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GRAPHS AND COOPERATION IN GAMES

by

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Abstract. Graph-theoretic ideas are used to analyze cooperation structures in games. A specific procedure is proposed for making a game's cooperation structure depend endogenously on choices by the players. Fair allocation rules, defined with reference to an equity criterion (that two players should gain equally from cooperation), are proven to be unique, closely related to the Shapley value, and stable for a wide class of games. These ideas are extended to games in "graph function" form, a new generalization of the characteristic function form.

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1. Introduction.

To build a theory of cooperative games, we must have a way to describe cooperation structures. That is, we need a vocabulary in which to talk about who is cooperating with whom among the players. In most studies of games with many players, the only concept of cooperation structure used is the coalition . In this paper we develop the concept of a cooperation graph , with which we can describe a much richer variety of cooperation structures than we could with only the concept of coalitions . We will explore a number of related applications of this new concept to games in characteristic function form, and to games in a new, more general graph function form.

To motivate the progression from coalitions to graphs, consider the problem of making the coalition structure an endogenous factor, to depend in some natural way on the player choices. There does not seem to be any satisfactory general solution to this problem. For example, suppose that, in a three-player game, player 1 wants to cooperate with player 2 but not with player 3, 3 wants to cooperate with 2 but not with 1, and 2 wants to cooperate with both 1 and 3. No coalition structure seems to naturally result from these player preferences. Whether we try $\{\{1,2\}, \{3\}\}$ or $\{\{1\}, \{2,3\}\}$ or $\{\{1,2,3\}\}$, each partition into coalitions seems to suppress some significant aspects of this cooperative situation.

If we talk about systems of bilateral cooperation links, instead of coalitions, the dilemma disappears. In this example, there should be a cooperation link between players 1 and 2, and between players 2 and 3, but there should be no link between players 1 and 3. So the following graph naturally summarizes the players' cooperation preferences:

2. Definitions and notation

For any set A, let R^A represent the set of all vectors of real numbers indexed on the members of A. (Equivalently, one may consider R^A as the set of all functions from A into the reals.)

Let N be a nonempty finite set, to be interpreted as the set of players. Let CL be the set of *coalitions* of players in N:

(1) $$CL = \{S \mid S \subseteq N,\ S \neq \emptyset\}.$$

The set of games in characteristic function form is R^{CL}. For any game $v \in R^{CL}$ and any coalition $S \in CL$, v_S (the S-component of v) is interpreted as the wealth (of transferable utility) which coalition S would have to divide among its members if it were to form. In Section 6 we will drop this implicit assumption that utility is transferable.

A *graph* on N is a set of unordered pairs of distinct members of N. We will refer to these unordered pairs as *links*, and we will denote the link between n and m by $\underline{n,m}$. Notice that $\underline{n,m} = \underline{m,n}$ because the link is an unordered pair. Let g^N be the complete graph of all links:

(2) $$g^N = \{\underline{n,m} \mid n \in N,\ m \in N,\ n \neq m\}.$$

Then let GR be the set of all graphs of N, so that:

(3) $\quad GR = \{g \mid g \subseteq g^N\}$.

Throughout this paper the symbol $\setminus$ will be used to denote removal of a member from a set. Thus:

(4) $\quad S\setminus n = \{i \mid i\in S,\ i \neq n\}, \quad g\setminus \underline{n,m} = \{\underline{i,j} \mid \underline{i,j}\in g,\ \underline{i,j} \neq \underline{n,m}\}$.

Suppose $S\in CL$, $g\in GR$, $n\in N$, and $m\in N$ are given. Then we say that n and m are connected in S by g iff $n = m$ or there is some integer $k\geq 1$ and a sequence $(n^0, n^1, \ldots n^k)$ such that $n = n^0$, $m = n^k$, and $n^i\in S$ and $\underline{n^{i-1},n^i} \in g$ for all i from 1 to k. There is a unique partition of S which groups players together iff they are connected in S by g, and we will denote this partition by S/g (read "S divided by g"). That is:

(5) $\quad S/g = \{\{i \mid \text{i and j are connected in S by g}\} \mid j\in S\}$.

We can interpret S/g as the collection of smaller coalitions into which S would break up, if players could only coordinate along links in g.

When we speak of connectedness without reference to any specific coalition, we will always mean connectedness in N.

For example, if $N = \{1, 2, 3, 4, 5\}$ and $g = \{\underline{1,2}, \underline{1,4}, \underline{2,4}, \underline{3,4}\}$, then $\{1, 2, 3\}/g = \{\{1, 2\}, \{3\}\}$ and $N/g = \{\{1, 2, 3, 4\}, \{5\}\}$.

Suppose that graph $g\in GR$ effectively represents the cooperation structure of a game, in that $\underline{n,m}\in g$ iff n and m have a bilateral cooperation agreement. Then the partition N/g is the natural coalition structure to associate with this cooperation structure. That is, if we must speak of a coalition structure for the game, then the most reasonable choice is the partition into the connected components of g in N. Letting links represent friendship, the idea is that friends of one's allies must in some sense also be one's allies. (As noted in the introduction, describing the cooperation structure by a partition might suppress some significant information, but now we can retain the option to recall the underlieing graph structure if necessary.)

3. Endogenous cooperation structures and stable allocation rules.

We now propose a specific procedure for making the graphical cooperation structure an endogenous function of player choices. Simply assume that a link, representing a bilateral cooperation relationship, will form between two players iff both players want it to form. To be precise, we allow each player n to choose a set $\sigma_n \subseteq N \setminus n$, representing the set of all players with whom he wants cooperate. Then the graph of all cooperation links which will result from this sequence $\sigma = (\sigma_n)_{n \in N}$ of player choices is:

(6) $$g^*(\sigma) = \{\underline{n,m} \mid n \in \sigma_m,\ m \in \sigma_n\}.$$

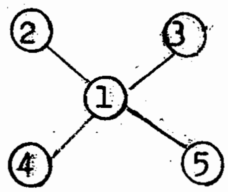
What criteria player n should use, in selecting his cooperation-offer set σ_n, must depend on how the cooperation structure $g^*(\sigma)$ will in turn influence the payoff to player n in the game. After all, each player is really only concerned with maximizing his final utility payoff, and the structure of cooperation is of interest only to the extent that cooperation influences the final outcome and allocation of utility.

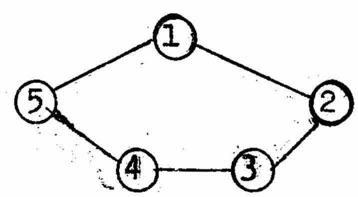
So we want to know how the payoff allocation in game $v \in R^{CL}$ will depend on the cooperation structure $g \in GR$. We hope to find some function $Y: GR \to R^N$, mapping cooperation structures into allocation vectors, such that $Y_n(g)$ (the n-component of $Y(g)$) is the utility payoff expected by player n when g is the set of bilateral cooperation links which have formed.

Formally, for any characteristic function game $v \in R^{CL}$, we define an <u>allocation rule</u> for v to be any function $Y\colon GR \to R^N$ such that:

(7) $$\forall\, g \in GR,\ \forall\, S \in N/g,\ \sum_{n \in S} Y_n(g) = v_S.$$

This condition (7) expresses the idea that, if S is a connected component of g , then the members of S ought to allocate to themselves the total wealth v_S which they can earn. Notice that the allocation within S still depends on the actual graph g. For example, an allocation rule might give higher payoff to player 1 in graph g_1 than in g_2 illustrated below, because in g_1 his position is more essential to coordinating the others. In each case, however, condition (7) requires that $\sum_{n=1}^{5} Y_n(g_1) = v_{\{1,2,3,4,5\}} = \sum_{n=1}^{5} Y_n(g_2)$.

g_1: 

g_2: 

Given an allocation rule $Y\colon GR \to R^N$, we can ask whether $\bar{g}^N$, the complete graph on N, is a Nash equilibrium outcome under the procedure for determining the cooperation structure described above (equation (6)). That is, if player n assumes that all other players will offer to cooperate with him and with each other, will he have any incentive to try to disrupt the universal cooperation by refusing to offer cooperation to certain others, say, those in some set S? If the answer is "no", then we can say that the allocation rule $Y(\cdot)$ is stable . Formally, we say that $Y\colon GR \to R^N$ is <u>stable</u> iff

(8) $$\forall\, n \in N,\ \forall\, S \subseteq N \setminus n,\ Y_n(\bar{g}^N) \geq Y_n(\{\underline{i,j} \mid \underline{i,j} \in \bar{g}^N,\ \underline{i,j} \notin \{\underline{n,m} \mid m \in S\}\})$$

An even stronger property is "total stability". We say that $Y\colon GR \to R^N$ is <u>totally stable</u> iff

(9) $$\forall\, g \in GR,\ \forall\, \underline{n,m} \in g,\ Y_n(g) \geq Y_n(g \setminus \underline{n,m}).$$

So total stability means that a player never loses by forming more cooperation relationships. Clearly total stability implies stability.

4. Fair allocation rules

In this section we will show how imposing an equity criterion leads to a unique allocation rule.

Consider a simple example: the "Divide the Dollar" game for two players. Let $N = \{1,2\}$, $v_{\{1\}} = 0 = v_{\{2\}}$, and $v_{\{1,2\}} = 1$. So players 1 and 2 can divide one unit of transferable utility among themselves if they work together; alone each can get at most zero. An allocation rule for this game must prescribe $Y_1(\emptyset) = 0 = Y_2(\emptyset)$ ($\emptyset$ is the empty graph, with no links) and some cooperative allocation $(Y_1(\{\underline{1{:}2}\}), Y_2(\{\underline{1{:}2}\}))$ such that

$$Y_1(\{\underline{1{:}2}\}) + Y_2(\{\underline{1{:}2}\}) = 1.$$

The allocation rule will be stable iff $Y_1(\{\underline{1{:}2}\}) \geq 0$ and $Y_2(\{\underline{1{:}2}\}) \geq 0$. Thus far, however, we have said nothing to suggest where the cooperative allocation $(Y_1(\{\underline{1{:}2}\}), Y_2(\{\underline{1{:}2}\}))$ should lie in the interval between (0, 1) and (1, 0). Any allocation in this interval has the same properties of stability and Pareto-optimality. But unequal allocations like (.9, .1) or (.05, .95) would seem unfair and therefore unlikely to most observers. If the players have an extra-utilitarian ethic against being exploited or taken advantage of in the cooperation process, then the equal-gains split of (.5, .5) must be the most likely outcome for this game. Certainly we would an expect an impartial arbitrator to suggest $Y(\{\underline{1{:}2}\}) = (.5, .5)$, based on considerations of symmetry or equity.

So some equity criterion is needed to derive a useful theory of cooperative allocation rules. The question is, how do we generalize to games with many players the equity criterion which solves the "Divide the Dollar" game, that players should gain equally from cooperation.

One of the great advantages of our graphical cooperation structures is that they automatically resolve a cooperation problem with many players into a system of two-person cooperation problems: simply consider one link at a time. Given graph $g \in GR$ and link $\underline{n,m} \in g$, each of players n and m has an equal opportunity to break the $\underline{n,m}$ link by refusing to offer cooperation to the other. So a reasonable equity criterion is that they should gain equally from their link. That is:

(10) $\quad \forall\, g \in GR,\ \forall\, \underline{n,m} \in g,\ Y_n(g) - Y_n(g \setminus \underline{n,m}) = Y_m(g) - Y_m(g \setminus \underline{n,m})$.

Given $v \in R^{CL}$, we define a <u>fair allocation rule</u> for v to be any function $Y: GR \rightarrow R^N$ satisfying both the equity condition (10) and the efficiency condition (7). Our main result is that there is a unique such fair allocation rule for each game.

THEOREM 1. For any characteristic function game $v \in R^{CL}$, there is a unique fair allocation rule $Y: GR \rightarrow R^N$ satisfying conditions (7) and (10).

An exact formula can be given for this fair allocation rule. For any game $v \in R^{CL}$ and any graph $g \in GR$, define $v/g \in R^{CL}$ so that:

(11) $\quad \forall\, S \in CL,\ (v/g)_S = \sum_{T \in S/g} v_T$.

(Recall the definition of S/g in (5).) So v/g can be interpreted as the characteristic function game which would result if we altered the situation represented by v, by requiring that players can only coordinate along links in g. Notice that $v/\bar{g}^N \equiv v$.

THEOREM 2. The unique fair allocation rule $Y: GR \to R^N$ for game $v \in R^{CL}$ satisfies:

$$Y(g) = \varphi(v/g),\ \forall\, g \in GR,$$

where $\varphi: R^{CL} \to R^N$ is the Shapley value operator. (See [SHAPLEY, 1953].) In particular, $Y(\bar{g}^N) = \varphi(v)$.

A game $v \in R^{CL}$ is superadditive iff:

(12) $\forall\, S \in CL,\ \forall\, T \in CL,$ if $S \cap T = \emptyset$ then $v_{S \cup T} \geq v_S + v_T$.

THEOREM 3. If $v \in R^{CL}$ is superadditive, then the fair allocation rule for v is totally stable.

5. Example.

Let $N = \{1, 2, 3\}$, and consider the game $v \in R^{CL}$ where:

$v_{\{1\}} = v_{\{2\}} = v_{\{3\}} = 0$, $v_{\{1,3\}} = v_{\{2,3\}} = 6$, and $v_{\{1,2\}} = v_{\{1,2,3\}} = 12$.

The fair allocation rule for this game is as follows:

$Y(\emptyset)$	$= (0, 0, 0)$;	$Y(\{\underline{1,2},\ \underline{1,3}\})$	$= (7, 4, 1)$;
$Y(\{\underline{1,2}\})$	$= (6, 6, 0)$;	$Y(\{\underline{1,2},\ \underline{2,3}\})$	$= (4, 7, 1)$;
$Y(\{\underline{1,3}\})$	$= (3, 0, 3)$;	$Y(\{\underline{1,3},\ \underline{2,3}\})$	$= (3, 3, 6)$;
$Y(\{\underline{2,3}\})$	$= (0, 3, 3)$;	$Y(\ \underline{1,2},\ \underline{1,3},\ \underline{2,3}\ )$	$= (5, 5, 2)$.

The Shapley value of v is $\varphi(v) = Y(\bar{g}^N) = (5, 5, 2)$.

This example was chosen because most other well-known solution concepts, the core and the nucleolus and the bargaining set, all select the single allocation (6, 6, 0) for this game.

These solution concepts are all based on ideas about what it means for the universal coalition N to be stable against objections. According to the argument for the core, (5, 5, 2) should be an unstable allocation because players 1 and 2 could earn 12 units wealth for themselves, which excedes the wealth 5+5 = 10 given to them. But when we shift our perspective from coalitions to cooperation graphs, this argument evaporates, and the value (5, 5, 2) actually is part of a totally stable fair allocation rule. If any one player were to break either or both of his cooperation links, then his fair allocation would decrease. (Check the table above.) To be sure, if both players 1 and 2 were to simultaneously break their links with 3, then both would benefit; but each would benefit even more if he continued to cooperate with player 3 while the other alone broke his link to player 3.

Thus, this example illustrates how our cooperation graph and fair allocation rule ideas give a new justification for the Shapley value.

6. Games in graph function form.

The ideas in this paper can be extended to games without transferable utility, and to games in "partition function form". To show the full generality of our ideas, we introduce a new and even more general game form: the graph function form.

An embedded subgraph is a pair (S,g) such that g is a graph and S is a connected component of g in N. Formally, we let ESG denote the set of all embedded subgraphs, and then by definition:

(13) $$\mathrm{ESG} = \{(S,g) \mid g \in GR,\ S \in N/g\}.$$

A set $W \subseteq R^S$ is <u>comprehensive</u> iff:

$\forall a \in W$, $\forall b \in R^S$, if $a_n \geq b_n$ $\forall n \in N$, then $b \in W$.

W is a <u>proper</u> subset of R^S iff $W \subseteq R^S$ and $\emptyset \neq W \neq R^S$.
Let ∂ denote the boundary operator, so that if $W \subseteq R^S$ then ∂W is the boundary of W in R^S.

A game in <u>graph function form</u> is a set-valued function $w(.)$ with domain ESG, such that $\forall$ $(S,g) \in$ ESG, $w(S,g)$ is a closed and comprehensive proper subset of R^S. We interpret $w(S,g)$ as the set of utility allocations which are feasible for the players in S when g is the set of bilateral cooperation links. If cooperation structure g is given then $\partial w(S,g)$ is the (weakly) Pareto-optimal frontier for the members of S.

A characteristic function game $v \in R^{CL}$ can be identified with a graph function game $w(.)$ iff $w(S,g) = \{r \in R^S \mid \sum_{n \in S} r_n \leq v_S\}$, $\forall$ $(S,g) \in$ ESG
All the ideas introduced in Sections 3 and 4 through Theorem 1 can be extended to graph function games, except that condition (7) must be replaced by:

(14) $\quad \forall g \in GR, \forall S \in N/g, \ (Y_n(g))_{n \in S} \in \partial w(S,g).$

So a fair allocation rule for a graph function game $w(.)$ must satisfy the efficiency condition (14) and the equity condition (10). Our final result generalizes Theorem 1.

THEOREM 4. For any graph function game $w(.)$, there is a unique fair allocation rule $Y: GR \rightarrow R^N$ satisfying conditions (14) and (10).

7. Proofs

We begin with Theorem 4.

PROOF OF THEOREM 4.

For any graph g, let $|g|$ be the number of links in g.

We will write $h \subset g$ iff $h \subseteq g$ and $h \neq g$.

Observe that, if $\underline{n,m} \in g$, then:

$$\sum_{\substack{h \subseteq g \\ \ni\, \underline{n,m} \in h}} (-1)^{|g|+|h|} (Y_n(h) - Y_n(h \setminus \underline{n,m})) = \sum_{h \subseteq g} (-1)^{|g|+|h|} Y_n(h)$$

$$= Y_n(g) - \sum_{h \subset g} (-1)^{|g|+|h|+1} Y_n(h).$$

So condition (10) will hold iff:

(15) $$\forall\, g \in GR,\ \forall\, \underline{n,m} \in g,\ Y_n(g) - \sum_{h \subset g} (-1)^{|g|+|h|+1} Y_n(h) = Y_m(g) - \sum_{h \subset g} (-1)^{|g|+|h|+1} Y_m(h).$$

Neither side of the equation in (15) actually depends on the link $\underline{n,m}$, so the equation will hold between all linked pairs of players iff it holds between all connected pairs of players. So condition (10) is also equivalent to:

(16) $\forall\, g \in GR$, $\forall\, S \in N/g$, there exists a number $d(S,g)$ such that $Y_n(g) - \sum_{h \subset g} (-1)^{|g|+|h|+1} Y_n(h) = d(S,g)$, $\forall\, n \in S$.

Given (16), condition (14) will be satisfied iff:

(17) $$d(S,g) = \max \left\{ x \,\middle|\, \left(x + \sum_{h \subset g} (-1)^{|g|+|h|+1} Y_n(h)\right)_{m \in S} \in w(S,g) \right\},$$

$\forall\, (S,g) \in ESG$.

But $\max \left\{ x \,\middle|\, (x + r_n)_{n \in S} \in w(S,g) \right\}$ is always uniquely defined, for any vector $(r_n)_{n \in S} \in R^S$, because $w(S,g)$ is closed, comprehensive, and a proper subset of R^S.

Thus, $Y: GR \to R^N$ will satisfy (10) and (14) iff it is constructed by the following equations:

$\forall\ g \in GR,\ \forall\ S \in N/g,\ \forall\ n \in S$:

(18a) $\quad t_n(g) = \sum_{h \subset g} (-1)^{|g|+|h|+1}\, Y_n(h), \qquad t_n(\emptyset) = 0;$

(18b) $\quad d(S,g) = \max \{x \mid (x + t_m(g))_{m \in S} \in w(S,g)\};$

(18c) $\quad Y_n(g) = t_n(g) + d(S,g).$

These equations can be solved in order of increasing $|g|$, and the solution is the unique fair allocation rule for $w(.)$.

PROOF OF THEOREM 1.

Theorem 1 follows immediately from Theorem 4, taking

$$w(S,g) = \{(r_n)_{n \in S} \mid \sum_{n \in S} r_n \leqslant v_S\}.$$

PROOF OF THEOREM 2.

We must prove that $Y(g) = \varphi(v/g)$, $\forall\ g \in GR$, implies conditions (7) and (10).

We show (7) first. Select any $g \in GR$. For each $S \in N/g$, define $u^S \in R^{CL}$ so that:

$$u_T^S = \sum_{R \in (T \cap S)/g} v_R\ ,\ \forall\ T \in CL.$$

Now, any two players connected in T by g are also connected in N by g, so

$$T/g = \bigcup_{S \in N/g} (T \cap S)/g\ .$$

Therefore $v/g = \sum_{S \in N/g} u^S$. But S is a carrier of u^S, because $u_T^S = u_{T \cap S}^S$. (The concept of a carrier is defined in [SHAPLEY, 1953].) So, using the carrier axiom in [SHAPLEY, 1953], for any $S \in N/g$ and any $T \in N/g$:

$$\sum_{n\in S}\varphi_n(u^T) = \begin{cases} u_N^T & \text{, if } S = T\,; \\ 0 & \text{, if } S\cap T = \emptyset. \end{cases}$$

Thus, by linearity of φ, if $S\in N/g$ then

$$\sum_{n\in S}\varphi_n(v/g) = \sum_{T\in N/g}\ \sum_{n\in S}\varphi_n(u^T) = u_N^S = \sum_{R\in S/g} v_R.$$

To show (10), recall the Shapley value formula:

$$\varphi_n(v) = \sum_{S\in CL} a_{n,S}\, v_S\,, \tag{19}$$

$$\text{where } a_{n,S} = \begin{cases} \dfrac{(|S|-1)!\,(|N|-|S|)!}{|N|!} & \text{, if } n\in S, \\ -\dfrac{|S|!\,(|N|-|S|-1)!}{|N|!} & \text{, if } n\notin S. \end{cases}$$

Let $b_{n,m,S} = a_{n,S} - a_{m,S}$. Notice that $b_{n,m,S} = 0$ if $\{n,m\}\subseteq S$. Notice also that $S/g = S/(g\setminus \underline{n,m})$ if $\{n,m\}\nsubseteq S$. Therefore:

$$\varphi_n(v/g) - \varphi_m(v/g) = \sum_{S\in CL} b_{n,m,S}\sum_{T\in S/g} v_T =$$

$$= \sum_{S\in CL} b_{n,m,S}\sum_{T\in S/g\setminus \underline{n,m}} v_T = \varphi_n(v/g\setminus \underline{n,m}) - \varphi_m(v/g\setminus \underline{n,m}).$$

PROOF OF THEOREM 4.

Observe that $S/(g\setminus \underline{n,m})$ always refines S/g as a partition of S, and if $n\notin S$ then $S/(g\setminus \underline{n,m}) = S/g$. So, if v is superadditive,

$$(v/g)_S = \sum_{T\in S/g} v_T \geq \sum_{T\in S/g\setminus \underline{n,m}} v_T = (v/g\setminus \underline{n,m})_S\,,$$

and the inequality becomes an equality if $n\notin S$. Therefore:

$$\varphi_n(v/g) - \varphi_n(v/g\setminus \underline{n,m}) = \sum_{S\in CL} a_{n,S}\,((v/g)_S - (v/g\setminus \underline{n,m})_S)$$

$$= \sum_{\substack{S\in CL\\ \ni n\in S}} \frac{(|S|-1)!\,(|N|-|S|)!}{|N|!}\,((v/g)_S - (v/g\setminus \underline{n,m})_S) \geq 0.$$

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