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TWO-PERSON BARGAINING PROBLEMS AND COMPARABLE UTILITY

by

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Abstract: Four conditions are shown to imply together that a solution function for two-person bargaining problems must equalize gains in some ordinal utility scales. These conditions are: weak Pareto-optimality, strong individual rationality, composition, and uniformity. The composition condition relates to sequences of bargaining problems. The uniformity condition requires that the solution function must be invariant under enough ordinal utility transformations to move any threat point to the origin.

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1. Introduction

SHAPLEY (1969) showed that a useful value or solution function for two-person bargaining problems cannot be invariant under all ordinal utility functions, so some restriction to cardinal utility would seem necessary. KALAI (1976) assumed cardinal utility and showed conditions under which a bargaining solution function must equalize all players' utility gains. Empirical studies by NYDEGGER and OWEN (1974) strongly suggested that people do make interpersonal comparison of gains in bargaining. In this paper we will build on these ideas to derive an equal-gains-in-bargaining condition, without any cardinality assumptions.

Consider an arbitrator responsible for helping two players to cooperate in game situations. For any bargaining problem which the players might face, he must be prepared to recommend a fair cooperative agreement for the two players. One simple procedure he could use is to propose the outcome which makes both players as happy as possible, subject to the constraint that the players should enjoy equal gains over their noncooperative alternative. To make such a comparison of gains, the arbitrator must measure the players' levels of happiness in some pair of scales for which he feels the differences are interpersonally comparable. That is, an equal-gains equity criterion is meaningful only if there is a pair of <u>comparable utility scales</u>.

The assumption that comparable utility scales exist has been generally avoided as much as possible in game theory. Interpersonal comparison has been sometimes considered theoretically defensible for scales having both transferability and von Neumann-Morgenstern risk-neutrality properties. However, utility comparison has significance quite independently of these other two properties.

In this paper we will show that, if an arbitration scheme satisfies certain desirable conditions, then it must be an equal-gains scheme for some pair of comparable utility scales. These comparable utility scales might be neither transferable nor risk-neutral von Neumann-Morgenstern utility scales.

2. Definitions and Notation

Let R be the set of real numbers. S is a proper subset of R^2 if and only if $S \subseteq R^2$ and $\emptyset \neq S \neq R^2$. S is a comprehensive subset of R^2 if and only if: $(u,v) \in S$ and $x \leq u$ and $y \leq v$ imply that $(x,y) \in S$.

A two-person bargaining problem is a triple (a,b,S) such that S is a closed comprehensive proper subset of R^2 and $(a,b) \in S$. Let BP be the class of all two-person bargaining problems. For any $(a,b,S) \in BP$, let ∂S be the boundary of S; that is:
$\partial S = \{(u,v) \mid \forall x > u,\ \forall y > v,\ (x,y) \notin S\}$.

An order-preserving transformation of the reals is a function $g\colon R \to R$ such that g is one-to-one, onto, and $x>y$ implies $g(x)>g(y)$. It can be easily shown that an order-preserving transformation must be a continuous function.

Suppose we have a function $F\colon BP \to R^2$ and an order-preserving transformation $g\colon R \to R$. Then we say that F is invariant under g applied to 1's utility if and only if, for any $(a,b,S) \in BP$, if $F(a,b,S)=(u,v)$ then $F(g(a),b,\{(g(x),y) \mid (x,y) \in S\}) = (g(u),v)$.

Similarly, F is <u>invariant under g applied to 2's utility</u> if and only if, for any $(a,b,S) \in BP$, if $F(a,b,S) = (u,v)$ then $F(a,g(b), \{(x,g(y)) | (x,y) \in S\}) = (u,g(v))$.

3. Solution Properties

We can now formally analyze the position of an arbitrator working to help some Player 1 cooperate with some Player 2. We assume that the arbitrator can measure each player's levels of happiness in a given real-valued utility scale. We do <u>not</u> assume any cardinal properties for the given utility scales; we only assume the ordinal property that higher utility numbers correspond to higher levels of happiness or preferred outcomes. Applying order-preserving transformations to the given utility scales would preserve this ordinal property.

With these given utility scales, the arbitrator can represent a bargaining situation for players 1 and 2 by a formal two-person bargaining problem (a,b,S), where S is the set of all <u>feasible utility allocations</u> available if the players cooperate, and (a,b) is the <u>threat point</u> or utility allocation which must result if they do not cooperate. (That is, Player 1 would get his utility level a and Player 2 would get his utility level b, if no cooperative agreement were reached.) The arbitrator must be prepared to propose some fair cooperative solution for any such bargaining problem. Thus, the arbitrator's behavior can be described by some function $F: BP \to R^2$, where $F(a,b,S)=(u,v)$ if the arbitrator would propose that Player 1 should get utility u and Player 2 should get utility v in a bargaining situation represented by (a,b,S). We can refer to $F: BP \to R^2$ as the arbitrator's <u>solution function</u>.

We list four conditions which a solution function might be expected to satisfy.

C1: (Weak Pareto-optimality.) For any $(a,b,S) \in BP$, $F(a,b,S) \in \partial S$.

C2: (Strong individual rationality.) For any $(a,b,S) \in BP$, if $(u,v) = F(a,b,S)$ then $u \geq a$ and $v \geq b$, and $u > a$ and $v > b$ if there exists any point $(x,y) \in S$ such that $x > a$ and $y > b$.

C3: (Composition.) For any $(a,b,S) \in BP$, if $T \supseteq S$ then $F(a,b,T) = F(F(a,b,S), T)$.

C4: (Uniformity.) For any $(a,b) \in R^2$, there exists order-preserving transformations g_1 and g_2 such that F is invariant under g_1 applied to 1's utility, F is invariant under g_2 applied to 2's utility, and $(g_1(a), g_2(b)) = (0,0)$.

C1 and C2 are similar to conditions in NASH (1950). C1 asserts that the arbitrator should select a feasible allocation on the weakly Pareto-optimal frontier, so that no other feasible outcome would, simultaneously make both players better off. C2 says that, if possible, both players should gain from their cooperation.

C3 is similar to the step-by-step negotiations axiom in KALAI (1976). Suppose that the players have faced the problem (a,b,S) and the arbitrator has gotten them to agree on $F(a,b,S)$. Suppose also that new opportunities for cooperative actions are now recognized, so that a larger feasible set T can be considered, and the bargaining has reopened. The new threat point must be $F(a,b,S)$, assuming that both players have to consent before the first session's agreement can be voided. Then C3 asserts that the outcome of the second bargaining session, $F(F(a,b,S),T)$, should be the same as if the bargaining had taken place in only one session with the problem (a,b,T).

There are two kinds of advantages for the arbitrator if F satisfies C3. It may be analytically easier to solve a complex bargaining problem if the negotiations can be broken up into several sessions (considering a few more cooperative opportunities at each session), as long as each player is confident that he will not lose by breaking up the problem this way. Also, C3 guarantees that neither player will have any incentive to try to change the order in which the cooperative opportunities are discussed.

C4 is a weaker version of the invariance axiom of NASH (1950), based on suggestions by SHAPLEY (1969). It assures us that any bargaining problem can be converted into an equivalent problem with threat point (0,0), by appropriate transformations of the players' utility scales. C4 puts an effective bound on the complexity of F, by requiring that F be qualitatively the same (modulo the given transformations) in all regions of R^2.

Notice that it would not be natural to require that F be invariant under affine transformations, since addition and multiplication may have no particular significance in the given ordinal utility scales.

Our principal result is that conditions C1 through C4 together imply the existence of comparable utility scales.

THEOREM. $F: BP \to R^2$ satisfies C1 through C4 if and only if there exist order-preserving transformations $U_1: R \to R$ and $U_2: R \to R$ such that, for any bargaining problem (a,b,S), F(a,b,S) is the point $(u,v) \in \partial S$ satisfying $U_1(u) - U_1(a) = U_2(v) - U_2(b)$.

4. Proof of the Theorem

In the numbered paragraphs of this section, the first sentence is a claim which is proven by the remainder of the paragraph.

The "if" part of the theorem, showing that an equal-gains scheme satisfies C1 through C4, is straightforward to check. Observe that, for any $(a,b,S) \in BP$, there is always a unique $(u,v) \in \partial S$ satisfying $U_1(u) - U_1(a) = U_2(v) - U_2(b)$. This is because the graph of the function $y = U_2^{-1}(U_1(x) - U_1(a) + U_1(b))$ has positive slope, passes through $(a,b) \in S$, and has $y \to +\infty$ as $x \to +\infty$. So the graph intersects ∂S at exactly one point (u,v), which is the point required. (Remember, S is comprehensive, so ∂S is a downward-sloping curve.)

Letting $F(a,b,S)$ be this point (u,v) will certainly satisfy C1 and C2. If $(w,z) \in \partial T$ satisfies $U_1(w)-U_1(u) = U_2(z)-U_2(v)$ then it also satisfies $U_1(w)-U_1(a) = U_2(z)-U_2(b)$, implying C3 for the equal-gains solution function. It can be checked that $g_1(x) = U_1^{-1}(U_1(x)-U_1(a)+U_1(0))$ and $g_2(y) = U_2^{-1}(U_2(y)-U_2(b)+U_2(0))$ will satisfy C4 for the equal-gains solution function.

The "only if" part of the theorem is much harder to prove, because we must construct the functions U_1 and U_2 for any given F. So henceforth assume that $F: BP \to R^2$ is given and satisfies C1 through C4.

For any number z, let $L_1(z) = \{(x,y) \mid x \leq z\}$, and let $L_2(z) = \{(x,y) \mid y \leq z\}$.

(1) If $(u,v) = F(a,b,S)$ and $(x,y) = F(a,b,T)$ and $u > x$, then $v \geq y$. Otherwise, with $u > x$ and $u < y$, consider $Q = S \cup T \cup L_1(\frac{x+u}{2}) \cup L_2(\frac{y+v}{2})$. C2 assures us that $F(x,y,Q)$ dominates (x,y) in both coordinates, so $F(x,y,Q)$ is not in T or $L_2(\frac{y+v}{2})$. Similarly, $F(u,v,Q)$ can not be in S or $L_1(\frac{x+u}{2})$. But C3 implies $F(x,y,Q) = F(a,b,Q) = F(u,v,Q)$. So $F(a,b,Q) \notin Q$, which violates C1.

(2) If $F(a,b,S) = (u,v)$ and $F(a,b,T) = (x,y)$ and $u > x$, then $v > y$. For, if $v=y$, then let $(w,z) = F(a,b,T \cup L_1(\frac{u+x}{2})) = F(x,y,T \cup L_1(\frac{u+x}{2}))$. By C3, $z>y$ and $w>x$, so $w \leq \frac{u+x}{2} < u$, which violates (1).

(3) If $F(a,b,S) = (u,v)$ then $F(a,b,L_2(v)) = (u,v)$. Let $(w,z) = F(a,b,L_2(v))$. If $w<u$ then $z<u$, by (2); and if $w>u$ then $z>v$. But $z \neq v$ would violate C1.

Let G_1 be the set of all order-preserving transformations under which F is invariant when applied to 1's utility; and let G_2 be the analogous set for 2's utility. In paragraphs (4)-(10) we develop many facts about G_1. All of these statements will also be applicable to G_2.

Composition and inversion are defined for order-preserving transformations by:

$(g \circ h)(x) = g(h(x))$, $g \circ g^{-1} = g^{-1} \circ g = e$, where e is the identity map on R $(e(x) = x)$. For any positive integer k, g^k is g composed with itself k times $(g \circ g \circ \ldots \circ g)$. For any negative integer k, $g^k = (g^{-1})^{-k}$, and $g^0 = e$.

(4) G_1 is a group; that is, G_1 is closed under composition and inversion. To see this, check that if F is invariant under g and h applied to 1's utility, then F is also invariant under g o h and g^{-1}.

(5) If $g \in G_1$ and $g(x)=x$, then $g(u)=u$ for all $u>x$. Suppose to the contrary that $g(u)>u$. Let $(u,v) = F(x,0,L_1(u))$ and let $(g(u),w) = F(u,v,L_1(g(u)))$.
By C2 $w>v$; but by C3 and invariance we get $(g(u),w) = F(x,0,L_1(g(u))) = (g(u),v)$, a contradiction. Also, $g(u)<u$ is impossible, because the same argument could be applied to g^{-1}.

(6) If $g \in G_1$ and $g(x) = x$, then $g(u) = u$ for all $u \in R$.
As in (5), it suffices to prove that $g(u)>u$ is impossible, for any $u<x$. If $g(u)>u$, then $\{g^k(u)\}_{k=1}^{\infty}$ is an increasing sequence bounded above by x. (It is increasing because g is order-preserving.) Let $z = \lim_{k\to\infty} g^k(u)$. By C4 there exists h_1 and h_2 in G_1 such that $h_1(g(u)) = 0$ and $h_2(z) = 0$. Let $h = h_2^{-1} \circ h_1$. Then $h(g(u)) = z$, and $h \in G_1$. Since h is order-preserving, $h(u)<z$, so $g^k(u)> h(u)$, for some k. Let $v_1=g^k(u)$ and $v_2= g^{k+1}(u)$. Then $h \circ g^{-k}(v_1) = h(u)<v_1$ and $h \circ g^{-k}(v_2) = z > v_2$. Since $h \circ g^{-k}$ is continuous, there is some v for which $h \circ g^{-k}(v) = v$, $v_1<v<v_2$. But G_1 is a group, so $h \circ g^{-k}$ is in G_1. This would contradict (5).

(6) For any numbers x and y, there is a unique $g \in G_1$ such that $g(x)=y$. There is at least one, since $h_1^{-1} \circ h_2(x)=y$, where $h_1 \in G_1, h_2 \in G_1, h_1(x) = 0$, and $h_2(y) = 0$. There is at most one, since $g(x) = \hat{g}(x)$ implies $g^{-1} \circ \hat{g}(x)=x$, so $g^{-1} \circ \hat{g} = e$ by (5).

(7) If $g \in G_1$ and $h \in G_1$ and $g(0)>h(0)$, then $g(x)>h(x)$ for all x. If not, then for some z between 0 and x, $g(z) = h(z)$, which contradicts (6).

For any order-preserving transformations, we write $g>h$ if and only if $g(0)>h(0)$.

(8) If g, $\hat{g}$, h, and $\hat{h}$ are in G_1, $g>\hat{g}$, and $h>\hat{h}$, then $g \circ h > \hat{g} \circ \hat{h}$. Because g is order-preserving, $g \circ h(0) = g(h(0)) > g(\hat{h}(0))$; and $g(\hat{h}(0)) > \hat{g}(\hat{h}(0)) = \hat{g} \circ \hat{h}(0)$ by (7).

(9) If $g \in G_1$ and $g > e$, then $\{g^k(0)\}$ is an increasing sequence in k, with $g^k(0) \to +\infty$ as $k \to +\infty$ and $g^k(0) \to -\infty$ as $k \to -\infty$. (Recall e is the identity on R). The sequence is increasing because $g > e$ implies $g^{k+1}(0) = g(g^k(0)) > g^k(0)$. If either end of the sequence had a finite limit, that limit would be a fixed point of g, since g is continuous, contradicting (6) and $g > e$.

(10) G_1 is a commutative (abelian) group. If not, suppose $g \circ h(0) < x < h \circ g(0)$. Find f_1 and f_2 in G_1 so that $f_1(g \circ h(0)) = x$ and $f_2(x) = h \circ g(0)$. Let f be the smaller of f_1 and f_2. So $f > e$ and $f^2(u) \leq h \circ g(0)$ if $u \leq g \circ h(0)$. Find i and j so that $f^i \leq g < f^{i+1}$ and $f^j \leq h < f^{j+1}$, using (9). By (8), $f^{i+j} \leq g \circ h$ and $h \circ g < f^{i+j+2}$. But $f^{i+j}(0) \leq g \circ h(0)$ implies $f^{i+j+2}(0) \leq h \circ g(0)$, contradicting $h \circ g < f^{i+j+2}$.

(11) Statements (4) through (10) are also true for G_2 as well as G_1. The proofs are symmetric, reversing the role of the two coordinates.

Define $\bar{g}_1$ and $\bar{g}_2$ so that $\bar{g}_1 \in G_1$, $\bar{g}_2 \in G_2$, $\bar{g}_2(0) = 1$, and $F(0,0,L_2(1)) = (\bar{g}_1(0), \bar{g}_2(0))$. Observe that $\bar{g}_1 > e$ and $\bar{g}_2 > e$.

For i=1 or 2, define $V_i: G_i \to R$ by $V_i(g) = \lim_{k\to\infty} \frac{p(k)}{2^k}$, where, for each nonnegative integer k, p(k) satisfies $\bar{g}_i^{\,p(k)} \le g^{(2^k)} < \bar{g}_i^{\,p(k)+1}$.

(12) $V_1: G_1 \to R$ is well defined. Given $g \in G_1$, the sequence p(k) can always be found, by (9). Also, $2p(k) \le p(k+1) < 2\cdot p(k)+1$, because $\bar{g}_1^{\,2p(k)} \le g^{(2^{k+1})} < \bar{g}_1^{\,2p(k)+2}$, by (8). So $\frac{p(k)}{2^k}$ is an increasing sequence converging to some limit between p(0) and p(0)+1.

(13) $V_1(g \circ h) = V_1(g) + V_1(h)$. If $\bar{g}_1^{\,p(k)} \le g^{2^k} < \bar{g}_1^{\,p(k)+1}$ and $\bar{g}_1^{\,r(k)} \le h^{2^k} < \bar{g}_1^{\,r(k)+1}$ then $\bar{g}_1^{\,p(k)+r(k)} \le (g \circ h)^{2^k} < \bar{g}_1^{\,p(k)+r(k)+2}$, by (8) and (10). So $V_1(g) + V_1(h) = \lim_{k\to\infty} \frac{p(k)+r(k)}{2^k} \le V_1(g \circ h) \le \lim_{k\to\infty} \frac{p(k)+r(k)+2}{2^k} \le V_1(g)+V_1(h)$.

(14) If $g > e$ then $V_1(g) > 0$. By (9), find K so that $g^{2^K} \ge \bar{g}_1$. Then $V_1(g) \ge \frac{1}{2^K}$.

(15) If $g > h$ then $V_1(g) > V_1(h)$. Notice $g > h$ implies $g \circ h^{-1} > e$, so $V_1(g) = V_1(g \circ h^{-1}) + V_1(h) > V_1(h)$.

(16) $V_1: G_1 \to R$ is onto. Given $r \in R$, let $X = \{h(0) \mid h \in G_1 \text{ and } V_1(h) \le r\}$. If $k \le r < k+1$ then $\bar{g}_1^{\,k}(0) \in X$ and $\bar{g}_1^{\,k+1}(0) \notin X$. Also, $x \in X$ and $y < x$ imply $y \in X$. So X is a lower half-line. Let u be the supremum of S, and let $g(0) = u$. We will show that $V_1(g) = r$. If not, find n so that $|V_1(g) - r| > \frac{1}{n}$.

Then $V_1(h_2)-V_1(h_1) > \frac{1}{n}$ for any $h_1 \in G_1$ and $h_2 \in G_1$ such that $h_2 > g > h_1$. Select any sequence of points so that $0 = x_0 < x_1 < x_2 < \dots < x_{2n-1} < x_{2n} = \bar{g}_1(0)$. Let $f_i \in G$ satisfy $f_i(x_{i-1}) = x_i$, and let f be the smallest of these $2n$ maps. So $f^{2n} < \bar{g}_1$, $f > e$, and $V_1(f) \leq \frac{1}{2n}$. By (4) find s so that $f^s \leq g < f^{s+1}$. Then $f^{s-1} < g < f^{s+1}$, but $V_1(f^{s+1}) - V_1(f^{s-1}) \leq \frac{1}{n}$.

(17) Statements (12) through (16) also hold for $V_2: G_2 \to R$.

(18) If $g \in G_1$, $h \in G_2$, $h \geq e$, and $F(0,0,L_2(h(0))) = (g(0),h(0))$ then $F(0,0,L_2(h^p(0))) = (g^p(1), h^p(0))$ for any positive integer p. Observe that $F(g^{p-1}(0), h^{p-1}(0), L_2(h^p(0))) = (g^p(0), h^p(0))$, because F is invariant under g^{p-1} and h^{p-1} applied to 1's utility and 2's utility respectively. So an induction hypothesis for $p-1$ together with C3 will prove the statement for p.

(19) If $g \in G_1$, $h \in G_2$, $h \geq e$, and $F(0,0,L_2(h(0))) = (g(0), h(0))$, then $V_1(g) = V_2(h)$. Given a positive integer k, let p satisfy $\bar{g}_2^{\,p} \leq h^{2^k} < \bar{g}_2^{\,p+1}$. Then $F(0,0,L_2(\bar{g}_2^{\,p}(0))) = (\bar{g}_1^{\,p}(0),\bar{g}_2^{\,p}(0))$ and $F(0,0,L_2(\bar{g}_2^{\,p+1}(0))) = (\bar{g}_1^{\,p+1}(0),\bar{g}_2^{\,p+1}(0))$, by (18) and the definition of $\bar{g}_1$ and $\bar{g}_2$. Also, (18) implies $F(0,0,L_2(h^{2^k}(0))) = (g^{2^k}(0),h^{2^k}(0))$. But $L_2(\bar{g}_2^{\,p}(0)) \subseteq L_2(h^{2^k}(0)) \subset L_2(\bar{g}_2^{\,p+1}(0))$, so $\bar{g}_1^{\,p}(0) \leq g^{2^k}(0) < \bar{g}_1^{\,p+1}(0)$ by C2 and C3. Thus $\bar{g}_2^{\,p} \leq h^{2^k} < \bar{g}_2^{\,p+1}$ implies $\bar{g}_1^{\,p} \leq g^{2^k} < \bar{g}_1^{\,p+1}$, so $V_1(g) = V_2(h)$, by the way V_1 and V_2 were constructed.

(20) If $g \in G_1$, $h \in G_2$, $h \geq e$, and $F(a,b,L_2(h(b))) = (g(a),h(b))$, then $V_1(g) = V_2(h)$. Let $f_1 \in G_1$ and $f_2 \in G_2$ satisfy $f_1(a) = 0$ and $f_2(b) = 0$. So $F(a,b,L_2(h(b))) = (g(a), h(b))$ implies

implies $F(0,0,L_2(f_2(h(b)))) = (f_1(g(a)), f_2(h(b)))$. But $f_2(h(b)) = f_2 \circ h \circ f_2^{-1}(0) = h(0)$ and $f_1(g(a)) = f_1 \circ g \circ f_1^{-1}(0) = g(0)$, because G_1 and G_2 are commutative. So (19) implies $V_1(g) = V_2(h)$.

Define U_1: $R \to R$ so that $U_1(g(0)) = V(g)$ for any $g \in G_1$. Similarly, define U_2: $R \to R$ so that $U_2(h(0)) = V_2(h)$ for any $h \in G_2$.

(21) If $F(a,b,L_2(v)) = (u,v)$ then $U_1(u) - U_1(a) = U_2(v) - U_2(b)$. Let $f_1 \in G_1$, $f_2 \in G_2$, $g \in G_1$ and $h \in G_2$ satisfy $f_1(0) = a$, $f_2(0) = b$, $g(a) = u$, and $h(b) = v$. Then $F(a,b,L_2(v)) = (u,v)$ implies $F(a,b,L_2(h(b))) = (g(a),h(b))$, so $V_1(g) = V_2(h)$. But $V_1(g) = V_1(g \circ f_1) - V_1(f_1) = U_1(u) - U_1(a)$, by (13). Similarly, $V_2(h) = V_2(h \circ f_2) - V_2(f_2) = U_2(v) - U_2(b)$. So $U_1(u) - U_1(a) = U_2(v) - U_2(b)$.

(22) U_1 and U_2 are order-preserving transformations. This follows from (15) and (16) and the definitions of the U_i.

So if F satisfies C1 through C4, $F(a,b,S) = (u,v)$ implies $U_1(u) - U_1(a) = U_2(v) - U_2(b)$, by (3) and (21). We have already observed that the condition $U_1(u) - U_1(a) = U_2(v) - U_2(b)$ must be satisfied by a unique $(u,v) \in \partial S$, given that U_1 and U_2 are order-preserving transformations and **S** is a closed comprehensive proper subset of R^2. This completes the proof of the theorem.

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