Discussion Paper #242

On Group Manipulability of Voting Procedures

by

Pierre Batteau and Jean-Marie Blin

September 1976

On Group Manipulability of Voting Procedures

by

Pierre Batteau and Jean-Marie Blin

I. Introduction

Organizations operate by a set of explicit or implicit rules known to and, possibly, understood by their members. Understanding the "rules of the game" can be profitable, for a clever agent may manipulate the actions of the organization without trespassing the legal boundaries defined by the rules. This possibility raises the following question: to what extent can the decision process be so designed as to make such manipulated outcomes impossible. While coercive solutions may appear promising, a moment's reflection suggests that they ignore the basic privacy of individual decision-making. As long as individual true motives remain inaccessible to outside observers, the search for incentive compatible decision processes will remain a fundamental issue. A number of recent contributions have focused on the incentive structure of organizations. [5], [10], [2]. In group choice theory, in particular, this research has led to an impossibility result: a theorem due to A. Gribbard and M. Satterthwaite shows that, in general, only dictatorial voting procedures are immune to individual strategic manipulations. As is well known, they pose the question of existence of voting rules such that no individual will ever find it advantageous to disguise his true preferences in his vote to secure a more favorable outcome. Alternatively in game theoretic terms they inquire about the existence of Nash equilibria in the set of sincere preferences (strategies). Their result states that only dictatorial procedures meet the individual strategy-proofness test, for a society of at least two individuals and three alternatives. In this context, however, it may seem that restricting the strategy-proofness issue to individuals is unnecessary; and one may pose the general question of existence of strong Nash equilibria in sincere strategies. As it turns out, of course, the answer is similar: dictatorial procedures being the only group - and individual - strategy-proof procedures. But as we shall show, the group strategy-proofness formulation leads to a unified view of the problem and a straightforward proof of this result. In fact, the structure we develop for this problem resembles the proofs of Arrow's theorem given by P. Fishburn [4] or A. Kirman and Sonderman [9]. Our approach is suggested by the fact that the Gibbard-Satterthwaite's result shows the existence of a dictator - as does Arrow's theorem. Thus, as we show, an ultrafilter must be defined over the set of subsets of voters by the strategy-proofness requirement. Specifically, we show that the strategy-proofness requirement induces a set of properties on the 'winning' subsets of voters. These properties characterize an ultrafilter over the set of subsets of voters, which implies the existence of a privileged voter - the basis of that ultrafilter - the dictator. This is formally established in Section III. Before proceeding with this proof, the basic concepts, and notations are defined in Section II. And, in a concluding section, we point out how a parallel proof can be constructed for individual strategy-proofness - i.e. the original Gibbard-Satterthwaite result - and discuss some implications of our approach. Finally it is interesting to note the formal connection between the proofs of the two impossibility results in light of the recent work by E. Kalai and E. Muller [8], and E. Maskin [10] establishing the exact relationship between Arrow type social welfare function and strategy-proof voting procedures.

II. Statement of the problem, Definitions and Notations

2.1. We consider a finite set N of n voters $(i=1,\ldots,n)$ and a set A of m proposals (candidates, motions, social states, etc.); m is assumed to be no less than three; $m \geqq 3$. A group decision involves the choice through some well-defined procedure,of a single proposal $a_j \in A$. A voting procedure picks a_j by combining individual votes over A. Each individual can express his preference through a complete strict order $P \in \mathcal{P}$(1) the set of all permutations on A. An individual vote amounts to stating some ranking P_i^o. A priori, this revealed ranking P_i^o may or may not coincide with the true feelings of Mr. i about the relative merits of the proposals. These sincere, but inaccessible, feelings about A are also represented by a strict ordering denoted P_i^*. In general, for any number of self-interested voters who understand the working of the voting procedure, there is no basis for assuming that $P_i^o = P_i^*$ for all i. Some group I of voters may find it in their best interest to reveal $P_i^o \neq P_i^* (i \in I)$. The strategy proofness issue inquires about the existence of voting procedures for which individual incentives always guarantee that the revealed opinion profile $P^o = (P_1^o, P_2^o, \ldots, P_n^o)$ is equal to $(P_1^*, P_2^*, \ldots, P_n^*) = P^*$. To guide their decisions in choosing P_i^o, each voter must know not only the voting procedure to be followed but also some, possibly all, of the other voters' true preferences.

---

(1) We limit ourselves to strict ordering for expositional simplicity only. The result do obtain with weak orderings as well as Satterthwaite [12] has shown for individual strategy-proofness.

2.2 Formally a voting procedure is a single valued mapping f: $\prod_{i=1}^{n} \theta_i \rightarrow A$.

Given some f and some sincere preference profile $P^* = (P_1^* \ldots P_n^*) \in \prod \theta_i^*$, we have a voting game (or game for m [5]) denoted (f, P^*) as originally defined by Farqugarson with individual strategies $P_i^o \in P_i$. For each strategy n-tuple P^o, the outcome $f(P^o)$ is a proposal in A and the relative desirability of these outcomes are ordinally defined by the sincere preference n-tuple P^*.

Definition: A voting procedure f is group stragegy-proof (or group non-manipulable) if and only if for all $(P_1^* \ldots P_n^*) \in \prod \theta_i$, the profile $(P_1^* \ldots P_n^*)$ is a strong Nash equilibrium of the game (f, P^*).

Individual strategy-proofness simply requires the existence of Nash equilibria in sincere strategies. Two well known examples of non manipulable voting procedures are:

(1) the dictatorial procedure in which there exists some privileged voter, the dictator $d \in N$ such that $f(P^o) = \text{Max } P_d^o$ for all profiles P^o

(2) the imposed procedure in which $f(P^o)$ = constant for all profiles P^o

Both of these prodedures are clearly group (and individual) non manipulable. A less trivial case would occur if we limit the image $f(\prod \theta_i)$ to a pair say (x,y) in A. For in such a case a vote for x is also a vote against y, and vice versa; and the rank of all other proposals in P^o is simply ignored; it does not affect the outcome. To rule out this case we assume in the sequel that $|f(\prod \theta_i)| > 2$ - where $| \quad |$ denotes the cardinal number of the image set $f(\prod \theta_i)$.

III. Impossibility theorem for group non manipulable voting procedures.

Under these assumptions we not establish the following theorem.

Theorem: Any group non manipulable voting procedure is dictatorial.

To prove this result we introduce the notion of a group $I \subset N$ of voters as a determining group for x (The reason for this name becomes clear later). It is defined as the set of all voters $i \in I$ who have not ranked x last in their revealed ordering P_i^o. We denote $I_x(P^o)$

(1) $I_x(P^o) = I_{xy}(P^o) \cup I_{xz}(P^o) \cup I_x \quad (P^o)$...etc.

where

(2) $I_{ab}(P^o) = \{ i \in N | a \, P_i^o \, b \} \quad \forall (a,b) \in A \times A$

We also denote P_I^o the restriction of P^o to group $I \subset N$:

(3) $P_I^o = \{ P_i^o | P_i^o \in P^o \text{ and } i \in I \}$

The proof consists in considering the families of determining groups for each proposal in A. We first show they are the same for all proposals; and then show that this single family is nothing but an ultrafilter on the set of subsets of N, $\{2^N\}$, which implies the existence of a singleton (the dictator) as a basis for that ultrafilter.

We note first that ΠP_i is partitioned into equivalence classes by the equivalence relation $P_I^o \overset{I}{\longleftrightarrow} P_I^{o\prime}$ defined as the set of all profiles with $P_{\bar{I}}^o$ fixed. We say that P_I^o is 'accessible' to group I whenever $P_{\bar{I}}^o = P_{\bar{I}}^{o\prime}$. (2)

Without loss of generality, let $A = \{x,y,z\}$. A necessary condition for group non-manipulability is now derived. Informally, whenever x wins for some profile P^o then it must still win for any other profile $P^{o\prime}$ if f is to be group strategy-proof. If this is the case, clearly their revealing a different ranking for x cannot make x lose.

Lemma 1: If f is group non manipulable then

$$[x = f(P^o), I_x(P^o) = I, J \subset \bar{I}, P^{o\prime} \overset{J}{\longleftrightarrow} P^o] \Rightarrow [x = f(P^{o\prime})$$

The proof is immediate

---

(2) $\bar{I}$ denotes the complement of I in N

Proof: By assumption $yP_j^o x$ and $zP_j^o x$ $\forall j \in J$. Letting $P^* = P^o$, if there exists $P^{o\prime}$ accessible to J from P^o such that $x \neq f(P^{o\prime})$ then $P^{o\prime}$ is preferable to P^o for J; and thus P^* could not be a strong Nash point. A contrario, we must have $x = f(P^{o\prime})$.

Q.E.D.

In such a case all voters $i \in I$ by revealing P_I^o can, in effect, impose x as the winner whatever the choice of all

see the rationale for calling the group I determining for x. Formally, this can be restated:

Definition: $I \subset N$ is determining for $x \in A$ if and only if $\exists\, P^o \ni f(P^o) = x$ and $I_x(P^o) = I$ (3)

Let F_x, F_y, F_z denote the families of determining sets for x, y and z respectively. These families are characterized by the following properties:

Property 1: $[I \in F_x \quad I' \supset I] \Rightarrow [I' \in F_x]$
(similarly for F_y and F_z)

Proof: It follows directly from Lemma 1 if we consider two subsets $I \in F_x$, $I' \supset I$. Then $(I' - I) \supset \bar{I}$ and by changing the profiles in $(I' - I)$ to get $P^{o\prime}$ so that $I_x(P^{o\prime}) = I \cup (I' - I) = I'$ we must still have $f(P^{o\prime}) = x$ by Lemma 1. Thus I' is determining for x: $I' \in F_x$

Property 2: $\phi \notin F_x$

---

(3) Note, that this is, not the same thing as Arrow's decisive sets although it plays an analogous role in the proof. The essential difference stems from the fact that a determining group (set) is defined for a single alternative - as f maps into A; whereas Arrow's decisive sets are defined for a pair as he is concerned with social welfare functions i.e. f maps into P the set of rankings.

Proof: Assume $\phi \in F_x$, $\exists P^o \ni f(P^o) = x$ and $I_x = \phi$ i.e. all voters rank the winner x last in their revealed profile P^o. Now let $P^o = P^*$. By our assumption we have $\left| f(\prod_{i=1}^{n} P_i) \right| > 2$; so $\exists P^{o\prime} \ni f(P^{o\prime}) = y$. Thus all prefer $P^{o\prime}$ to P^o which means that P^* cannot be a strong Nash equilibrium. A contrario we must have $\phi \notin F_x$.

Property 3: If f is group non manipulable then

$$[I_1 \in F_x, I_2 \in F_y, I_3 \in F_z] \Rightarrow [I_1 \cap I_2 \cap I_3 \neq \phi]$$

Proof:

(i) Consider $I_1 \in F_x$ $I_2 \in F_y$ and assume $I_1 \cap I_2 = \phi$. Then by property 1 there exists $I_1' \supset I_1$ and $I_2' \supset I_2$ partitioning N so that I_1' is determining for x and I_2' determining for y. Now let $P^o = (P^o_{I_1'} ; P^o_{I_2'})$ with $P^o_{I_1'} = x\ y\ z$ and $P^o_{I_2'} = y\ z\ x$. And let $P^o = P^*$

- either x wins but then I_2' can impose y by casting an insincere ballot
- or y wins but then I_1' can impose x
- or z wins but then I_1' can impose x or I_2' can impose y.

In any case P^o is not a strong Nash equilibrium and we must conclude $I_1 \cap I_2 \neq o$.

(ii) Now consider $I_3 \in F_z$ and assume $I_3 \cap (I_2 \cap I_1) = \phi$. That is $I_3 \subset (\overline{I_1 \cap I_2})$ But then by property 1 $(\overline{I_1 \cap I_2}) \in F_z$. If we then set $P^* = P^o$ and pick P^o as follows

$$\left.\begin{array}{l} -\ x\ P_{I_1}\ y \\ -\ y\ P_{I_2}\ z \end{array}\right\} \Rightarrow x\ P_{I_1 \cap I_2}\ z \text{ by transitivity}$$

- $z\ P_{\overline{I_1 \cap I_2}}\ x$

Now if x wins, $I_1 \cap I_2$ which prefers z is determining for z

if y wins I_1 which prefers x is determining for x

if z wins I_2 which prefers y is determining for y.

We conclude that P^* cannot be a strong Nash equilibrium i.e. f is group-manipulable. A contrario we must have $I_3 \cap (I_2 \cap I_1) \neq \emptyset$.

Q.E.D.

Thus any time we take any three groups I_1, I_2, I_3 in F_x F_y and F_z respectively they must have a common intersection. Then, there must exist some group D contained in any one of the groups of any one of these families. Furthermore, and in view of property 1, $F_x = F_y = F_z = F$ with $D \in F$. It follows immediately:

Property 4: $I \in F \Rightarrow \bar{I} \notin F$ for any two determining coalitions have a nonempty intersection (property 3) and one of the two has to be determining.

We can now conclude: as we know properties 1,2,3 and 4 characterize an ultra-filter F on 2^N. And we also know that on the finite set N, this ultra filter has a singleton as a basis. This singleton is the dictator. This completes the proof of the theorem: in general any group non manipulable voting procedure is a dictatorial procedure.

III. Some remarks on individual non manipulability of voting procedures

3.1 As stated previously, the original result of A. Gibbard and M. Satterthwaite dealt with the more restricted issue of individual strategy proofness. Our method of proof can be readily transposed to deal with this case. For brevity we simply outline the general strategy of this proof. In the case of individual non manipulability (simple rather than strong Nash equilibria) coalitions are not allowed. Thus, rather than considering the group of voters who do not rank x last in P^o, $I_x(P^o)$, we consider

$$I^x(P^o) = I_{yx}(P^o) \cup I_{zx}(P^o) \ldots$$

i.e. the set of voters who do not rank x first in P^o. Then $\bar{I}^x(P^o)$ ranks x first in P^o. We now restate Lemma 1 as

Lemma 1$'$: If f is individual non manipulable then

$$[f(P^o) \neq x,\ I^x(P^o) = I \quad i \in \bar{I},\ P^{o'} \overset{i}{\longleftrightarrow} P^o] \Rightarrow [f(P^{o'}) \neq x]$$

The proof is analogous to that of Lemma 1. Using Lemma 1$'$ with $I^x(P^o) = I \cup \{i\}$ $i \in \bar{I}$, we conclude $f(P^{o'}) \neq x \quad \forall J \subset \bar{I}$. In other words for any profile $P^{o'}$ where $P_I^{o'} = P_I^{o}$, x cannot win; we can then say that the I set is "blocking for x" (rather than determining for x as before). By considering three families B_x, B_y, B_z of blocking sets for x, y and z respectively we can establish properties 1 and 2. Similarly property 3 is derived by considering the same profile P^o as before and property 4 follows immediately. Again, we conclude that $B_x = B_y = B_z = B$ an ultrafilter on 2^N. Its basis is a singleton, the dictator who can block any alternative in A.

3.2. The approach followed in this paper parallels the approach used in the direct proof of Arrow's theorem via an ultrafilter argument. Given the conclusion of both theorems - existence of a dictator - this similarity is not surprising. The essential difference is the way we define the winning sets in 2^N: here a set is determining for some alternative x over all others whereas in Arrow's theorem a decisive set can impose the outcome on a pair.

References

[1] Arrow, K.J., Social Choice and Individual Values, 2nd Ed., J.Wiley and Sons, N.Y., 1962.

[2] Batteau, P., and J.M. Blin, "Sur le Problème des Procèdures de Scrutin Garantissant une Expression Sincère des Opinions", Mathèmatiques et Sciences Humaines, No. 58, Summer 1976.

[3] Farquharson, R., Theory of Voting, Yale University Press, New Haven, 1969.

[4] Fishburn, P., "Arrow's Impossibility Theorem: Concise Proof and Infinite Voters", Journal of Economic Theory, 2, 1974, 103-106.

[5] Gibbard, A., "Manipulation of Voting Schemes: A General Result", Econometrica, 41, 587-602.

[6] Groves, T. and J. Ledyard, "Optimal Allocation of Public Goods: A solution to the 'Free Rider Problem'", Econometrica, forthcoming.

[7] Guilbaud, G.Th., "Les Théories de l'Interêt General et le Problème Logique de l'Agrégation", Economie Appliqueé, 1952, 501-584.

[8] Kalai, E., and E. Muller, "Characterization of Domains Admitting Nondictatorial Social Welfare Functions and Non manipulable Voting Procedures", 1976 Math Center paper #234.

[9] Kirman, A.P., and D. Sondermann, "Arrow's Theorem, Many Agents and Invisible Dictators", Journal of Economic Theory, October 1972, 267-77

[10] Maskin, E., "Social Welfare Functions on Restricted Domains" and "On Strategy-Proofness and Social Welfare Functions when Preferences are Restricted", working paper Harvard University and Darwin College, Cambridge.

[11] Roberts, D.J., and A. Postlewaite, "The Incentives for Price-Taking in Large Exchange Economies", Econometrica, 1976, 44, 1, 115-127.

[12] Satterthwaite, M.A., "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions", Journal of Economic Theory, 10, 187-217.