second; but the converse is not necessarily true. We shall discuss the case where $\tau_u \cap \mathring{\Omega} \neq \emptyset$.

If $a = 1$, then this submanifold is zero dimensional. That is, it is the union of isolated points. However, if $a = 1$ there is only one agent, and the LPO is a maximum point. Consequently, this result is not surprising. It merely states that in the general case maximum points are isolated.

If $a = 2$, then $\tau_u \cap \mathring{\Omega}$ is an one-dimensional submanifold. But one dimensional submanifolds are the disjoint union of images of an open interval and the unit circle. On the other hand, τ_u is the inverse image of a zero condition. To see this, recall that $\lambda_1 \nabla u_1(p) + \lambda_2 \nabla u_2(p) \in N_p$. Therefore the projection, $\mathscr{P}$, of this vector onto $\mathring{\Omega}$ is zero. Namely, $\theta_u = \{p \mid$ there exist $\lambda_1, \lambda_2 > 0$ such that $\mathscr{P}(\lambda_1 \nabla u_1(p) + \lambda_2 \nabla u_2(p)) = \underline{0}\}$. Therefore, $\tau_u \cap \mathring{\Omega}$ is closed. This means that any images of line intervals in τ_u must be both open and closed in $\mathring{\Omega}$. This is possible only if the line interval terminates on the boundary of $\mathring{\Omega}$.

We now show that this one dimensional submanifold cannot contain images of a circle. (On the other hand, for the location problem, Example 2b, it can only be a circle.) This is a by product of the following facts to be found in Simon and Titus [11]. a) If p is a LPO for the pure exchange problem without externalities, and

$u_1(p) = D$ then p is a local maximum for u_2 when restricted to the level surface $\{u_1 = D\} \cap \mathring{\Omega}$. The converse is also true. If $u_1(p) = D_1$ and $u_2(p) = E_1$, p is a local maxima for u_1 when restricted to $\{x \in \mathring{\Omega} | u_2(x) = E_1\}$ and a local maxima for u_2 when restricted to $\{x \in \mathring{\Omega} | u_1(x) = D_1\}$, then p is a LPO.

(Notice, for a = 2 part of this result follows almost directly from the Lagrange multiplier theorem, where we treat a level set of u_1 as a constraint.)

b) In some neighborhood of p in $\mathring{\Omega}$, there exists a change of coordinates such that $u_1(x)$ can be represented as $u_1(x) = x_1$.

From these we prove the following two Lemmas.

Lemma 7.1. <u>Let</u> $p \in \mathring{\Omega}$ <u>be a non-degenerate</u> LPO, <u>and assume that in some neighborhood</u> $\mathscr{V}$ <u>of</u> p, $u_1(x) = x_1$. <u>Let</u> $u_1(p) = D_1$. <u>Assume that</u> p <u>is a strict local maximum for</u> u_2 <u>when restricted to</u> $\{x \in \mathring{\Omega} | u_1(x) = D_1\}$. <u>Then there is a cylinder</u> $\mathscr{D} \subset \mathring{\Omega}$, <u>which is the product of a sufficiently small disk centered at</u> p <u>in the hyperplane</u> $\{x | u_1(x) = D_1\} \cap \mathring{\Omega}$ <u>times the</u> x_1 <u>axis</u>, <u>with the following property</u>: <u>For any value of</u> D <u>sufficiently close to</u> D_1, u_2 <u>has a local maximum in</u> $\{x | u_1(x) = D\} \cap \mathscr{D}$.

This shows that the circle cannot intersect the level sets of either utility function in isolated points. If it did, then there would exist p on the circle and constant D_1 such that, at least locally, the circle is on one side of the level set $\{x | u_1(x) = D_1\}$, say for $D \geq D_1$. But according to the above lemma and sentence a, there are LPOs arbitrarily close to p on the half space $D < D_1$. This would lead to a structure in τ_u which is not generically admissible.

Lemma 7.2. <u>Generically</u>, <u>any</u> LPO <u>on the level set</u> $\{x|u_1(x) = D_1\}$ <u>is isolated</u>.

This states that generically we can expect τ_u to intersect level sets in an isolated fashion. The combination of these statements proves the assertion. Only the proof of the lemma remains.

The proof of Lemma 7.2 is most easily obtained by referring to Saari and Simon [10], where, essentially, they showed for this example that at a LPO $p \in \mathring{\Omega}$, u_2 can be expressed as a Morse function relative to the level set $\{x|u_1(x) = D_1\}$. This means the point p is isolated [8].

Proof of Lemma 7.1. For a sufficiently small disk about p in $\{x|u_1(x) = D_1\} \cap \mathring{\Omega}$, the gradient (more precisely - the projection of the gradient) of u_2 is pointing inward along the boundary. Take the product of this disk with the x_1 axis and call it $\mathscr{D}$. For all D sufficiently close to D_1, and have that $\{x|u_1(x) = D\} \cap \mathscr{D}$ is a disk. By continuity considerations, for sufficiently small $D - D_1$, the gradient of u_2 points inward along the boundary of this disk. According to the Brouwer fixed point theorem, or the Hopf index theorem, [7], the gradient of u_2 has a zero in the interior of the disk. Clearly this is a local maximum point for u_2 relative to the level surface of u_1. This completes the proof.

Thus, for a = 2, the generic structure of τ_u (and, by a similar argument, of θ_u) is a union of disjoint lines terminating at the boundaries.

For a > 2, τ_u is an a - 1 dimensional manifold. Of course here the classification becomes much harder, and we stop at this point. However, extensions of the above lemmas exist.

We continue with a = 2, but now we relax the condition that $\nabla u_i(p) \neq \underline{0}$ for i = 1,2. From Corollary 4.2 we see that if $p \in \mathring{\Omega}$ is a LPO and $\nabla u_1(p) \neq \underline{0}$, $\nabla u_2(p) = 0$, $K = \{\underline{0}\}$, then $D^2 u_p$ is negative definite. That is, p is a satiation point for u_2. Furthermore, it is easy to see that the generic structure of such points form a zero dimensional submanifold of $\mathring{\Omega}$. Namely, it is the union of isolated points. Let τ_u' contain such LPOs.

The generic structure of τ_u' can now be determined. The part of τ_u' corresponding to LPOs where neither gradient is zero consists of unions of disjoint lines in $\mathring{\Omega} - \{p \mid \nabla u_1(p) = 0$ or $\nabla u_2(p) = 0\}$, and each of these lines terminates at the boundary. The boundary now <u>may</u> be interior points of $\mathring{\Omega}$. The closure of these lines <u>need not</u> be part of τ_u'. For an example where this is the case let c = 2, $p \in \mathring{\Omega}$, $u_1(x) = (x_1^2 - p_1^2)^2 - (x_1^1 - p_1^1)^2$ and $u_2(x) = \alpha x_2^1 + \beta x_2^2$ where $\alpha^2 + \beta^2 \neq 0$. (See Figure 1). Indeed, recall that if $\nabla u_1(p) = 0$, $D^2 u_1(p)$ is not positive definite, and $\nabla u_2(p) \neq 0$, then p is not a LPO (Corollary 4.2).

The reader may be interested in comparing the structure of Θ_u versus that of τ_u' for this example. In this case a branch of Θ_u, corresponding to non-pareto points, meets p. Assuming $K_1 \cap L_{1,p} = \{0\}$, and $\nabla u_2(p) \neq 0$, one can show this always occurs.

On the other hand, if $p \in \tau_u'$ is such that either $\nabla u_1(p)$ or $\nabla u_2(p)$ is zero, say $\nabla u_1(p) = 0$, then it is easy to show that it <u>cannot</u> be an isolated point of τ_u'. Namely, at least one (actually, precisely two), one dimensional arcs of LPOs terminates at p. This is because the level sets of u_1 near p are essentially ellipses. (The second order term dominates near p. The ellipse statement follows from the Jordan canonical representation of $D^2 u_1$).

We cannot expect, at least generically, that there is some

$p \in \mathring{\Omega}$ which is a LPO such that $\nabla u_1(p) = \nabla u_2(p) = 0$. The representation of such a point in $J^1_{\mathcal{U}}$ would be $\mathring{\Omega} \times \mathbb{R}^a \times \underline{0}$ which has codimension 2c. Thus, generically, such points in Θ_u or τ_u' would form a submanifold of dimension $c(a - 1) - 2c = -c < 0$.

We can ask other questions about the structure of τ_u'. For example, what part of τ_u' has the gradient of all utility functions equal to zero in the first component. It follows from Theorem 8 that it is an a - 2 dimensional submanifold of Θ_u.

A description similar to the one given above holds for $\mathring{\Omega}$ where M_Z or $M_{Z,P}$ are non-empty for $Z \neq \emptyset$. The set τ_u in $M_\emptyset$ consists of manifolds which may need to terminate at the boundary of $M_\emptyset$. However, this boundary may contain M_Z for $Z \neq \emptyset$. Thus the manifolds join on M_Z with the appropriate dimension.

We conclude this discussion by briefly discussing the non-generic LPOs. There exist smooth utility functions such that their set of LPOs does not match the above description. Such utility functions lie in the complement of some open-dense set $\mathscr{B} \subset C^3_{\mathcal{U}}(M,\mathbb{R}^a)$. Consequently an arbitrarily small perturbation of such a utility function can change the structure into one corresponding to the above description. An example given by H. Sonnenschein is given in Figure 2. Notice that the Pareto set is <u>not</u> the <u>disjoint</u> union of one dimensional manifolds terminating at the boundary. Since $\nabla u_i(p) \neq 0$ for $p \in \mathring{\Omega}$, this structure cannot be explained in terms of satiation points. However, a small perturbation of this utility function splits the Pareto set into two disjoint lines. They are given by the dotted lines in the figure.

In this figure the two maxima are converging to a single point. A small perturbation, say a $(x-p)^2$ term added locally, keeps the maxima from converging and the Pareto set splits as indicated.

The natural problem is to determine a systematic way of determining how these non-generic solution concepts split under small perturbations. The answer is known, and it can be found in the study of unfolding of singularities and in Catastrophe Theory [3,14,15]. We will not carry out all the details, but we will show for an example how Catastrophe Theory gives the answer. The example we use is somewhat more complicated than the previous one in that ∇u_2 vanishes in $\mathring{\Omega}$.

On some open set containing the origin of $\mathbb{R}^2$, let $u_1(x_1,x_2) = x_1$ and $u_2(x_1,x_2) = \frac{x_2^4}{4} - \frac{x_1 x_2^2}{2}$. The gradients are $\nabla u_1 = (1,0)$, $\nabla u_2 = (\frac{-x_2^2}{2}, x_2^3 - x_1 x_2)$. Since the normal space is $\{\underline{0}\}$, (x_1,x_2) is in θ if and only if $x_2^3 - x_1 x_2 = 0$. Notice $(x_1,x_2) = 0$ is a critical point where the null space of $D^2 u_2$ is non-empty. The solid line in Figure 3 shows the set θ_u. τ_u can be seen to be the Y formed by the outside branch.

In summary, we are interested in the zero set of the gradient of u_2 with respect to its second variable. At (0,0) this is not a Morse function, which is a non-generic situation for a critical point ([4]). We would like to perturb this potential function. Catastrophe theory states that for potential function $\frac{x_2^4}{4}$, the generic perturbation is of the form $\alpha \frac{x_2^2}{2} + \beta x_2$. That is, $\frac{x_2^4}{4} + \alpha \frac{x_2^2}{2} + \beta x_2$ for $\alpha^2 + \beta^2 \neq 0$ is in $\mathscr{B}$. Clearly $\alpha = -x_1$. So, a small perturbation of u_2 is $u_2(x) + \beta x_2$, and the gradient of the perturbed u_2 with respect to x_2 is $x_2^3 - x_1 x_2 + \beta$. The new pareto set is given in Figure 4. The dashed line corresponds to τ_u', while the dotted plus the dashed line gives θ_u. The point where $\nabla u_2 = 0$ has moved, and it is not part of τ_u', although its null space is empty. The reader familiar with catastrophe theory will recognize that the two representations of θ_u, for $\beta = 0$ and for $\beta \neq 0$, are merely two cross-sections of the cusp fold. The point where $\nabla u_2(p) = 0$ corresponds to the fold point in one of the surfaces.

This is not an isolated example. Its resolution depends upon the fact that the order of contact of u_2 with respect to the level surface of u_1 is of degree 3. For any problem where this is the case, there is a change of coordinates such that the problem is reduced to a similar formulation. The main difference being that the choice of parameter α may not be x_1. This also works for finite $c \geq 1$.

If the singular point is x_2^n, then a choice of the local perturbations is $\alpha_1 x_2^{n-2} + \alpha_2 x_2^{n-3} + \cdots + \alpha_1 x_2$. As long as $\nabla u_i(p) \neq 0$ for all the other agents, this type of a program can be carried out using basic ideas from catastrophe theory. Higher order singularities are more complicated.

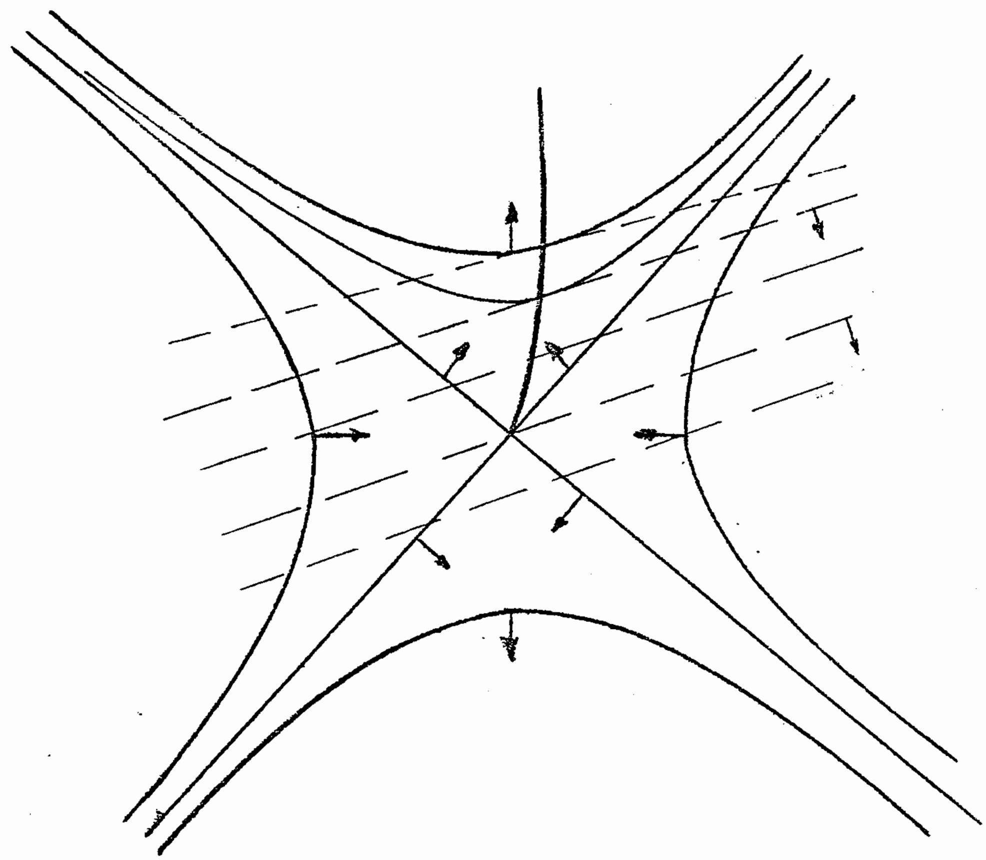

Figure 1. The heavy solid line shows the set of local pareto points. Notice that it terminates at an interior point. Set Θ_u continues into the opposite quadrant, but it consists of points which are not pareto points.

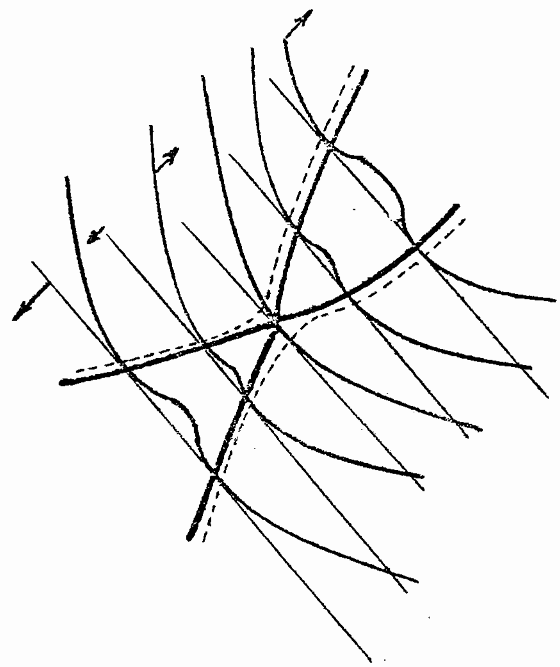

Figure 2. The heavy solid line is the set of local pareto points. Since it is not a manifold, it does not represent the generic case. The dotted line represents how the pareto set splits under a general perturbation.

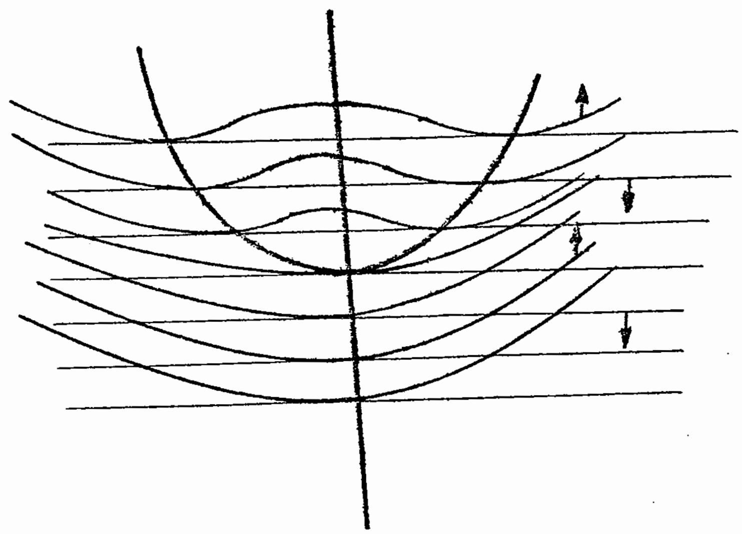

Figure 3. The heavy solid line represents the set Θ_u, and it includes non-pareto points. At the point where the two lines meet, the gradient of one of the level curves is equal to zero.

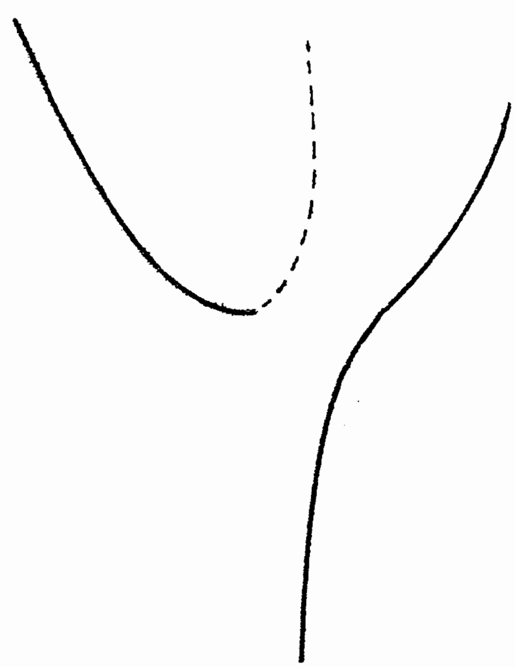

Figure 4. Under a general perturbation, the set Θ_u given in Figure 3 divides as given here. The heavy solid line is the set of pareto points. The dotted line represents that part of Θ_u which contains the non-pareto points.

References

1. Debreu, G., 1959, Theory of Value (Yale University Press, New Haven, Conn.)
2. Dierker, E, 1974, Topological Methods in Walrasian Economics (Springer-Verlag, New York)
3. Golubitsky, M., 1976, An Introduction to Catastrophe theory and its applications, (Preprint, dated July, 1976.)
4. Golubitsky, M., and V. Guillemin, 1973, Stable Mappings and Their Singularities (Springer, New York).
5. Hurwicz, L., 1972, On informational decentralized systems, in McGuire, C. B., and R. Rachner, ed., Decision and Organization (North Holland, Amsterday), Chapter 4.
6. Milner, J., 1974, Characteristic Classes, Annals of Mathematic Studies No. 76 (Princeton University Press, Princeton).
7. Milner, J., 1965, Topology from the Differentiable Viewpoint, (University of Virginia Press; Charlottesville, Va.)
8. Milner, J., 1963, Morse Theory, Annals of Mathematic Studies No. 51 (Princeton University Press, Princeton).
9. Mount, K., and S. Reiter, 1974, The informational size of message spaces, Journ. Econ. Theory, 161-192.
10. Saari, D. and C. Simon, 1975, Singularity theory of utility mappings I: Degenerate maxima and Pareto optima. Discussion paper No. 180, Center for Mathematical Studies, Northwestern, and, to be published elsewhere.
11. Simon, C. and C. Titus, 1975, Characterization of optima in smooth Pareto economic systems, Jour. of Math Econ., 2.
12. Smale, S., 1973, Global analysis and economics I, Pareto optimum and a generalization of Morse theory, in: M. Peixoto,

ed., Dynamical Systems (Academic Press, New York).

13. Smale, S., 1974, Global analysis and economics III, Pareto optima and price equilibria, Journal of Math. Econ. 1, 107-117.

14. Thom, R., 1975, Structural Stability and Morphogenesis (Benjamin, Reading, Massachusetts).

15. Zeeman, C., 1974, The classification of elementary catastrophes of codimension ≤ 5. (University of Warwick preprint, dated June, 1974.)