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VALUES OF GAMES IN PARTITION FUNCTION FORM

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Games in partition function form were introduced in [LUCAS and THRALL, 1963] to generalize the characteristic function form. The Shapley value for games in characteristic function form was developed from three simple axioms in [SHAPLEY, 1953]. In this paper we introduce the natural extension of the Shapley value to the partition function form.

Let N be a fixed nonempty finite set, and let the members of N be interpreted as players in some game situation. Given N, let CL be the set of all coalitions (nonempty subsets) of N,

(1) $$CL = \{S \mid S \subseteq N,\ S \neq \emptyset\}.$$

Let PT be the set of partitions of N, so

(2) $\{S^1,\ldots,S^\ell\} \in PT$ iff:

$$\bigcup_{i=1}^{\ell} S^i = N,\ \forall j\ S^j \neq \emptyset,\ \forall k\ S^j \cap S^k = \emptyset \text{ if } k \neq j.$$

And let ECL be the set of <u>embedded coalitions</u>, that is the set of coalitions together with specifications as to how the other players are aligned. Formally:

(3) $$ECL = \{(S,Q) \mid S \in Q \in PT\}.$$

For any finite set L, let R^L denote the set of real vectors

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indexed on the members of L.

We define a <u>game in partition function form</u> (or a <u>partition function game</u> for short) to be any vector in R^{ECL}. For any such $w \in R^{ECL}$ and any embedded coalition $(S,Q) \in ECL$, $w_{S,Q}$ (the (S,Q)-component of w) is interpreted as the wealth, measured in units of transferable utility, which the coalition S would have to divide among its members if all the players were aligned into the coalitions of partition Q. We assume that, in any game situation represented by such a vector $w \in R^{ECL}$, the universal coalition N (embedded in $\{N\}$) will actually form, so that the players will have $w_{N,\{N\}}$ to divide among themselves. But we also anticipate that the actual allocation of this wealth will depend on all the other potential wealths $w_{S,Q}$, as they influence the relative bargaining strengths of the players.

So we are interested in <u>value functions</u>, which are functions from the space of partition function games R^{ECL} into the space of payoff allocations R^N. If $\Phi(\cdot)$ is a value function and $w \in R^{ECL}$ is a partition function game, then for any $n \in N$ we would like to interpret $\Phi_n(w)$ (the n-component of $\Phi(w) \in R^N$) as the utility payoff which player n should expect from the game w. If the value function is to be realistic for these purposes, then it ought to satisfy certain properties, which we will develop.

We will need some additional notation. For any $Q \in PT$ and $\tilde{Q} \in PT$, define $Q \wedge \tilde{Q} \in PT$ by:

(4) $$Q \wedge \tilde{Q} = \{S \cap \tilde{S} \mid S \in Q,\ \tilde{S} \in \tilde{Q},\ S \cap \tilde{S} \neq \emptyset\}.$$

Given any (S,Q) and $(\hat{S},\hat{Q})$ in ECL, we write:

(5) $$(S,Q) \gtrsim (\hat{S},\hat{Q}) \text{ iff } S \supseteq \hat{S} \text{ and } Q \wedge \hat{Q} = \hat{Q}.$$

(Read: "(S,Q) covers $(\hat{S},\hat{Q})$".)

For any $S \subseteq N$, let $|S|$ be the number of players in S; for any $Q \in PT$, let $|Q|$ be the number of coalitions in Q. For any $w \in R^{ECL}$ and any $Q \in PT$, we adopt the convention that $w_{\emptyset,Q} = 0$, where $\emptyset$ is the empty set.

Now, suppose $\pi: N \to N$ is any permutation of the set of players. Then without danger of confusion, we can also let π act as a permutation on CL and on ECL in the natural way:

$$\pi(S) = \{\pi(n) \mid n \in S\}, \qquad \forall S \in CL, \text{ and}$$

$$\pi(S^1, \{S^1,\ldots,S^k\}) = (\pi(S^1), \{\pi(S^1),\ldots,\pi(S^k)\}),$$

$$\forall (S^1, \{S^1,\ldots,S^k\}) \in ECL.$$

Then, for any $w \in R^{ECL}$, define $\pi \circ w \in R^{ECL}$ so that

(6) $$(\pi \circ w)_{\pi(S,Q)} = w_{S,Q}, \quad \forall (S,Q) \in ECL.$$

(equivalently, $(\pi \circ w)_{S,Q} = w_{\pi^{-1}(S,Q)}$.) So $\pi \circ w$ is the partition function game which would result from w if we relabeled the players by permutation π. The first axiom asserts that the value should be invariant under such relabelling.

VALUE AXIOM 1: For any permutation $\pi : N \to N$, $\forall w \in R^{ECL}$, $\forall n \in N$, $\Phi_n(w) = \Phi_{\pi(n)}(\pi \circ w)$.

Given $w \in R^{ECL}$ and $S \in CL$, we say that S is a carrier of w iff $w_{\tilde{S},\tilde{Q}} = w_{\tilde{S} \cap S, \tilde{Q} \wedge \{S, N \setminus S\}}$, $\forall (\tilde{S},\tilde{Q}) \in ECL$. (Recall $w_{\emptyset,Q} \equiv 0$.) Note that N is always a carrier. The second axiom suggests that all available wealth should be divided among the members of a carrier.

VALUE AXIOM 2: $\forall w \in R^{ECL}$, $\forall S \in CL$, if S is a carrier of w, then $\sum_{n \in S} \Phi_n(w) = w_{N,\{N\}}$. (Note that $w_{N,\{N\}} = w_{\tilde{S},\tilde{Q}}$ whenever S is a carrier of w.)

Using the real vector space nature of R^{ECL}, we can talk about adding games. Our third axiom suggests that addition partition function games, which could be interpreted as merging games to be played simultaneously, should cause addition to their values,

VALUE AXIOM 3: $\forall w^1 \in R^{ECL}$, $\forall w^2 \in R^{ECL}$, $\Phi(w^1 + w^2) = \Phi(w^1) + \Phi(w^2)$.

These three axioms can be recognized as adaptations of the Shapley value axioms (see [SHAPLEY, 1953]) to the partition function game. Our main result is that these three axioms determine a unique value function, which is the analogue of the Shapley value for partition function games.

THEOREM 1: There exists exactly one value function $\Phi : R^{ECL} \to R^N$ which satisfies Value Axioms 1, 2, and 3.

Henceforth, we will always use Φ to represent the value function of Theorem 1. The following theorem shows how to compute our value function.

THEOREM 2: The value function $\Phi : R^{ECL} \to R^N$ defined in Theorem 1 satisfies the following formula, $\forall n \in N$, $\forall w \in R^{ECL}$:

$$\Phi_n(w) = \sum_{(S,Q) \in ECL} (-1)^{|Q|-1} \cdot (|Q|-1)! \left(\frac{1}{|N|} - \sum_{\substack{\tilde{S} \in Q \\ \ni \tilde{S} \neq S \\ n \notin \tilde{S}}} \frac{1}{(|Q|-1)(|N|-|\tilde{S}|)} \right) w_{S,Q}.$$

For example, if $N=\{1,2,3\}$, then the formula for $\Phi_1(w)$ is:

(7)
$$\begin{aligned}
\Phi_1(w) = {} & \tfrac{1}{3} \cdot w_{\{1,2,3\},\{\{1,2,3\}\}} \\
& + \tfrac{1}{6} \cdot w_{\{1,2\},\{\{1,2\},\{3\}\}} - \tfrac{1}{3} \cdot w_{\{3\},\{\{1,2\}\{3\}\}} \\
& + \tfrac{1}{6} \cdot w_{\{1,3\},\{\{1,3\},\{2\}\}} - \tfrac{1}{3} \cdot w_{\{2\},\{\{1,3\},\{2\}\}} \\
& + \tfrac{2}{3} \cdot w_{\{1\},\{\{1\},\{2,3\}\}} - \tfrac{1}{3} \cdot w_{\{2,3\},\{\{1\},\{2,3\}\}} \\
& + \tfrac{1}{6} \cdot w_{\{2\},\{\{1\},\{2\},\{3\}\}} + \tfrac{1}{6} \cdot w_{\{3\},\{\{1\},\{2\},\{3\}\}} \\
& - \tfrac{1}{3} \cdot w_{\{1\},\{\{1\},\{2\},\{3\}\}}.
\end{aligned}$$

In a corollary, we note that a stronger version of Value Axiom 2 is true. Given $Q \in PT$ and $w \in R^{ECL}$, we say that w is <u>Q-decomposable</u> iff:

$$\forall (\tilde{S},\tilde{Q}) \in ECL,\ w_{\tilde{S},\tilde{Q}} = \sum_{\tilde{S} \in \tilde{Q}} w_{\tilde{S} \cap S,\ \tilde{Q} \wedge Q}.$$

(Recall $w_{\emptyset,Q} = 0$.)

COROLLARY 1: If $w \in R^{ECL}$ is Q-decomposable, then, for any $S \in Q$, $\sum_{n \in S} \Phi_n(w) = w_{S,Q}$.

A second corollary notes that our value is a consistent extension of the Shapley value.

COROLLARY 2: Suppose $w \in R^{ECL}$ and $v \in R^{CL}$ satisfy $w_{S,Q} = v_S$, $\forall (S,Q) \in ECL$. Then $\Phi(w) = \varphi(v)$, where $\varphi: R^{CL} \to R^N$ is the Shapley value operator.

<u>Example.</u> Suppose $w_{\{2,3\},\{\{1\},\{2,3\}\}} = 1$, $w_{\{1\},\{\{1\},\{2,3\}\}} = -1$, and all other embedded coalitions have wealth zero, in a partition function game w on the set of players $N = \{1,2,3\}$. The idea is that every coalition earns wealth zero if players 2 and 3 are separated, but if 2 and 3 are grouped together then they cooperate to earn one unit of wealth at a cost of one unit to the coalition containing player 1 (so $w_{\{1,2,3\},\{\{1,2,3\}\}} = 1 + (-1) = 0$). Our value function prescribes $\Phi_1(w) = -1$, $\Phi_2(w) = 1/2 = \Phi_3(w)$. This is a reasonable result, because this game is $\{\{1\},\{2,3\}\}$ - decomposable and $\Phi_2 = \Phi_3 = 1/2$ is the reasonable allocation for two players who earn zero

apart but earn one unit wealth together.

The Shapley value for this game depends on how we try to reduce the partition function game to a characteristic function game. If we use the rule $v_S \equiv w_{S,\{S,N\setminus S\}}$, then the Shapley value is $\varphi(v) = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$. If we use the rule $v_S \equiv w_{S,\{S\}\cup\{\{n\}|n\notin S\}}$ then the Shapley value is $\varphi(v) = (-\frac{1}{3}, \frac{1}{6}, \frac{1}{6})$. And if we use the rule $v_S = \min_Q w_{S,Q}$, then the Shapley value is $\varphi(v) = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$. All of these values would be inconsistent with our insight that the game is decomposable, with no effective cooperation between player 1 and the other two players.

Proofs.

THEOREM 1:

For any $(S,Q)\in ECL$ and any real number t, define the partition function game $x^{t,S,Q}$ by

$$(x^{t,S,Q})_{\tilde{S},\tilde{Q}} = \begin{cases} = t, & \text{if } (\tilde{S},\tilde{Q}) \gtrsim (S,Q) \\ = 0, & \text{otherwise}. \end{cases}$$

(Recall equation (5).) Observe that $\pi\circ x^{t,S,Q} = x^{t,\pi(S,Q)}$, because $(\tilde{S},\tilde{Q})\gtrsim(S,Q)$ iff $\pi(\tilde{S},\tilde{Q})\gtrsim\pi(S,Q)$.

Suppose $\{n,m\}\subseteq\hat{S}\in Q$. Let $\pi:N\to N$ switch n and m, leaving all other players fixed. Then Axiom 1 requires that $\Phi_n(x^{t,S,Q}) = \Phi_{\pi(n)}(\pi\circ x^{t,S,Q}) = \Phi_m(x^{t,\pi(S,Q)}) = \Phi_m(x^{t,S,Q})$.

Also, if $\hat{S}\in Q$, then both S and $S\cup\hat{S}$ are carriers of $x^{t,S,Q}$, so Axiom 2 requires that:

$$\sum_{n\in\hat{S}} \Phi_n(\mathbf{x}^{t,S,Q}) = \begin{cases} = t, \text{ if } \hat{S} = S; \\ = 0, \text{ for any other } \hat{S}\in Q. \end{cases}$$

Thus Value Axioms 1 and 2 require that

$$\Phi_n(\mathbf{x}^{t,S,Q}) = \begin{cases} = t/|S|, \text{ if } n\in S; \\ = 0, \text{ if } n\notin S. \end{cases}$$

Note that, in terms of R^{ECL} as a real vector space, $\mathbf{x}^{t,S,Q} = t\cdot\mathbf{x}^{1,S,Q}$. So $\Phi(t\cdot\mathbf{x}^{1,S,Q}) = \Phi(\mathbf{x}^{t,S,Q}) = t\cdot\Phi(\mathbf{x}^{1,S,Q})$. For simplicity, henceforth let $\mathbf{x}^{S,Q} = \mathbf{x}^{1,S,Q}$.

The next claim is that the collection $\{\mathbf{x}^{S,Q} | (S,Q)\in ECL\}$ forms a basis for R^{ECL} considered as a real vector space. There are $|ECL|$ games in the collection, and $\dim(R^{ECL}) = |ECL|$. So it suffices to show that the games in the collection are linearly independent.

Suppose that there were some sequence of coefficients $r_{S,Q}$, not all zero, such that $\sum_{(S,Q)\in ECL} r_{S,Q}\cdot\mathbf{x}^{S,Q} = 0$ in R^{ECL}. Let $(\hat{S},\hat{Q})$ be any embedded coalition such that $|\hat{Q}| = \max\{|Q| \,|\, r_{S,Q} \neq 0\}$ and $r_{\hat{S},\hat{Q}} \neq 0$. Consider now that $\sum_{(S,Q)\in ECL} r_{S,Q}\cdot(\mathbf{x}^{S,Q})_{\hat{S},\hat{Q}} = 0$. But $(\mathbf{x}^{S,Q})_{\hat{S},\hat{Q}} = 0$ unless $(\hat{S},\hat{Q}) \gtrsim (S,Q)$, and $r_{S,Q} = 0$ unless $|\hat{Q}| \geq |Q|$. There is only one embedded coalition (S,Q) which satisfies both $(\hat{S},\hat{Q}) \gtrsim (S,Q)$ and $|\hat{Q}| \geq |Q|$, and that is $(S,Q) = (\hat{S},\hat{Q})$. We can thus conclude that $r_{\hat{S},\hat{Q}} = 0$, a contradiction of the way $(\hat{S},\hat{Q})$ was chosen. Therefore the collection of games $\mathbf{x}^{S,Q}$ is linearly independent, and forms a basis for R^{ECL}.

By induction in k, the number of summands, it easily follows from Axiom 3 that $\Phi(\sum_{i=1}^{k} w^i) = \Phi(\sum_{i=1}^{k-1} w^i + w^k) = \Phi(\sum_{i=1}^{k-1} w^i) + \Phi(w^k) = \sum_{i=1}^{k} \Phi(w^i)$.

Now for any $w \in R^{ECL}$, there is a unique collection of coefficients $b_{S,Q}$ such that $w = \sum_{(S,Q)\in ECL} b_{S,Q} \cdot \mathbf{x}^{S,Q}$, because the $\mathbf{x}^{S,Q}$ games form a basis. But then we must have $\Phi(w) = \Phi(\sum_{(S,Q)} b_{S,Q} \cdot \mathbf{x}^{S,Q}) = \sum_{(S,Q)} \Phi(b_{S,Q} \cdot \mathbf{x}^{S,Q}) = \sum_{(S,Q)} b_{S,Q} \cdot \Phi(\mathbf{x}^{S,Q})$. This formula, together with

$$\Phi_n(\mathbf{x}^{S,Q}) = \begin{cases} 1/|S|, & \text{if } n \in S, \\ 0, & \text{if } n \notin S, \end{cases}$$

determines the unique function Φ which can satisfy the three Axioms.

To show that this value function actually satisfies the Axioms, henceforth let Φ be the function defined in the above paragraph. Because Φ is a linear map, it must satisfy Axiom 3.

To prove Axiom 1, consider any permutation $\pi: N \to N$ and any $n \in N$. Define the function $f: R^{ECL} \to R$ by $f(w) = \Phi_n(w) - \Phi_{\pi(n)}(\pi \circ w)$. Then $f(\cdot)$ is a linear function, because $\Phi(\cdot)$ and $(\pi \circ \cdot)$ are linear. For any $(S,Q) \in ECL$, $\Phi_n(\mathbf{x}^{S,Q}) = \Phi_{\pi(n)}(\mathbf{x}^{\pi(S,Q)})$, because $n \in S$ iff $\pi(n) \in \pi(S)$. So $f(\mathbf{x}^{S,Q}) = 0$, because $\mathbf{x}^{\pi(S,Q)} = \pi \circ \mathbf{x}^{S,Q}$. But these $\mathbf{x}^{S,Q}$ games span R^{ECL}, so $f(w) = 0$ for any w.

To prove Axiom 2, consider any $S \in CL$. Define the function $g: R^{ECL} \to R^{ECL}$ by $g_{\tilde{S},\tilde{Q}}(w) = w_{\tilde{S}\cap S, \tilde{Q}\wedge\{S, N\backslash S\}}$. Notice that g is linear, and

$$g(x^{\bar{S},\bar{Q}}) = \begin{cases} = x^{\bar{S},\bar{Q}}, \text{ if } (S,\{S,N\backslash S\}) \succsim (\bar{S},\bar{Q}); \\ = 0, \text{ otherwise.} \end{cases}$$

Thus games of the form $x^{\bar{S},\bar{Q}}$ such that $(S,\{S,N\backslash S\}) \succsim (\bar{S},\bar{Q})$ span the range of g. But for any game of this form, $\sum_{n\in S} \Phi_n(x^{\bar{S},\bar{Q}}) = \sum_{n\in\bar{S}} 1/|\bar{S}| = 1 = (x^{\bar{S},\bar{Q}})_{N,\{N\}}$. So by linearity of Φ, if w is in the range of g then $\sum_{n\in S} \Phi_n(w) = w_{N,\{N\}}$. But if S is a carrier of w then $w = g(w)$, so $\sum_{n\in S} \Phi_n(w) = w_{N,\{N\}}$ follows.

THEOREM 2:

For any finite set K, let PT(K) be the set of partitions of K. We will use the following combinatorial fact:

$$\sum_{P\in PT(K)} (-1)^{|P|-1} (|P|-1)! = \begin{cases} 1 & \text{if } |K| = 1 \\ 0 & \text{if } |K| > 1. \end{cases}$$

(If $|K| = 1$, then the unique partition has $|P| = 1$, so the equation is obvious.) (If $|K| > 1$, select any $k\in K$, and observe that $K\backslash\{k\} \neq \emptyset$. For each partition $\hat{P}\in PT(K\backslash\{k\})$, we can make $|\hat{P}|$ different partitions of K by adding k to any of the $|\hat{P}|$ sets in $\hat{P}$, and we can make one other distinct partition of K by putting k alone in $\{K\}$. Each $P\in PT(K)$ is made from exactly one $P\in PT(K\backslash\{k\})$ by one of these procedures. Therefore, $\sum_{P\in PT(K)} (-1)^{|P|-1}(|P|-1)! =$

$$= \sum_{\hat{P}\in PT(K\backslash\{k\})} ((-1)^{|\hat{P}|-1}(|\hat{P}|-1)!\cdot|\hat{P}| + (-1)^{|\hat{P}|+1-1}(|\hat{P}|+1-1)!)$$

$$= \sum_{\hat{P}\in PT(K\backslash\{k\})} (-1)^{|\hat{P}|-1} (|\hat{P}|!-|\hat{P}|!) = 0.)$$

The value function of Theorem 1 was characterized in the proof as being the unique linear map $\Phi: R^{ECL} \to R^N$ such that

$$(*) \qquad \Phi_n(x^{\overline{S},\overline{Q}}) = \begin{cases} 1/|\overline{S}| & \text{if } n \in \overline{S}, \\ 0 & \text{if } n \notin \overline{S}, \end{cases}$$

$$\text{where } (x^{\overline{S},\overline{Q}})_{S,Q} = \begin{cases} 1 & \text{if } (S,Q) \gtrsim (\overline{S},\overline{Q}) \\ 0 & \text{otherwise.} \end{cases}$$

So it suffices to show that the formula in Theorem 2 defines a functim which satisfies these conditions as well. The function is obviously linear in w, so it remains to show that (*) is satisfied for every $(\overline{S},\overline{Q}) \in ECL$ and every $n \in N$.

Consider any $n \in N$ and $(\overline{S},\overline{Q}) \in ECL$. Let $\hat{S}$ be the coalition such that $n \in \hat{S} \in \overline{Q}$. Then the formula in Theorem 2 implies that:

$$\Phi_n(x^{\overline{S},\overline{Q}}) = \sum_{(S,Q) \gtrsim (\overline{S},\overline{Q})} (-1)^{|Q|-1} \cdot (|Q|-1)!$$

$$\cdot \left(\frac{1}{|N|} - \sum_{\substack{\tilde{S} \in Q \\ \ni\, n \notin \tilde{S} \\ S \neq \tilde{S}}} \frac{1}{(|Q|-1)(|N|-|\tilde{S}|)}\right)$$

$$= \sum_{P \in PT(\overline{Q})} (-1)^{|P|-1} (|P|-1)!$$

$$\cdot \left(\frac{1}{|N|} - \sum_{\substack{T \in P \\ \ni\, \hat{S} \notin T \\ \overline{S} \notin T}} \frac{1}{(|P|-1)(|N|-|\bigcup_{S \in T} S|)}\right) =$$

$$= \sum_{T \subseteq \overline{Q} \setminus \{\hat{S}, \overline{S}\}} \frac{1}{|N| - |\bigcup_{S \in T} S|} \left(\sum_{P \in PT(\overline{Q} \setminus T)} (-1)^{|\hat{P}|-1} (|\hat{P}|-1)! \right) =$$

$$= \begin{cases} \dfrac{1}{|N| - |\bigcup_{S \in \overline{Q} \setminus \{\overline{S}\}} S|} & \text{if } \overline{S} = \hat{S} \\ 0 & \text{if } \overline{S} \neq \hat{S} \end{cases}$$

$$= \begin{cases} 1/|\overline{S}| & \text{if } n \in \overline{S}, \\ 0 & \text{if } n \in \overline{S}. \end{cases}$$

COROLLARY 1:

We are given a partition Q, and a Q-decomposable partition function game w. For each $S \in Q$, define $w^S \in R^{ECL}$ by $(w^S)_{\tilde{S},\tilde{Q}} = w_{\tilde{S} \cap S, \tilde{Q} \wedge Q}$. Then $w = \sum_{\hat{S} \in Q} w^{\hat{S}}$, by Q-decomposability. For any S and $\hat{S}$ in Q, both $\hat{S}$ and $\hat{S} \cup S$ are carriers of $w^{\hat{S}}$, so

$$\sum_{n \in S} \Phi_n(w^{\hat{S}}) = \begin{cases} = (w^{\hat{S}})_{N,\{N\}} = w_{S,Q}, & \text{if } S = \hat{S}; \\ = 0, & \text{for any other } S \in Q. \end{cases}$$

Thus, for any $S \in Q$, $\sum_{n \in S} \Phi_n(w) = \sum_{n \in S} \Phi_n(w^S) = w_{S,Q}$.

COROLLARY 2:

Define the linear map $\theta : R^{CL} \to R^{ECL}$ by $\theta_{S,Q}(v) = v_S$ for any $v \in R^{CL}$ and $(S,Q) \in ECL$. Then our Value Axioms 1, 2, and 3 for $\Phi(\cdot)$ directly imply the Shapley value axioms for $\Phi(\theta(\cdot))$. So by uniqueness of the Shapley value for these axioms ([SHAPLEY, 1953]), $\varphi(v) \equiv \Phi(\theta(v))$.

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