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INFORMATION SYSTEMS FOR OBSERVING INVENTORY LEVELS

by

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Abstract

This paper analyzes some problems in the design of information systems for observing and reporting inventory levels. The designer has simultaneously to choose a system for observing inventory and a method for incorporating the information obtained into an estimate of inventory levels. With regard to the observation system the designer can choose to use 'physical stocktaking' or the 'perpetual inventory' method in any period. With regard to the estimation system the designer can choose to use the observations themselves in forming the estimates of the inventory level (conventional system) or he can process the observations to form estimates of inventory levels which are optimal in a least-squares sense. Techniques for information system design are developed and numerical examples are provided.

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There are two possible systems for observing the status of inventory: (a) direct observation of the inventory levels of various items by a 'physical count' and (b) observation of the transactions which affect inventory levels and subsequent computation of the number of items of each type which are in stock. Method (a) is commonly referred to as 'physical stocktaking' and method (b) as 'perpetual inventory bookkeeping'. In general, management can decide to use method (a) or method (b) exclusively in every time period or, alternatively, to use the less expensive perpetual inventory method in each period and to employ physical stock-taking at less frequent intervals as a means for correcting for cumulative errors from 'shrinkage' of perishable items and from 'pilfering' or 'shoplifting'. In current information systems the observations are used directly without further processing as the estimates of inventory status. However, an alternative exists in that it is possible to use the observations to form linear least-squares estimates of inventory status using some adaptations of Kalman filtering theory [9]. Thus choices are available both with respect to the type of observation system and with respect to the way the observations are used to form the estimates of inventory levels. This paper examines these alternatives and develops a method for determining the optimal 'accuracy' of observation systems of types (a) and (b) together with an optimal strategy for their development over time. The extent of the inaccuracies in stock records and the importance of the problem discussed in this paper are well-documented in [4]. The model should be most applicable to department stores and supermarkets where inventory losses are large relative to profit margins.

Previous research on inventory information systems has concentrated mainly on forecasting aspects (see for example [3]) and on the data structure and processing requirements for 'material requirements planning' [13]. Discussion concerning perpetual inventories versus physical stocktaking can be found in the accounting literature (for example [6]). Analytical studies are contained in [3] and [16] in terms of economic lot size formulae and under the assumption of perfect knowledge of inventory levels. Iglehart and Morey [7] consider the imperfect information case. For a single item of inventory and given accuracy of the physical counts they provide formulae for the time interval between physical counts which minimizes the long-run average cost subject to a constraint on the probability of stock outs. For comparison, the approach discussed in this paper encompasses both finite and infinite horizon problems, allows for statistical and other interactions between inventory items, and considers the possibility of processing the inventory observations to form optimal least-squares estimates of the inventory levels. Tapiero [18] also discusses the application of Kalman filtering techniques in inventory systems, and derives formulae for the optimal time between inventory measurements that are similar to those derived here. However, Tapiero does not consider the optimal combination of perpetual inventory and physical stocktaking observation systems and only treats the case of a single inventory item.

In Section 1 of this paper the general model and notation is developed. The theory of the Kalman filter is briefly described and extended to include the special types of observation system associated with method (b) above. In Section 2 some methods for evaluating the performance of the alternative observation systems are given. Section 3 states the information systems design problem in general terms. Section 4 characterizes the solution and develops computational procedures for the time-invariant case. A computational example is given in Section 5. Finally, applications of the techniques developed to other areas of management information systems are discussed briefly in the conclusion.

1. Alternative Estimation Schemes

This section develops procedures for measuring the performance of the various observation systems in terms of the associated covariances of the observation errors in the system state. These results might be used without further analysis to make decisions with respect to the observation system to be employed. Alternatively, as discussed in Section 2, the covariance matrices can be used as inputs to a simulation program or mathematical model in which some criterion function is to be optimized.

Let $x_{i,t} \in R^1$ represent the number of units of inventory level of item i at time t, where i = 1,2,...,n. Depending on the inventory system employed $x_{i,t}$ might represent inventory-on-hand or inventory-on-hand plus on-order minus backorders. In the latter case the model below accommodates non-zero lead-times and negative values for $x_{i,t}$. The state vector $\mathbf{x}_t \in R^n$ can easily be augmented to include components representing other quantities of interest such as work force level as is done in [5] and [10]. The present interpretation of x_t has been chosen to simplify the exposition. Let $x \in R^n$ represent the initial status of inventory at some reference time t = 0. In general x will not be known with certainty and it will be assumed to be a random vector with prior $x_{0|-1}$ and covariance matrix $X_{0|-1}$. Let $a_{i,t} \geq 0$ represent the production or purchasing decision with respect to item i and $d_{i,t} \geq 0$ be a random variable representing the demand for item i at time t. The corresponding action and demand vectors are given by $a_t \in R^n$ and $d_t \in R^n$ respectively. The random vectors d_t are assumed to have covariance matrices, Δ_t, t = 0,1,2,... Due to pilfering and/or natural shrinkage the level of inventory diminishes over time (in the absence of both replenishment action and random demands) according to the difference equations:

$$x_0 = x$$

$$x_{t+1} = D_t x_t - p_t;\ t = 0,1,2,\ldots$$

where $p_t \in R^n$, is a random disturbance term with $E[p_t] \geq 0$ and covariance Π_t. In many applications the matrices D_t will be diagonal. If the diagonal elements are less than or equal to one the above system represents exponential decay of the inventory items over time with additive noise. However, in contrast to the perishable inventory models treated in [12] and [14] the model does not account for the inventory levels of different age groups for each item. Let the decrease in inventory due to d_t and p_t in period t be given by $e_t = d_t + p_t$. It is assumed that $x_0, e_0, e_1, \ldots$ are uncorrelated. Note that it is not necessary to assume that the demand, d_t, and random shrinkage and pilfering, p_t, are independent. The dynamic system describing the progression of inventory levels over time is then given by:

$$x_0 = x \tag{1}$$

$$x_{t+1} = D_t x_t + E_t a_t + f_t + u_t; \ 0 \leq t \leq T-1 \tag{2}$$

where $f_t = -E[e_t]$; $t = 0,1,2,\ldots$ and $u_0, u_1, u_2, \ldots$ are uncorrelated zero-mean random vectors with covariance matrices U_t, $t = 0,1,2,\ldots$ The term $E_t a_t$ represents the increase in inventory levels during the t^{th} time period due to the ordering action a_t taken at the beginning of the period. The n x n matrices, E_t can be used (possibly in conjunction with a cost function) to impose restrictions on the pattern of purchasing or production. For example, it may only be possible to purchase some items in certain time periods. Alternatively, consider a three-item inventory system in which one unit of item 3 is used in each of items 1 and 2, but is never ordered separately. This situation is captured by letting $E_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $t = 0, 1, 2, \ldots$ where the second element, a_{2t}, of the decision vector represents the amounts of items 2 and 3 which are to be ordered.

At time t = 1,2,..., information $y_t \in R^m$ becomes available from which an estimate of the state, x_t, can be constructed. Denote the history of past actions at time $t \geq 0$ by $a^t = (a_0,a_1,\ldots,a_{t-1})$ with $a^t = \emptyset$ when $t < 0$ (a 'null' action) and let the history of past observations be given by $y^t = (y_0,y_1,\ldots,y_t)$. Also let the conditional expectations and covariances of the system state be given by:

$$x_{t|s} = E[x_t|y^s,a^{t-1}], \quad t = 0,1,2,\ldots \tag{3}$$

$$X_{t|s} = \text{cov}[x_t,x_t|y^s,a^{t-1}], \quad t = 0,1,2,\ldots \tag{4}$$

where $t-1 \leq s \leq t$. In particular, $x_{t|t-1}$ can be regarded as the 'prior' and $x_{t|t}$ as the 'posterior' expected level of inventories.

The 'physical stocktaking' and 'perpetual inventory' information systems are assumed to produce observations y_t^a and y_t^b respectively at time t according to the following 'observation equations':

$$y_t^a = H_t^a x_t + v_t^a \qquad ; t = 0,1,2,\ldots \tag{5}$$

$$y_t^b = H_t^b(E_{t-1}a_{t-1} + d_{t-1}) + v_t^b \; ; t = 1,2,\ldots \tag{6}$$

where $H_t^a \in R^{m_a \times n}$ and $H_t^b \in R^{m_b \times n}$ are constant matrices and $v_t^a \in R^{m_a}$ and $v_t^b \in R^{m_b}$ are zero mean random 'noise' vectors with covariance matrices V_t^a and V_t^b respectively. Note that d_{t-1} and not e_{t-1} is observed in (6). The matrices H_t^a,H_t^b can be used to model cases where aggregates of inventory items rather than individual items are observed. For example, if the perpetual inventory system only tracks the sum of transactions for all items we set $H_t^b = (1\,1\,1\ldots1)$. The noise vectors v_t^a, $t \geq 0$ model sources of error in the observation of the physical quantities on hand while the vectors v_t^b, $t \geq 0$, model sources of error in the observation of the transactions which affect inventory. Possible reasons for inaccuracy in these observations include time lags between the flows of information and material, clerical errors and physical measurement errors (for example, the stock-on-hand might be estimated by weighing it or by a quick visual inspection rather than by an explicit count). In addition, the perpetual inventory system accumulates errors in the estimates of x_t because the random term, p_t, is never observed. Cases where selected items are not observed at all at time t can be accommodated by setting the appropriate diagonal elements of the matrices V_t^a, V_t^b to be very large.

It will be assumed in the remainder of the paper that the disturbance terms, u_t, v_t^a, v_t^b and the initial state x have Gaussian probability distributions. The normal probability distribution is often assumed for inventory systems (see [3].) However, strictly speaking this assumption is only necessary for the 'Linear Quadratic Gaussian' problem described in Section 2. All other results hold for more general distributions with the interpretation that $x_{t|s}$ is the linear least squares estimator of x_t given information y^s or the projection of x_t on the space spanned by y^s. When the probability distributions are sufficiently symmetric the least squares estimates and resulting cost calculations will be satisfactory.

Methods for processing the observations (5) and (6) to form optimal estimates of x_t are now described. The following theorem, due to Kalman [9], is a basic result.

Theorem 1. Physical Stocktaking. For the dynamic system described by (1), (2), and (5), the conditional probability density functions, $P(x_t|y^t, a^{t-1})$ and $P(x_{t+1}|y^t, a^t)$ are normal with means and covariances defined by equations (7) to (12):

$$x_{0|-1} = E[x];\ X_{0|-1} = \mathrm{cov}[x,\ x] \tag{7}$$

$$x_{t|t} = x_{t|t-1} + K_t^{a}(y_t - H_t^{a}x_{t|t-1}), \quad t = 0,1,2,\ldots, \tag{8}$$

$$X_{t|t} = X_{t|t-1} - K_t^a H_t^a X_{t|t-1} \quad t = 0,1,2,\ldots, \tag{9}$$

where

$$K_t^a = X_{t|t-1}H_t^{a\prime}(H_t^a X_{t|t-1}H_t^{a\prime} + V_t^a)^{-1} \quad t = 0,1,2,\ldots, \tag{10}$$

$$x_{t+1|t} = D_t x_{t|t} + E_t a_t + f_t, \quad t = 0,1,2,\ldots, \tag{11}$$

$$X_{t+1|t} = D_t X_{t|t} D_t' + U_t, \quad t = 0,1,2,\ldots \quad // \tag{12}$$

A similar result is obtained here for the 'perpetual inventory' observation system:

Theorem 2. Perpetual Inventory. For the dynamic system defined by (1), (2) and (6) the conditional probability density functions $P(x_{t|}y^{t},a^{t-1})$ are normal with means and covariances defined by equations (11) and (12) above together with (13 to (16):

$$x_{0|0} = x_{0|-1} = E[x];\quad X_{0|0} = X_{0|-1} = \mathrm{cov}[x;x] \tag{13}$$

$$x_{t|t} = x_{t|t-1} + K_t^b(y_t - H_{t-1}^b(E_{t-1}a_{t-1} + E[d_{t-1}])) \quad t = 1,2,\ldots, \tag{14}$$

$$X_{t|t} = X_{t|t-1} - K_t^b H_t^b \Delta_{t-1} \quad t = 1,2,\ldots, \tag{15}$$

where:

$$K_t^b = \Delta_{t-1}H_t^{b\prime}(H_t^b\Delta_{t-1}H_t^{b\prime} + V_t^b)^{-1} \quad t = 1,2,\ldots, \quad // \tag{16}$$

A proof of Theorem 2 is given in the Appendix. Theorems 1 and 2 provide a simple means for computing the conditional means, $x_{t|t}$, and covariances, $X_{t|t}$, for any alternating sequence of observations of types a and b. For example, suppose inventory status reports are issued monthly using a physical count every quarter and that the perpetual inventory method is used in the intervening months. Also let D_t, H_t^a and H_t^b be identity matrices, and $\Delta_t = \Delta$, $U_t = U$, $V_t^a = V^a$, $V_t^b = V^b$, $t = 0,1,2,\ldots$ Then $K_t^b = K^b = \Delta(\Delta + V^b)^{-1}$, $t = 1,2,4,5,\ldots$, and the sequence of computations would be:

$$K_0^a = X_{0|-1}(X_{0|-1} + V^a)^{-1}; \; X_{0|0} = X_{0|-1} - K_0^a X_{0|-1}; \; X_{1|0} = X_{0|0} + U.$$

$$X_{1|1} = X_{1|0} - K^b\Delta; \; X_{2|1} = X_{1|1} + U$$

$$X_{2|2} = X_{2|1} - K^b\Delta; \; X_{3|2} = X_{2|2} + U$$

$$K_3^a = X_{3|2}(X_{3|2} + V^a)^{-1}; \; X_{3|3} = X_{3|2} - K_3^a X_{3|2}; \; X_{4|3} = X_{3|3} + U, \text{ etc.}$$

An alternative means for carrying-out these calculations which involves augmenting the state space (and a heavier computational burden) is described in [17]. The method in [17] also allows for the case where observations of types a and b are made simultaneously.

As far as is known no existing inventory data processing system attempts to compensate for errors due to inventory shrinkage and measurement noise in the manner implied by Theorems 1 and 2. Observations of types a and b are made according to (5) and (6), but this data is not further processed to form optimal (least squares) estimates of the inventory levels. This contrasts with the considerable efforts which have been made to provide processing systems which will accurately forecast demand [3]. To analyze the 'conventional' data processing system, let $H_t^a = H_t^b = I_n$, $t = 0,1,2,\ldots$ where I_n is the $n \times n$ identity matrix. Thus all inventory items and transactions are observed. The conventional estimators corresponding to direct observation and observation of the transactions are given by (17) and (18) respectively:

$$x_{t|t} = y_t^a \tag{17}$$

$$x_{t|t} = D_{t-1}x_{t-1|t-1} + E[p_t] + y_t^b \tag{18}$$

where y_t^a is given by (5) with $H_t^a = I_n$ and y_t^b is given by (6) with $H_t^b = I_n$. Theorems 3 and 4 summarize the error covariances for conventional estimating systems:

<u>Theorem 3</u>: Physical Stocktaking. For the dynamic system given by (1), (2) and (5) the error covariances of the estimator (17) are:

$$X_{t|t} = V_t^a \quad , t = 0,1,2,\ldots \qquad // \tag{19}$$

<u>Theorem 4</u>: Perpetual Inventory. For the dynamic system given by (1), (2) and (6) the error covariances of the estimator (18) are given recursively by:

$$X_{t|t} = D_{t-1}X_{t-1|t-1}D'_{t-1} + \pi_{t-1} + V_t^b \quad , t = 1,2,\ldots \quad // \tag{20}$$

Theorem 3 is obvious; the proof of Theorem 4 is given in the Appendix. The covariance matrices, $X_{t|t}$, $t = 0,1,\ldots$, for any alternating sequence of observations of types a and b can easily be found using (19) and (20).

2. Evaluation of Observation System Performance

Depending on the circumstances involved there are several ways of evaluating the performance of the alternative observation systems in terms of the error covariance matrices. As a simple example consider a finite horizon periodic review 'order-up-to-z' inventory system with perfect observations (see, for example, [1], Chapter 9). If $x_{i,t} \leq z_{i,t}$ the order quantity for item i is $z_i - x_{i,t}$ (the case $x_{i,t} > z_{i,t}$ will not normally arise and is ignored). The expected cost in period t if inventory levels x_t are observed is assumed to be given by $PC'(z_t - x_t) + L(z_t)$ where $L:R^n \rightarrow R$ is the penalty and holding cost function and $PC \in R^n$ is the vector of procurement costs for the items. Note that L may simply be the sum of the corresponding functions for the individual items. Now if $x_{t|t}$ rather than x_t is known, the expected cost will be $PC'(z_t - x_{t|t}) + L(z_t + x_t - x_{t|t})$ if the possibility, $x_{i,t|t} > z_{i,t}$ is ignored. The expected cost of imperfect information at time, t, will be:

$$E[PC'(x_t - x_{t|t}) + L(z_t + x_t - x_{t|t}) - L(z_t)] = E[L(z_t + x_t - x_{t|t}) - L(z_t)]$$

$$\cong \tfrac{1}{2}E[(x_t - x_{t|t})' \frac{\partial^2 L(z_t)}{\partial z^2}(x_t - x_{t|t})] = \tfrac{1}{2}\, tr[\frac{\partial^2 L(z_t)}{\partial z^2} X_{t|t}]$$

where tr[X] denotes the trace of the matrix X and it has been assumed that L can be approximated by the first two terms in its Taylor series expansion about z_t.

For a time horizon of T periods the cost of imperfect information is:

$$\sum_{t=0}^{T-1} tr\, F_t X_{t|t} \quad \text{where} \quad F_t = \frac{1}{2}\frac{\partial^2 L\,(z_t)}{\partial Z^2} \,. \tag{21}$$

The values of z_t in (21) should be chosen to minimize the expected inventory cost taking into account the existence of imperfect information. However, this refinement is not pursued here.

Similar approximations can be found for other fixed review period inventory policies including those with set up costs. Alternatively, the cost performance of the observation systems might be evaluated using a Linear Quadratic Gaussian (LQG) control model [2]. Define the decision function at time t by $a_t = \alpha_t(x_{t|t})$ and a policy for the T horizon problem by $\alpha = (\alpha_0, \alpha_1 \ldots \alpha_{T-1})$. Assume that the objective of the inventory control system is to find the policy, α^* which minimizes (22) below subject to (1) and (2):

$$V(\alpha^*) = \min_{\alpha} E[\sum_{t=0}^{t=T-1} \gamma_t(a_t,x_t) + \gamma_T(x_T)] \tag{22}$$

The cost functions, γ_t, $t = 0,1,\ldots,T$ are assumed to be quadratic and convex in the states and actions. They would include the cost of ordering inventory and of deviations of the inventory levels from design target levels in each time period. The model (1), (2) and (22) is quite general and can be adapted to many different situations by giving different interpretations to the vectors x_t and a_t. For example, the production planning and inventory model of Holt et al, [5], in which the decision variables are aggregate production rate and level of workforce, can be written in this form. Specifically, let:

$$\gamma_t(a_t,x_t) = k_t + c_t'a_t + a_t'C_ta_t + a_t'M_tx_t + b_t'x_t + x_t'B_tx_t, \quad 0 \leq t \leq T-1, \tag{23}$$

$$\gamma_T(x_T) = k_T + b_T'x_T + x_T'B_Tx_T.$$

The parameters, k_t, c_t, C_t, b_t, B_t, and M_t will be assumed to be known constants. The $m \times m$ matrices, C_t, $0 \leq t \leq T-1$, are assumed to be symmetric positive definite and the $n \times n$ matrices, B_t, $0 \leq t \leq T$, are symmetric positive semi-definite. By augmenting the state space and redefining variables it is possible to simplify the algebra by eliminating f_t, k_t, c_t, b_t. However, it will be computationally advantageous to work in the original state space in section 4. The following theorem is a version of the Kalman "Separation Theorem", (see [2]), for (1), (2), and (23).

<u>Theorem 5:</u> The minimum expected cost of the LQG problem is:

$$V(\alpha^*) = x_{0|-1}'G_0x_{0|-1} + h_0'x_{0|-1} + g_0 + tr[G_0X_{0|-1}] + \sum_{t=0}^{T-1} tr[G_{t+1}U_t] + \sum_{t=0}^{T-1} tr[F_tX_{t|t}]. \tag{24}$$

The optimal decision function at time t, $0 \leq t \leq T-1$, is given by

$$\alpha_t^*(x_{t|t}) = -\tfrac{1}{2}(C_t + E_t'G_{t+1}E_t)^{-1}(c_t + E_t'h_{t+1} + 2E_t'G_{t+1}f_t + [M_t + 2E_t'G_{t+1}D_t]x_{t|t}), \tag{25}$$

where:

$$F_t = \tfrac{1}{4}(M_t + 2E_t'G_{t+1}D_t)'(C_t + E_t'G_tE_t)^{-1}(M_t + 2E_t'G_{t+1}D_t)$$

and the constants, h_t, g_t, G_t, $0 \leq t \leq T-1$, are recursively generated by the following equations:

$$g_T = k_T, \quad h_T = b_T, \quad G_T = B_T;$$

and for $0 \leq t \leq T-1$:

$$g_t = g_{t+1} + k_t + h_{t+1}'f_t + f_t'G_{t+1}f_t$$

$$- \tfrac{1}{4}(c_t + E_t'h_{t+1} + 2E_t'G_{t+1}f_t)'(C_t + E_t'G_{t+1}E_t)^{-1}(c_t + E_t'h_{t+1} + 2E_t'G_{t+1}f_t)$$

$$+ \mathrm{tr}[B_tX_{t|t} + G_{t+1}(X_{t+1|t} - X_{t+1|t+1})]$$

where $X_{T|T} = 0$

$$h_t' = b_t' + h_{t+1}'D_t + 2f_t'G_{t+1}D_t$$

$$- \tfrac{1}{2}(c_t + E_t'h_{t+1} + 2E_t'G_{t+1}f_t)'(C_t + E_t'G_{t+1}E_t)^{-1}(M_t + 2E_t'G_{t+1}D_t);$$

$$G_t = B_t + D_t'G_{t+1}D_t - F_t; \qquad //$$

A proof of these results is given in [17]. Theorem 5 summarizes a number of important properties of the LQG problem. First, the control action, a_t, is calculated using a decision rule which is a linear function of the estimated status of inventory. This decision rule is also optimal for the deterministic problem obtained by setting all the random disturbance terms to their expected values and assuming perfect observation of the system states (the "certainty equivalence" principle, [15]). Second, the matrices, $X_{t|t}$, are independent of the actual evolution of the process and can be computed from prior knowledge. Furthermore, $x_{t+1|t+1}$ can be computed knowing only the value of $x_{t|t}$ and the intervening action, a_t, and observation, y_{t+1}. Finally, the expression (24) separates-out the effects of the three sources of uncertainty in the inventory management problem. The first three terms in (24) give the minimum expected cost of the equivalent deterministic problem obtained by setting all random parameters of the problem to their expected values and assuming perfect observation of the inventory levels. The next term, $tr[G_0 X_{0|-1}]$, is the additional expected cost caused by the uncertainty concerning the initial inventory level. The term, $\sum_{t=0}^{T-1} tr[G_{t+1} U_t]$, measures the additional expected cost caused by the Gaussian random vectors u_t i.e., by the randomness in the demand and inventory shrinkage. Finally, the term, $\sum_{t=0}^{T-1} tr[F_t X_{t|t}]$, is the additional expected cost caused by the uncertainty in the observations of the inventory levels. Note that this has the same form as (21). In the perfect observation case, $y_t = x_t$, $0 \leq t \leq T-1$, and this term is zero. The problem of selecting an optimal information system involves trading off the cost of different observation and estimation systems against the corresponding cost of the term $\sum_{t=0}^{T-1} F_t X_{t|t}$.

3. The Information System Design Problem

An information system in period t is characterized by the observation method (defined either by (5) or (6)) and the estimation method employed (described by Theorems 1,2,3 and 4). Let i_t be the information system chosen in period t, $i^t = (i_0, i_1, \ldots, i_t)$ and I_{T-1} be the set of all feasible sequences i^{T-1}. An information system for the T-period problem is a particular choice of $i^{T-1} \in I_{T-1}$. Let the cost of an information system, i^{T-1}, be given by $\Gamma(i^{T-1})$. The dependence of the covariance matrix $X_{t|t}$ on the sequence of observations up to and including time t will be recognized by the notation $X_{t|t}(i^t)$. If the cost of imperfect observation at time t is given by (21), the information system design problem is:

$$\min_{i^{T-1} \in I_{T-1}} \{\Gamma(i^{T-1}) + \mathrm{tr}[\sum_{s=0}^{T-1} F_s X_{s|s}(i^s)]\} \tag{26}$$

where the n $\times$ n matrices, $F_s, 0 \leq s \leq T-1$, are calculated using the methods given in Section 2 and, for each sequence, i^s, the n $\times$ n covariance matrix $X_{s|s}(i^s)$ can be calculated using the methods described in Section 1. Define the "state transition matrices":

$$\Phi(t,s) = D_{t-1}D_{t-2}\cdots D_s, \; t > s; \qquad \Phi(s,s) = I_n$$

where $s = 0,1,2,\ldots$ The minimum possible value for the second term in (26) is zero which occurs when $y_t = x_t$, $0 \leq t \leq T-1$ (perfect observation) while for the LQG problem the maximum possible value can be shown to be

$$\mathrm{tr}[\sum_{t=0}^{T-1} F_t(\Phi(t,0)X_{0|-1}\Phi'(t,0) + \sum_{s=1}^{t} \Phi(t,s)U_{s-1}\Phi'(t,s)\;)]$$

which occurs if there are no observations at all and which provides an upper bound for the expected value of perfect information.

Evaluation of (26) for any particular information system, i^{T-1}, is relatively easy. For the inventory problem it is assumed that management can choose to observe inventory in any period by physical stocktaking (see (5) or by the perpetual inventory method (see (6)). The solution to the information system design problem will trade-off the greater accuracy of physical stocktaking against the lower cost of the perpetual inventory method. The alternative observation systems of each type are characterized by the matrices (H_t^a, V_t^a)

and (H_t^b, V_t^b) respectively, available in each time period. To solve the information system design problem, the matrices, F_t, are first generated for the underlying control problem using the methods discussed in Section 2. Then for any given observation sequence, i^{T-1}, the optimal covariance matrices $X_{t|t}(i^{T-1})$ are found using Theorems 1 and 2, and (26) is evaluated. This process is repeated for the conventional observation systems for which the covariance matrices $X_{t|t}(i^{T-1})$ are found using Theorems 3 and 4.

4. Properties of the Time-Invariant Case

The number of feasible observation sequences may be extremely large so that optimization of (26) by complete enumeration may be impossible. However, the computational problem is greatly simplified if it is assumed that the data for the problem (as defined by equations (2) through (6) and (21) and the covariance matrices of the noise terms) is constant over time and that the objective is to minimize the long-run average cost per time period:

$$\lim_{T\to\infty} \frac{1}{T}(\Gamma\ (i^{T-1}) + \sum_{s=0}^{T-1} F_s X_{s|s}(i^s)$$

In this case it is shown below that it is only necessary to consider observation sequences in which the time interval between observations of type a is a constant k. In the time-invariant case the state transition matrix, $\Phi(t,s) = D^{t-s-1}$. If r is a time period in which an observation of type a is made we obtain for the optimal estimation system by application of Theorems 1 and 2:

$$X_{r+k|r+k-1} = D^{k-1} X_{r|r} D'^{k-1} + \sum_{s=r+1}^{r+k-1} D^{k-1}(U - K^b H\Delta)\, D'^{k-1} + U \qquad (27)$$

where $K^b = \Delta H^{b'} (H^b \Delta H^{b'} + V^b)^{-1}$

$$X_{r+k|r+k} = (I_n - K^a_{r+k} H^a)\,[\,D^{k-1} X_{r|r} D'^{k-1} + \sum_{s=r+1}^{r+k-1} D^{k-1}(U - K^b H\Delta) D'^{k-1} + U\,] \qquad (28)$$

where $K^a_{r+k} = X_{r+k|r+k-1} H^{a'} (H_a X_{r+k|r+k-1} H^{a'} + V^a)^{-1}$

Conditions for convergence of the covariance matrices $X_{r|r}$, $r = 0,k,2k,\ldots$ can be derived by the methods given in [8]. These conditions will generally be satisfied for the type of problem considered here. For a constant observation cycle length, k, (28) therefore provides a system of simultaneous equations which can be used in conjunction with Theorem 2 to compute the steady state covariances $\tilde{X}_o(k)$, and $\tilde{X}_s(k)$, $s = 1,2,\ldots,k-1$ corresponding to the error covariance matrices just after the observation of type a and after each observation of type b during the 'observation cycle' of length k:

$$\tilde{X}_s(k) = D^s \tilde{X}_o(k)\, D^{s'} + \sum_{j=0}^{s-1} D^j (U - K^b H\Delta) D^{j'}, \quad s = 1,2,\ldots,k-1 \tag{29}$$

For the conventional system the steady state covariances are given by:

$$\tilde{X}_o(k) = V^a \tag{30}$$

and:

$$\tilde{X}_s(k) = D^s V^a D^{s'} + \sum_{j=0}^{s-1} D^j (\Pi + V^b) D^{j'}, \qquad s = 1,2,\ldots,k-1 \tag{31}$$

<u>Theorem 6</u>: Conventional Estimation. For the system defined by (1), (2), (17) and (18), the long-run average cost per time period in the time-invariant case is minimized if observations of type a are made at regularly spaced intervals of time.

<u>Proof</u>:

From (30) and (31) the cost for an observation cycle of length k is given by:

$$w(k) = \gamma^a + (k-1)\gamma^b + \mathrm{tr}[F\{\sum_{s=0}^{k-1} (D^s V^a D'^s) + \sum_{s=0}^{k-2} D^s (\Pi + V^b) D'^s\}] \tag{32}$$

where γ^a is the cost of making an observation of type a and γ^b the cost of making an observation of type b.

Note that the cost of a cycle of length k is independent of the previous sequence of observation lengths because of (30). Let $\hat{k}$ minimize $\frac{w(k)}{k}$, $k = 1,2,\ldots$ Consider an arbitrary sequence $k(1),k(2),\ldots,k(n)$ of observation intervals. Over the n observation cycles the average cost per time period is:

$$W(k(1),k(2),\ldots,k(n)) = \sum_{j=1}^{n} w(k(j)) / \sum_{j=1}^{n} k(j)$$

It can be shown that W is reduced by replacing k(j) by $\hat{k}$, $1 \leq j \leq n$. Taking the limit as $n \to \infty$ proves the theorem.//

If optimal state estimation procedures are used the cost of a cycle of length k depends on the previous history of observations and a result similar to Theorem 6 is difficult to prove. However, a constant observation length policy will be at least approximately optimal and will often be desirable from management considerations.

<u>Theorem 7</u>: The constant observation cycle length which minimizes the average cost per period in the time-invariant case with $D = I_n$ is given by:

Optimal Estimation:

$$k' = \sqrt{2(\gamma^a-\gamma^b)/(\frac{d}{dk}\, tr[F\tilde{X}_o(\tilde{k})] + tr[F(U - K^b H\Delta)]} \tag{33}$$

Conventional Estimation:

$$k' = \sqrt{2(\gamma^a-\gamma^b)/tr[F(\Pi + V^b)]} \tag{34}$$

<u>Proof</u>:

From (29) the cost in each period increases linearly over the length of the cycle. The combined cost of observations and information over a cycle of length k is:

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[1]For simplicity k is treated as a continuous variable.

$$\gamma^a + (k-1)\gamma^b + tr[F\tilde{X}_o(k)] + \sum_{s=1}^{k-1}(tr[F\hat{X}_o(k)] + \sum_{j=0}^{s-1} tr[F(U - K^b H\Delta)]$$

Collecting terms, dividing by k and differentiating gives (33). The proof of (34) is similar.//

Choosing the optimal value $\hat{k}$ equal to $\lceil k' \rceil$ or $\lceil k' \rceil - 1$ where k' is given by (34) gives the optimal observation strategy for the conventional estimation case. In the optimal estimation case $\frac{d}{dk}\, tr[F\tilde{X}_o(k')]$ is not known a priori. However, since this term is positive (and is usually small), enumeration for several values of $k \leq \lceil k' \rceil$ rapidly yields the optimal observation cycle length $\hat{k}$. Ceteris paribus the value of $\hat{k}$ for the optimal estimation scheme will not be less than that for the conventional scheme because of its increased accuracy. Note that the minimum average cost of observation and imperfect information per time period for the conventional estimation case is:

$$\gamma^b + \frac{\gamma^a - \gamma^b}{\hat{k}} + tr[FV^a] + \frac{(\hat{k}-1)}{2}\, tr[F(\Pi + V^b)]$$

which gives an upper bound for the cost in the optimal estimation case.

When inventory shrinkage is important the eigenvalues of D will be less than or equal to one in absolute value and the cost of imperfect information will increase in a concave fashion over the duration of the observation cycle. (see for example (32)). As a result the optimal constant observation length $\hat{k}$ will exceed k' as given by (33) or (34).

5. Numerical Example

An infinite period problem is assumed with discount factor, DF = .995 and three inventory items having characteristics as shown in Table 1.

i	PC_i	HC_i	SC_i	$E[e_i]$	U_{ii}
1	15	.10	5	500	305
2	30	.20	10	200	205
3	40	.30	13	250	255

Table 1

In the table PC_i is the procurement cost of a unit of item i per time period, HC_i is the holding cost and SC_i is the stock-out cost per unit of item i per time period. It is assumed that the matrix $D = I_3$ (the identity matrix) and thus $e_i = d_i + p_i$ is the effective demand for item i per time period with expected values $E[e_i]$ and variances U_{ii} as given in the table. It is assumed that the probability distributions of effective demand are normal and independent from period-to-period. Under these conditions an 'order-up-to z_i' policy is optimal for item i (see [1], Chapter 9). If back-ordering is allowed, the optimal value of z_i is found by minimizing the function $(1-DF)p_i z_i + L_i(z_i)$ where

$$L_i(z_i) = HC_i \int_0^{z_i} (z_i - \delta)\omega_i(\delta)d\delta + SC_i \int_{z_i}^{\infty} (\delta - z_i)\omega_i(\delta)d\delta \qquad (35)$$

is the expected holding and shortage costs and ω_i is the density function for demand of item i. The optimal order-up-to values, z_1^*, satisfy:

$\Omega_i(z_i^*) = \dfrac{SC_i - PC_i(1 - DF)}{SC_i + HC_i}$ where Ω_i is the cumulative probability distribution function for demand of item i.[2] For the data in Table 1 the optimal values are $z_1^* = 531.8$, $z_2^* = 226.1$, $z_3^* = 278.4$. Finally, $\dfrac{\partial^2 L_i}{\partial z_i^2}(z_i) = (HC_i + SC_i)\omega_i(z_i)$, and assuming no interaction between items, the matrix, F_t, in (21) is as shown in Table 2.

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[2]A similar formula applies if it is assumed that unfilled demand is lost.

To complete the specification of the data it is assumed that $H^a = H^b = I_3$ and the covariance matrices Δ, P, U describing the random elements in the dynamics and V^a, V^b describing the noise in the two different types of observation systems are as shown in Table 2.

$$\Delta = \begin{bmatrix} 300 & -100 & 50 \\ -100 & 200 & 150 \\ 50 & 150 & 250 \end{bmatrix} ; \quad P = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 5 \end{bmatrix} ; \quad U = \Delta + P$$

$$V^a = \begin{bmatrix} 25 & 15 & -15 \\ 15 & 25 & 10 \\ -15 & 10 & 25 \end{bmatrix} ; \quad V^b = \begin{bmatrix} 5 & 4 & -3 \\ 4 & 5 & 3 \\ -3 & 3 & 5 \end{bmatrix} ; \quad F_t = \begin{bmatrix} .011 & 0 & 0 \\ 0 & .027 & 0 \\ 0 & 0 & .034 \end{bmatrix}$$

$$\gamma^a = 200; \qquad \gamma^b = 2$$

Table 2

In each time period (say 1 week) ordering actions will be taken based on the estimated inventory levels and the values of z_i. In some time periods the estimates of the inventory levels are obtained via perpetual inventory bookkeeping at a cost $\gamma^b = \$2$ and at other time periods by actual inspection of the amounts of each item on hand at cost $\gamma^a = \$200$. The same observation scheme (values for $\gamma^a, H^a, V^a, \gamma^b, H^b, V^b$) is assumed for both the optimal and conventional estimation schemes.

The computational results are shown in Table 3.

	Observation Cycle Length		
a. Optimal Estimation:	$k = 12$	$\hat{k} = 27$	$k = 52$
1. Holding and shortage costs (35)	23.29	23.29	23.29
2. Costs of Observation $(\frac{\gamma^a + (k-1)\gamma^b}{k})$	22.67	11.19	6.77
3. Cost of Imperfect Information (21)	5.42	10.67	19.36
Total Cost (1. +2. +3.)	51.38	45.15	49.41
b. Conventional Estimation:	$k = 12$	$\hat{k} = 26$	$k = 52$
1. Holding and Shortage Costs (35)	23.29	23.29	23.29
2. Costs of Observation $(\frac{\gamma^a + (k-1)\gamma^b}{k})$	22.67	11.54	6.77
3. Cost of Imperfect Information (21)	5.78	10.85	20.25
Total Cost (1. +2. +3.)	51.74	45.68	50.31

Table 3

Average Costs Per Week for Three Inventory Items

In the cases tested the optimal value of $\hat{k}$ for the optimal estimation case was the closest integer to the value of k' as given by (33) with $\frac{d}{dk}\,\mathrm{tr}[F\tilde{X}_o(\tilde{k})] = 0$ and hence a minimal amount of search was required. Note that $\hat{k}$ is approximately 6 months for both the optimal and conventional estimation schemes. The variability of costs with the choice of k is indicated by the columns for $k = 12$ (3 months) and $k = 52$ (1 year).

Although the problem size is small the results show significant differences in relative costs for different observation cycle lengths and to a lesser extent for the two estimation schemes. The imperfect information costs rise rapidly with k even when P is relatively small. The value of using an optimal estimation scheme rather than the point values of the observations (conventional scheme) increases with the noisiness of observation system as determined by V^a and V^b. In the example the cost saving is approximately 18¢ per item per week. Aggregated over many items the annual cost reduction could easily justify the increased computer storage and computation involved in implementing the optimal estimation scheme.

7. Conclusion

Computational studies including the results just cited indicate first, that it might sometimes be beneficial to include optimal state estimation techniques as an integral part of the data processing associated with an inventory information system. If the stochastic components of the system are relatively invariant over time the steady state Kalman gain matrices $\tilde{K}_o^a$, $\tilde{K}_s^b$, $1 \leq s \leq k-1$ can be precomputed and stored between successive estimations of the state vector x. If the stochastic elements vary in an unknown fashion over time 'adaptive' Kalman filtering techniques can be utilized (see [11]). The updated estimates (via (8),(11) and (14)) are simple to obtain however it is obvious that the storage and computational repuirements would be excessive for a large inventory of (say) several thousand items. Several approaches to this problem are possible including: (a) using the technique only for a limited number (say 20 or 30) of key items, (b) dividing the inventory into a large number of small groups of related items or (c) treating each item individually. In alternative (c) the information obtainable through knowledge of the correlations between items is lost but only a few additional data items ($x_{i,t|t-1}, X_{i,t|t-1}, \tilde{K}_i^a, \tilde{K}_i^b$) need be retained in the inventory record. An additional important advantage to be gained from implementing such a system is that it automatically provides a means for detecting and reporting observations which are unexpectedly high or low (and might therefore be in error or represent an environmental change worthy of immediate management attention.)

A second conclusion from the examples cited in the previous section (and other studies) is that the choice of a correct observation cycle length, k can be economically important. The computations involved in determining a good value of k are straightforward involving only the solutions of systems of simultaneous equations and the iterative computations defined by Theorems 1 and 2 or 3 and 4. The major difficulty with the technique is the requirement for special studies to determine the covariance matrices.

Although this paper has concentrated on a particular area of application of optimal state estimation procedures the techniques developed have potential in many other areas of management information systems design and implementation. The updating of estimates of the state of the system (e.g. the balance sheet accounts of a financial information system) by observation of transactions rather than direct observation of the system state itself is a pervasive phenomenon in information systems.

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APPENDIX

Derivation of Theorems 2 and 4

In the following the superscript 'b' will be dropped unless needed and the notation $\sigma \sim N(\mu,\Sigma)$ will mean that σ is a Gaussian random vector with mean, μ, and covariance matrix Σ.

Proof of Theorem 2

Equations (13) follow since there can be no observation at t = 0. To prove (11) and (12) use (2) and take conditional expectations:

$$E[x_{t+1}|y^t,a^{t-1}] = E[D_t x_t + E_t a_t + f_t + u_t | y^t,a^{t-1}] = D_t x_{t|t} + E_t a_t + f_t$$

$$\begin{aligned}\text{cov}[x_{t+1};x_{t+1}|y^t,a^{t-1}] &= E[(D_t x_t - D_t E[x_t|y^t,a^{t-1}] + u_t)\\ &\quad (D_t x_t - D_t E[x_t|y^t,a^{t-1}] + u_t)'|y^t,a^{t-1}]\\ &= D_t X_{t|t} D_t' + U_t\end{aligned}$$

where the results follow because a_t is a function of y^t and u_t is independent of x_t, y^t and a^{t-1}.

To prove (14) and (15) let:

$$w_t = y_t - H_t(E_{t-1}a_{t-1} + E[d_{t-1}]) = H_t g_{t-1} + v_t \tag{36}$$

where $g_{t-1} = d_{t-1} - E[d_{t-1}]$, t = 1,2,... Using Bayes' Theorem:

$$P(g_{t-1}|w_t) = \frac{P(w_t|g_{t-1})P(g_{t-1})}{P(w_t)} \tag{37}$$

where $P(w_t|g_{t-1}) \sim N(H_t g_{t-1}; V_t)$, $g_{t-1} \sim N(0, \Delta_{t-1})$, and $w_t \sim N(0; H_t \Delta_{t-1} H_t' + V_t)$. Substituting in (37), $p(g_{t-1}|w_t$ = constant. $\exp\{W\}$ where:

$$W = (w_t - H_t g_{t-1})' V_t^{-1}(w_t - H_t g_{t-1}) + g_{t-1}' \Delta_{t-1}^{-1} g_{t-1} - w_t'(H_t \Delta_{t-1} H_t' + V_t)^{-1} w_t.$$

Temporarily dropping the subscripts and using the matrix identity:

$$(H\Delta H' + V)^{-1} = V^{-1} - V^{-1}H(\Delta^{-1} + H'V^{-1}H)^{-1}H'V^{-1}, \tag{38}$$

$$W = w'V^{-1}w + g'(\Delta^{-1} + H'V^{-1}H)g - g'H'V^{-1}w - w'v^{-1}Hg$$

$$-w'V^{-1}w + w'V^{-1}H(\Delta^{-1} + H'V^{-1}H)^{-1}H'V^{-1}w$$

$$= (g - (\Delta^{-1} + H'V^{-1}H)^{-1}H'V^{-1}w)'(\Delta^{-1} + H'V^{-1}H)(g - (\Delta^{-1} + H'V^{-1}H)^{-1}H'V^{-1}w)$$

Thus, $E[g_{t-1}|w_t] = (\Delta_{t-1}^{-1} + H_t'V_t^{-1}H_t)'^{-1}H_t'V_t^{-1}w_t$

$$= K_t^b(y_t - H_t(E_{t-1}a_{t-1} + E[d_{t-1}]))$$

where the second equality is obtained from (16), (36) and the matrix identity:

$$\Delta H'(H\Delta H' + V)^{-1} = (\Delta^{-1} + H'V^{-1}H)H'V^{-1}. \tag{39}$$

Also using (16) and (38):

$$\mathrm{cov}[g_{t-1}, g_{t-1}|w_t] = (\Delta_{t-1}^{-1} + H_t' V_t^{-1} H_t)^{-1} = \Delta_{t-1} - K_t^b H_t \Delta_{t-1} \tag{40}$$

It then follows from the independence assumptions and (36) that:

$$E[d_{t-1}|y^t, a^{t-1}] = E[d_{t-1}|y_t, a_{t-1}] \tag{41}$$

$$= E[g_{t-1}|w_t] + E[d_{t-1}]$$

$$= K_t^b(y_t - H_t(E_{t-1}a_{t-1} + E[d_{t-1}])) + E[d_{t-1}]$$

Equation (14) can now be obtained:

$$x_{t|t} = D_{t-1} E[x_{t-1}|y^t, a^{t-1}] + E_{t-1}a_{t-1} + E[d_{t-1} + p_{t-1}|y^t, a^{t-1}]$$

$$= x_{t|t-1} + K_t^b(y_t - H_t(E_{t-1}a_{t-1} + E[d_{t-1}]))$$

where the second equality follows from (11) and (41), because y_t contains no new information about x_{t-1} and because p_{t-1} is independent of y^t and a^{t-1}.

From (2) and (14):

$$x_t - E[x_t|y^t, a^{t-1}] = D_{t-1}x_{t-1} + E_{t-1}a_{t-1} + d_{t-1} + p_{t-1}$$

$$-D_{t-1}x_{t-1|t-1} - E_{t-1}a_{t-1} - E[d_{t-1}] - E[p_{t-1}] - K_t^b(y_t - H_t(E_{t-1}a_{t-1} + E[d_{t-1}]))$$

Hence collecting terms and using (6), (12) and (16):

$$X_{t|t} = E[(x_t - E[x_t|y^t,a^{t-1}])(x_t - E[x_t|y^t,a^{t-1}])'|y^t,a^{t-1}]$$

$$= D_{t-1}X_{t-1|t-1}D'_{t-1} + (I - K_t^b H_t)\Delta_{t-1}(I - K_t^b H_t)' + \Pi_{t-1} + K_t^b V_t K_t^{b\prime}$$

$$= X_{t|t-1} - K_t^b H_t \Delta_{t-1}$$

which proves (15). //

Proof of Theorem 4

From (18), $x_{t|t} = D_{t-1}x_{t-1|t-1} + E[p_{t-1}] + E_{t-1}a_{t-1} + d_{t-1} + v_t^b$

and using (2):

$$x_t - x_{t|t} = D_{t-1}(x_{t-1} - x_{t-1|t-1}) + p_{t-1} - E[p_{t-1}] - v_t^b$$

Thus

$$X_{t|t} = E[(x_t - x_{t|t})(x_t - x_{t|t})'|y^t,a^{t-1}]$$

$$= D_{t-1}X_{t-1|t-1}D'_{t-1} + \Pi_{t-1} + V_t^b$$

where the second equality follows from the independence of p_{t-1}, v_t^b and x_{t-1}. //

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