Discussion Paper No. 245

AN EXTENSION OF THE KAKUTANI FIXED POINT THEOREM

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September 1976

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In arguments using the Kakutani Fixed Point Theorem, special difficulties often arise for points on the boundary of the domain. In this paper we present an extension of the Kakutani Theorem which relaxes the restrictions on the map at such points.

Let S be a nonempty, convex, closed and bounded subset of a finite dimensional Euclidean space. Let S* be the set of all subsets of S. Let S** be the set of all subsets of S which are nonempty, convex, and closed. Let S^o be a nonempty subset of S such that

(1) $\forall\, x \in S,\ \forall\, y \in S^o,\ \frac{1}{2}(x+y) \in S^o.$

For example, S^o may be the relative interior of S, which will always satisfy (1).

A point-to-set map $F: S \to S^*$ is <u>upper-semicontinuous</u> iff $x_n \to \bar{x}$, $y_n \in F(x_n)$, and $y_n \to \bar{y}$ imply that $\bar{y} \in F(\bar{x})$.

KAKUTANI [1941] showed that, if $F: S \to S^{**}$ is upper-semicontinuous, then there exists some $\bar{x} \in S$ such that $\bar{x} \in F(\bar{x})$. Our extension of his theorem weakens the convexity assumptions on $F(x)$.

<u>Theorem</u>. If $F: S \to S^*$ is upper-semicontinuous, and if $F(y) \in S^{**}$ for all $y \in S^o$, then there exists an $\bar{x} \in S$ such that $\bar{x} \in F(\bar{x})$.

<u>Proof</u>. Select any $\bar{y} \in S^o$. For any positive integer n,

define $F_n: S \to S^*$ by $F_n(x) = F(\frac{2^n-1}{2^n} x + \frac{1}{2^n} \bar{y})$.

F_n is clearly upper-semicontinuous. For any $x \in S$, $\frac{2^n-1}{2^n} x + \frac{1}{2^n} \bar{y} = \frac{1}{2}(x + \frac{1}{2}(x + \ldots + \frac{1}{2}(x + \bar{y}) \ldots))$ must be in S^o, so $F_n(x) \in S^{**}$. Hence F_n satisfies Kakutani's conditions; and so for each n there exists some $x_n \in S$ such that $x_n \in F_n(x_n)$.

Since S is compact, we can find some $\bar{x}$ and a convergent subsequence such that $N(i) \to \infty$ and $x_{N(i)} \to \bar{x}$ as $i \to \infty$.

Thus, $x_{N(i)} \to \bar{x}$, $x_{N(i)} \in F(\frac{2^{N(i)}-1}{2^{N(i)}} x_n + \frac{1}{2^{N(i)}} \bar{y})$, and $(\frac{2^{N(i)}-1}{2^{N(i)}} x_n + \frac{1}{2^{N(i)}} \bar{y}) \to \bar{x}$. By upper-semicontinuity of F, we conclude that $\bar{x} \in F(\bar{x})$. Q.E.D.

Suppose $F^o: S^o \to S^{**}$ is an upper-semicontinuous map. F^o has a unique <u>minimal upper-semicontinuous extension</u> $F: S \to S^*$ defined by:

(2) $$F(x) = \left\{ z \;\middle|\; \begin{array}{l} \text{there exist sequences } \{x_n\} \text{ and } \{z_n\} \text{ such} \\ \text{that } x_n \in S^o,\ x_n \to x,\ z_n \in F^o(x_n), \text{ and } z_n \to z \end{array} \right\}.$$

(Upper-semicontinuity assures us that $F(x) = F^o(x)$ if $x \in S^o$.) Then F may not be convex-valued outside S^o, but it will satisfy the hypotheses of our theorem. A fixed point of F can be interpreted as a limit of almost-fixed points of F^o.

<u>Corollary</u> For any upper-semicontinuous map $F^o: S^o \to S^{**}$, there exist sequences $\{x_n\}$ and $\{z_n\}$ and an $\bar{x} \in S$ such that $x_n \in S^o$, $z_n \in F^o(x_n)$, $x_n \to \bar{x}$, and $z_n \to \bar{x}$.

REFERENCES

Kakutani, S., "A Generalization of Brouwer's Fixed Point Theorem," <u>Duke Mathematical Journal</u> 8 (1941), pages 457 - 459.

ABSTRACT

An extension of the Kakutani Fixed Point Theorem is presented, in which convexity restrictions are relaxed for points on the boundary of the domain.