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ON THE RELATIONSHIP BETWEEN COMPLETE AND INCOMPLETE FINANCIAL MARKET MODELS

by

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ABSTRACT

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Incomplete Financial Market Models

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A number of recent papers have dealt with firms whose securities are traded in an incomplete financial market and the results obtained are similar to those in an Arrow-Debreu complete markets model in the sense that shareholders either unanimously support or reject any production proposed by a firm. This paper considers the reformulation of an incomplete market model proposed by Radner and examines the sense in which that reformulation is equivalent to the models of Leland, Ekern, and Wilson.

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I. Introduction

A number of recent papers[1] have dealt with models of firms under uncertainty in which the securities of the firms are traded in an incomplete security market. Although the securities market is incomplete in the sense that the number of independent securities is less than the number of states of nature, the results obtained are similar to those derived in an Arrow-Debreu [1][6] complete market model. In these incomplete market models shareholders unanimously support or oppose any proposed production plan, and given a "price-taking" assumption, all shareholders prefer that a firm maximize its market value. In spite of the similarity in results, the relationship between the complete and incomplete market models must be interpreted with care.

Incomplete financial market models have been analyzed under two different classes of assumptions. Leland [16][17], Ekern and Wilson [13], and Ekern [10][11][12] have analyzed firm behavior under the assumption that investors hold optimal portfolios for the currently proposed production plans of firm. Then, if the vector of marginal returns for a firm is contained in the subspace spanned by the return vectors for all securities traded in the market, a proposed change in the production plan of a firm is unanimously supported or rejected by all shareholders. Leland [18] has referred to this as <u>ex post</u> unanimity in the sense that the portfolios held by shareholders are optimal for the currently proposed production plans. Ex post unanimity, however, does not answer the question of how investors arrived at a position in which the optimal portfolios are held.

In a manner analogous to an Arrow-Debreu model Radner [22] considered the case in which shareholders do not hold optimal portfolios for the currently proposed production plans of firms. He then demonstrates that all ex ante or initial shareholders prefer the firm to maximize its value and also unanimously support or reject any proposed production plan if the consumption sets of consumers and production sets of firms are contained in the subspace spanned by the return vectors available in the securities market. Radner states that "The situation described in the first Ekern-Wilson proposition[1] on unanimity of stockholders' preferences among production plans[2] can be formally reduced to an Arrow-Debreu model with complete markets." Radner's argument that an incomplete financial market equilibrium reduces to an Arrow-Debreu equilibrium rests on the assumption that consumption sets are spanned by the return vectors of securities, but such an assumption has no apparent justification in an incomplete market model. The standard interpretation of a consumption set is as the commodity bundles that are physically possible for consumption (Debreu [6,p.57]). The assumption that the consumption set is spanned is not required for unanimity and value maximization results, and its only role is to render an incomplete financial market model equilvalent to an Arrow-Debreu model. Since no contribution is made by this assumption, it will not be employed here. The assumption that the production sets are spanned is more natural and is satisfied in the original work of Diamond [7] and in the above-mentioned articles. This condition permits consumers to use the stock market values of firms to evaluate production plans, and unaminity can be then be demonstrated.

This paper is concerned with the relationships among these results and argues that:

1) An Arrow-Debreu equilibrium is not, in general, an equilibrium of the corresponding incomplete financial market model unless, for example, the consumption sets of consumers are spanned by the return vectors of firms.

2) An incomplete financial market equilibrium is not, in general, an equilibrium of an Arrow-Debreu model.

3) The difference between an Arrow-Debreu model and an incomplete financial market model with spanning is that in the latter model consumers do not have sufficient opportunities to trade to enable them to equate their marginal rates of substitution (or implicit prices), and hence, firm-specific price vectors are required to guide firms in the selection of production plans.

4) As in a complete market model *ex ante* unanimity results and all initial shareholders prefer that the firm maximize its net market value.

5) Radner's *ex post* unanimity analysis is equivalent to the maximization of the market value of the firm because he assumes that consumers use their implicit prices determined

at a portfolio equilibrium to forecast the changes in the market value resulting from a change in a production plan. The analysis of Ekern, Ekern and Wilson and Leland differs from that of Radner in that their *ex post* unanimity is not equivalent to market value maximization.

II. The Model

A. Notation

To analyze the complete and incomplete market models, the notation used by Radner will be followed. There are two dates (1 and 2) and one commodity that may be consumed at each date. The notation is:

I consumers, $i=1,\dots,I$

J firms, $j=1,\dots,J$

K states of the world at date 2, $k=1,\dots,K$

$S \subset \mathbb{R}^K$ is the set of "state-distributions of returns" at date 2 provided by the production plans of firms[2]

$S' = \mathbb{R} \times S$ is the set of returns at dates 1 and 2

$y_j = (y_{jo},(y_{jk}))$ is a production plan for firm j and $Y_j \subset S'$ is the production set for firm j

$r_{jk} = y_{jk}$, $k=1,\dots,K$, is the return in state k for firm j, R is the KxJ matrix of returns at time 2 for the firms, and R' is the K+1xJ matrix incorporating the input at time 1

$c_i = (c_{io},(c_{ik}))$ is the consumption of the i^{th} consumer and C_i is the consumption set

$w_i = (e_i,(0))$ is i's initial endowment of the commodity

$\bar{s}^i_j$ is i's initial endowment of shares of firm j

s_{ij} is the share of firm j purchased by consumer i

$z_{ik} = \sum_j s_{ij} r_{jk}$ is i's portfolio return in state k

$q' = (q_o, (q_k))$ is an equilibrium price vector in the Arrow-Debreu model, $q' \in \mathbb{R}^{K+1}$, $q = (q_k) \in \mathbb{R}^K$

$p_j^* =$ equilibrium stock market value of firm j's output (or return) (y_{jk})

$V_i(c_{io}, c_{i1}, \ldots, c_{iK})$ consumer i's strictly increasing, strictly quasiconcave, continuously differentiable utility function.[3]

B. A Complete Market Model

With a complete set of contingent claims markets the consumer has the program

$$\max_{(c_{ik}),(s_{ij})} V_i(c_{io}, c_{i1}, \ldots, c_{iK})$$

$$\text{S.T.} \quad q'c_i \leq q'w_i + \sum_j \bar{s}_j^i (y_{jo} + p_j), \; c_i \in C_i \qquad (1)$$

An Arrow-Debreu equilibrium is a consumption plan $c_i^* \in C_i$ for each consumer, a production plan $y_j^* \in Y_j$ for each firm, and a price vector $(q_o, (q_k))$ such that c_i^* solves (1), y_j^* maximizes the net market value $(y_{jo} + p_j^*) = q_o y_{jo} + \sum_{k=1}^{K} q_k y_{jk}^*$, and supply equals demand in each state i.e. $\sum_i c_{ik}^* = \sum_j y_{jk}^* + \sum_i w_{ik}$, $k=0,1,\ldots,K$. A central result of this model is that each consumer equates his marginal rate of substitution or implicit price $\rho_{ik}^* \equiv \left(\partial V_i / \partial c_{ik}\right) \big/ \left(\partial V_i / \partial c_{io}\right)$ between a dollar received in state k and a dollar received at date one to the market price q_k (q_o normalized to equal 1). It is evident from the budget constraint that all <u>ex ante</u> shareholders ($\bar{s}_j^i > 0$) prefer that the firm maximize its net market value, so shareholder unanimity is attained.

C. An Incomplete Financial Market Model

In an incomplete financial market model a consumer can generate consumption plans only through the purchase of securities. Consequently, consumption is $c_{ik}=z_{ik}$, where the portfolio return z_{ik} in state k is given by $z_{ik} = \sum_j s_{ij} r_{jk}$. The incomplete financial market model may be formulated as

$$\max_{c_{io},(s_{ij})} V_i(c_{io}, z_{i1}, \ldots, z_{iK})$$

$$\text{S.T.} \quad c_{io} + \sum_j s_{ij} p_j \leq e_i + \sum_j \bar{s}_j^i (y_{jo} + p_j)$$

$$z_{ik} = \sum_j s_{ij} r_{jk}, \quad k=1,\ldots,K$$

$$(c_{io}, (z_{ik})) \in C_i \tag{2}$$

Substituting z_{ik} into the utility function, the necessary optimality conditions, given return vectors (r_j^*), are

$$\frac{\partial V_i}{\partial c_{io}} - \lambda_i^+ = 0 \tag{3}$$

$$\sum_k \frac{\partial V_i}{\partial z_{ik}} r_{jk}^* - \lambda_i^+ p_j^+ = 0, \quad j=1,\ldots,J, \tag{4}$$

where λ_i^+ is the Lagrangian multiplier and p_j^+ is the market value of the output of firm j in the incomplete market model. The market value of the firm's output is thus

$$p_j^+ = \sum_k \rho_{ik}^+ r_{jk}^*, \tag{5}$$

where

$$\rho^{+}_{io} \equiv 1 \text{ and } \rho^{+}_{ik} \equiv \frac{\partial V_i}{\partial z_{ik}} \bigg/ \frac{\partial V_i}{\partial c_{io}}, \ k=1,\ldots,K, \tag{6}$$

is consumer i's implicit price for a dollar of return in state k. Since p^{+}_j is a market price that is the same for all consumers, each consumer imputes the same value to the firm even though their implicit prices for a state k may differ.

In order to relate the Arrow-Debreu model to the Ekern-Wilson model, Radner states that "given the plans $y^*_j = (y^*_{jo}, r^*_j)$," "$\{(c_{io}, [s^*_{ij}]), (p^*_j)\}$ is an Ekern-Wilson equilibrium" where $(c_{io}, (s^*_{ij}))$ solves (2) and $p^*_j = \sum_k q_k r^*_{jk}$. An Ekern-Wilson equilibrium is not however a complete specification of an equilibrium of the incomplete market model because the firm's production plans are taken as given. A full incomplete market equilibrium involves both the choice of portfolios by consumers and the choice of production plans by firms, so an equilibrium concept encompassing both choice problems is needed. The equilibrium concept that will be employed here is that used by Drèze [8] who defines a stockholders' equilibrium as a Lindahl equilibrium for firms and a portfolio equilibrium for the securities market.[4] A portfolio equilibrium is what Radner had referred to as an Ekern-Wilson equilibrium. More specifically, a stockholders' equilibrium is a vector

$$((c^+_{io}),\ (z^+_{ik}),\ (y^+_j = (y^+_{jo}, r^+_j)),\ (s^+_{ij}),\ (\rho^+_i), (p^+_j))$$

such that

a) $\rho^+_i \in S'$ for all i where ρ^+_i is defined in (6) given (y^+_j)

b) for all i $(c^+_{io}, (s^+_{ij}))$ solves program (2) and $(z^+_{ik}) = (\sum_j s^+_{ij} y^+_{jk})$

c) for all j $\sum_i s^+_{ij} = 1$

d) for all j $y^+_j \in Y_j$ maximizes $\sum_k y_{jk} q^+_{jk} + y_{jo}$

where $q^+_{jk} = \sum_i s^+_{ij} \rho^+_{ik}$

e) supply equals demand,[5] i.e., $\sum_i z^+_{ik} = \sum_j y^+_{jk}$, k=1,...,K, and

$\sum_i c^+_{io} = \sum_i e_i + \sum_j y^+_{jo}$

Drèze has given conditions under which a stockholders' equilibrium exists, demonstrates that every constrained Pareto optimum is a stockholder's equilibrium, and that the equilibrium production plans are efficient.

In general, a stockholder's equilibrium in an incomplete financial market model will not be the same as an Arrow-Debreu equilibrium because there are insufficient markets to permit the trades necessary to equate the marginal rates of substitution among consumers. The marginal rates of substitution are thus not equated to a price vector q′ that can be used to evaluate the production plans of a firm. Instead, a firm-specific "market aggregate price" vector (q^+_{jk}) in (d) is used to evaluate firm j's production plans.

In spite of these differences, the principal results that *ex ante* shareholders are unanimous in their evaluations of production plans and in their preference for value maximization still obtain.[6] Prior to demonstrating these results, three special cases in which an incomplete financial market equilibrium is an Arrow-Debreu equilibrium will be considered.

Radner considers the case in which the consumption sets of consumers are spanned by the return vectors of firms or $C_i \subset S' = \mathbb{R} x S$. In this case, the incomplete financial market model may be formulated to show its equivalence to an Arrow-Debreu model. Since consumption z_{ik} at time 2 is given by $z_{ik} = \sum_j s_{ij} r_{jk}$, for the case in which the (r_j^+) are linearly independent[7] let M denote the JxK matrix that takes z_i into s_i. Since $s_i = Mz_i = MR\overset{+}{s}_i$, $MR^+ = I$. Then, the budget constraint in (2) may be rewritten as

$$c_{io} + \sum_j \sum_k m_{jk} z_{ik} p_j \leq e_i + \sum_j \bar{s}_j^i (y_{jo} + p_j)$$

or

$$c_{io} + \sum_k q_k z_{ik} \leq e_i + \sum_j \bar{s}_j^i (y_{jo} + p_j), \qquad (7)$$

where $q_k \equiv \sum_j m_{jk} p_j$. Maximizing in (2) with respect to $(c_{io}, (z_{ik}))$ and using (3) yields the condition

$$\frac{\partial V_i}{\partial z_{ik}} \bigg/ \frac{\partial V_i}{\partial c_{io}} - q_k = 0.$$

Using (6) indicates that $\overset{+}{\rho}_{ik} = \overset{+}{q}_k = \sum_j m_{jk} \overset{+}{p}_j$ for all i, so all consumers have the same implicit prices for a dollar received in state k. It follows that an incomplete financial market equilibrium is an Arrow-Debreu equilibrium.[8] Given the price vector $q'^{+} = (1, q^+) =$

$(1,q_1^+,\dots,q_K^+)$, the feasible consumption set in the Arrow-Debreu model in (1) is the same as the feasible consumption set an incomplete market model, so $(z_{ik}^+) = (\sum_j s_{ij}^+ \, r_{jk}^+)$ is optimal in the Arrow-Debreu model. Consequently, given (r_j^+) and q'^+, $(c_{io}^+, (z_{ik}^+ = \sum_j s_{ik}^+ \, r_{jk}^+), (r_j^+), q'^+)$ is an Arrow-Debreu equilibrium. Radner has also shown that when the consumption set is spanned an Arrow-Debreu equilibrium is sustainable in an incomplete financial market model in the sense of an Ekern-Wilson equilibrium.[9] As previously stated, however, there is little reason to make the assumption that the consumption sets are spanned.

Another case in which the two models are equivalent is when there are as many firms with linearly independent return vectors as there are states (J=K), since then the entire space $\mathbb{R}^{K+1}$ is spanned. Then, the matrix $M = R^{-1}$, and the argument procedes as above.

A third case in which an incomplete market equilibrium is also an Arrow-Debreu equilibrium is when the utility function V_i can be represented as a separable von Neumann-Morgenstern utility function $V_i = U_{io}(c_{io}) + \sum_{k=1} \pi_{ik} \, U_i(c_{ik})$ where the U_i are quadratic. Mossin [20] has shown that in this case every consumer will hold the same proportion of every risky firm so that $s_{ij}^+ = s_i^+$ for all j.[10] Then, the market aggregate prices q_{jk}^+ are the same for all j, since

$$q_{jk}^+ = \sum_i s_{ij}^+ \rho_{ik}^+ = \sum_i s_i^+ \rho_{ik}^+ = q_k^+ .$$

The vector $(1,(q_k^+))$ then is an equilibrium price vector in an Arrow-Debreu model as previously argued.

III. Shareholder Unanimity and Value Maximization

A. *Ex Ante* Unanimity

A central result of an Arrow-Debreu model is that all *ex ante* shareholders ($\bar{s}^i_j > 0$) prefer that the net market value $(y_{jo} + p_j)$ of a firm be maximized. This result is immediate from (1), since an increase in that value expands the consumer's budget set. In an incomplete financial market model the same result apparently obtains by inspection of (2), but since their implicit prices are not the same, consumers may value proposed production plans differently. If, however, the production sets of firms are spanned by the return vectors R', all consumers will evaluate production plans identically. This follows because for any proposed production plan $(y^o_{ho}, r^o_h) \in Y_h$, there exists numbers α_{jh} such that

$$(y^o_{ho}, r^o_h) = \sum_{j=1}^{J} \alpha_{jh}(y^+_{jo}, r^+_j). \tag{8}$$

Since the proposed production plan does not alter the set of returns available to investors, consumers are assumed to act as price takers with respect to their implicit prices in the sense that they use their implicit prices determined at a portfolio equilibrium, given the plans (y^+_{jo}, r^+_j), to evaluate proposed production plans. A consumer thus forecasts the value p^o_{ih} of the proposed production plan as

$$y^o_{ho} + p^o_{ih} = y^o_{ho} + \sum_k \rho^+_{ik}\, r^o_{hk}, \tag{9}$$

where the subscript i on p^o_{ih} indicates that the forecast is made with consumer i's implicit prices.

Given the spanning property, it may be shown that all consumers forecast the same value for the firm. That is, substituting from (8) into (9) yields

$$y_{ho}^{o} + p_{ih}^{o} = y_{ho}^{o} + \sum_k \rho_{ik}^{+} r_{hk}^{o} = \sum_j \alpha_{jh} y_o^{+} + \sum_k \rho_{ik}^{+} \sum_j \alpha_{jh} r_{jk}^{+}$$

$$= \sum_j \alpha_{jh} (y_{jo}^{+} + \sum_k \rho_{ik}^{+} r_{jk}^{+}) = \sum_j \alpha_{jh}(y_{jo}^{+} + p_j^{+}), \text{ for all } i, \quad (10)$$

where the last equality follows from (5). Since the net market values $(y_{jo}^{+} + p_j^{+})$ are observable, all consumers forecast the same value for the production plan (y_{ho}^{o}, r_h^{o}), so $p_{ih}^{o} = p_{\ell h}^{o}$ for all i and ℓ. All <u>ex ante</u> shareholders are thus unanimous in their preferences for production plans. Leland [18] has shown that spanning in also a necessary condition for unanimity for general preferences, expectations, and initial endowments.

To illustrate the process by which consumers evaluate production changes, consider a variation (g_{ho}, g_h) in the production plan of firm h such that $(y_{ho}^{o}, r_h^{o}) = (y_{ho}^{+}, r_h^{+}) + (g_{ho}, g_h) \in Y_h$. Prior to making a portfolio allocation, a consumer will evaluate the change both in terms of the effect on the returns and the effect on his budget. The evaulation of the variation in return is

$$s_{ih}^{+} \sum_k \rho_{ik}^{+} (g_{ho} + g_{hk}),$$

and the variation in the budget depends on the change in the market value of the firm. Consumers are assumed to forecast this value change using their current implicit prices, so the budget effect is

$$(\bar{s}_h^{i} - s_{ih}^{+})(p_{ih}^{o} + y_{ho}^{o} - p_h^{+} - y_{ho}^{+}) = (\bar{s}_h^{i} - s_{ih}^{+})(\sum_k \rho_{ik}^{+} (g_{ho} + g_{hk})), \quad (11)$$

where s_{ih}^{+} is the share of the firm that the consumer would plan to purchase given a production plan (y^+_{ho}, r^+_h). The value of the variation in return is thus exactly offset by the forecast $(-s^+_{ih} \sum_k \rho^+_{ik}(g_{ho} + g_{hk}))$ in the amount required to purchase the s^+_{ih} share of the firm, since all gains to arbitrage are eliminated at a portfolio equilibrium. A consumer thus evaluates a variation in terms of the change $\bar{s}^i_h \sum_k \rho^+_{ik}(g_{ho} + g_{hk})$ in the value of his endowment. Because of spanning all consumers have the same valuation of this change as indicated in (10), and thus all ex ante shareholders are unanimous in evaluating changes in production plans.

To demonstrate the relationship between a consumer's evaluation of a production plan and the firm-specific market aggregate price vector $(1, (q^+_{hk}))$, note that

$$\begin{aligned} y^o_{ho} + \sum_k q^+_{hk} r^o_{hk} &= y^o_{ho} + \sum_k \sum_i s^+_{ih} \rho^+_{ik} r^o_{hk} \\ &= \sum_j \alpha_{jh} y^+_{jo} + \sum_i s^+_{ih} \sum_k \rho^+_{ik} \sum_j \alpha_{jh} r^+_{jk} \\ &= \sum_j \alpha_{jh} y^+_{jo} + \sum_i s^+_{ih} \sum_j \alpha_{jh} \sum_k \rho^+_{ik} r^+_{jk} \\ &= \sum_j \alpha_{jh}(y^+_{jo} + p^+_j), \end{aligned} \tag{12}$$

which is equivalent to (10). The firm may thus evaluate its production plans using the prices p^+_j established by consumers' trades in the securities market. This evaluation is equivalent to the maximization of the stock market value, since the right side of the first equality equals the market value of the firm.

Mark Satterthwaite has pointed out that while the (q^+_{jk}) vectors will in general differ among firms a firm j may use any of these J vectors to evaluate its production plans. This can be seen by using

$(q^{+}_{\ell k})$ instead of q_{hk} in (12) and is equivalent to observing from (5) that the implicit prices for any consumer can be used to compute the market value of the firm. Consequently, given spanning, each of the vectors (q^{+}_{jk}) conveys equivalent information regarding the valuations that consumers place on returns.

One special case of the spanning condition that has received attention (see Diamond) is when any proposed plan (y^{o}_{ho}, r^{o}_{h}) of firm h is spanned by any return vector (y^{+}_{ho}, r^{+}_{h}) of that same firm.[11] Then, $(y^{o}_{ho}, r^{o}_{h}) = \alpha_h (y^{+}_{ho}, r^{+}_{h})$ and the production possibility set is a ray in $\mathbb{R}^{K+1}$. With the price-taking assumption the proposed plan is evaluated as $p^{o}_{ih} = \alpha_h \, p^{+}_{h}$ using only the market price of the h^{th} firm. When this self-spanning property is satisfied and the price-taking assumption is made, shareholder unanimity regarding the production plans of firm h will obtain even if the production possibility sets or other firms are not spanned by the return vectors of all firms.

If the firm's output is thought of as a local public good,[12] the difference between an Arrow-Debreu equilibrium and an incomplete financial market equilibrium becomes evident. A Lindahl equilibrium in an economy with public goods involves a personalized price (for each public good) for each consumer, and in an incomplete market model the corresponding personalized prices are the implicit prices which may differ among consumers. Firms that produce public goods seek to maximize the value of their output of those public goods using the sum of the personalized prices to value the output (Roberts [23]).

In an incomplete market model the firm maximizes its value using the weighted sum of the implicit prices where the weights are the shareholdings as indicated in (d). In general, these market aggregate prices are different for each firm, but with spanning they each convey the same information regarding consumers' valuations.

B. Radner's *Ex Post* Unanimity

After consumers have had an opportunity to trade shares for a given set of proposed production plans and the input has been made, the preferences of *ex post* shareholders ($s^+_{ij} > 0$) are relevant in guiding the firm. If shareholders use their implicit prices in (6) to evaluate changes in production plans, all *ex post* shareholders can be shown to be unanimous in their preferences regarding local changes in production plans. If $y^+_h + tf_h \in Y_h$, for some t small, is a proposed change in production for firm h and $z^+_i + ts^+_{ih}g_h$ is the corresponding consumption for consumer i at time 2 where $f_h = (g_{ho}, g_h)$, the investor will prefer the change if $\sum_k \rho^+_{ik}(z^+_{ik} + ts^+_{ih}g_{hk}) >$ $\sum_k \rho^+_{ik}z^+_{ik}$, or if

$$ts^+_{ih} \sum_k \rho^+_{ik}g_{hk} > 0.$$

The spanning property implies that there exist (α_{jh}) such that $g_h = \sum_j \alpha_{jh}r^+_j$, i.e., that

$$ts^+_{ih} \sum_k \rho^+_{ik}g_{hk} = ts^+_{ih} \sum_k \rho^+_{ik} \sum_j \alpha_{jh}r^+_{jk} = ts^+_{ih} \sum_j \alpha_{jh} \sum_k \rho^+_{ik}r^+_{jk} = ts^+_{ih} \sum_j \alpha_{jh}p^+_j.$$

All *ex post* shareholders ($s^+_{ij} > 0$) will thus unanimously approve or disapprove the change if $\sum_j \alpha_{jh}p^+_j$ is positive or negative, respectively. As Radner indicates, after the input y^+_{ho} is fixed, *ex post* and *ex ante* shareholders evaluate production plans similarly.

When a consumer uses his current implicit prices to forecast the market value of the firm as in (9), the condition in (13) also implies that all *ex post* shareholders prefer that the market value of the firm be maximized, since the change p_h^t in the market value for the proposed production plan is evaluated as

$$p_h^t = t \sum_k \rho_{ik}^{+} g_{hk}. \tag{14}$$

An *ex ante* shareholder who uses his current implicit prices ρ_i^{+} to evaluate the production plans thus prefers a variation g_h that has the greatest market value.

C. The Market-Value Forecasting Assumption

In the incomplete market model, the price-taking assumption that consumers use their implicit prices established in a portfolio equilibrium to forecast changes in the market value of the firm is necessary for the unanimity and value maximization results as indicated in (11), but this assumption may not be a reasonable one. In a complete market it seems more reasonable that consumers use their implicit prices to forecast changes in the market values of firm, since each implicit price is equal to an observable market price. The argument for the forecasting assumption in an incomplete market model is weaker, however, since implicit prices differ among consumers. The basic argument is that since a proposed production plan does not alter the space of returns available in the market, consumers will be willing to use their portfolio equilibrium implicit prices to predict the change in market values.

But, for example, if the return vectors $r_1^*,\ldots,r_J^*$ are linearly independent and firm h proposes a change from r_h^* to r_h such that r_h is not linearly dependent on $r_1^*,\ldots,r_{h-1}^*$, $r_{h+1}^*,\ldots,r_J^*$, the space of available returns may be different and the forecasting assumption may not be reasonable. Ekern and Wilson and Leland, however, provide an ex post analysis that establishes unanimity when the forecasting assumption is not made.

D. Ekern, Wilson and Leland Ex Post Unanimity

Ekern, Ekern and Wilson, and Leland [17] do not make the price-taking assumption, so consumers are not assumed to use their portfolio equilibrium implicit prices to predict the value of the firm resulting from a proposed change in a production plan. In order to procede further in this case, they make the additional assumption that trades have already taken place in the securities market so that consumers hold an optimal portfolio, given the currently proposed production plans (y_j^+) (with $y_{jo}^+ = 0$). Consequently, consumers will not prefer to trade shares and $\bar{s}_j^i = s_{ij}^+$ for all j, so changes in the value of the firm do not affect the consumer's budget set. From (2) evaluated at a portfolio equilibrium the consumer will prefer a change in the production plan of firm h from r_h^+ to $r_h^+ + tg_h \in Y_h$ if and only if the value of the return for that firm increases, since the budget effect in (11) is zero. The change in the market value of the firm h does not affect the shareholder in this case because any additional cost of purchasing the s_{ih}^+ share is exactly offset by the additional income from the sale of the $\bar{s}_h^i$ share. An ex post shareholder ($s_{ih}^+ > 0$) thus prefers the change if and only if

$$s^{+}_{ih} t \sum_k \rho^{+}_{ik} g_{hk} > 0, \quad t \text{ small}$$

The spanning property may be used as in (10) to show that the sign of this term is the same for all consumers.

While the valuation is the same as that in (13), maximization of the market value of the firm, however, is not implied in the Ekern-Wilson-Leland model because consumers are not assumed to use their current implicit prices as in (13) to forecast the change in the market value of the firm. The forecast of the change Δp^t_h in the market value is given by

$$\Delta p^t_h = t \sum_k \rho^{+}_{ik} g_{hk} + t \sum_k \Delta\rho^{+}_{ik} g_{hk},$$

where $\Delta\rho^t_i$ denotes the change in the implicit prices. As Leland has shown, _ex post_ shareholder unanimity is not in general equivalent to value maximization unless the implicit prices are unchanged. The _ex post_ analysis of Ekern, Wilson, and Leland is thus of a different nature than the _ex post_ analysis of Radner.

Given that shareholders do not prefer that firms maximize their market values, the definition of an incomplete market equilibrium must be revised. One definition of an equilibrium in this case is to use a stockholders' equilibrium with $\bar{s}^i_j = s^{+}_{ij}$ and with condition d) replaced by

d') for all j, y^{+}_j is such that there exists no direction g_j and no $t > 0$, t small, such that $y^{+}_j + tg_j \in Y_j$ and $\sum_k q_{jk} g_{jk} > 0$.

This condition implies that at an equilibrium no firm can find a (small) feasible change in its production plan that will be preferred by its *ex post* shareholders. Spanning assures that all *ex post* shareholders evaluate the production plan identically.

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1. In addition to the papers cited below, the incomplete market methodology has been applied in papers in the study of international trade by Helpman and Razin [15], Baron [3], and Baron and Forsythe [5], in the theory of the firm by Leland [16], Baron and Taggart [4], and Drèze and Hagen [9], and in finance by Baron [2], Hagen [14], Milne [19] and Nielsen [21].

2. See Ekern and Wilson [12,p.175].

3. If preferences satisfy the von Neumann-Morgenstern axioms, the utility function may be expressed as

$$V_i(c_{io},c_{i1},\ldots,c_{iK}) = \sum_{k=1}^{K} \pi_{ik} U_{ik}(c_{io},c_{ik}),$$

where $U_{ik}(c_{io},c_{ik})$ is a state dependent von Neumann-Morgenstern utility function and π_{ik} is a subjective probability.

4. Drèze refers to a Lindahl equilibrium as a pseudo equilibrium and a portfolio equilibrium as a price equilibrium.

5. The condition $\sum_i z^+_{ik} = \sum_j y^+_{jk}$ is implied by c), since $z^+_{ik} = \sum_j s^+_{ij} y^+_{jk}$, and summing over i yields $\sum_i z^+_{ik} = \sum_j (\sum_i s^+_{ij})\, y^+_{jk} = \sum_j y^+_{jk}$.

6. One difference between the two models is that in an incomplete financial market model consumers must know the state distribution of returns for a firm in order to determine their preferred share purchases, while in an Arrow-Debreu model consumers are indifferent to their share purchases.

7. If the (r_j^+) are not linearly independent, the argument procedes in the same manner using the maximal linearly independent subset.

8. The author wishes to thank a referee for this argument.

9. Radner shows that an Arrow-Debreu equilibrium is also an Ekern-Wilson equilibrium in the sense that given the equilibrium outputs $y_j^* = (y_{jo}^*, r_j^*)$ and the stock market value vector (p_j^*), there exists a portfolio (s_{ij}^*) for each consumer such that (s_{ij}^*) solves the program (2) and $z_{ik}^* \equiv \sum_j s_{ij}^* \; r_{jk}^*$ equals c_{ik}^* for all k and i. Given (y_j^*) and the Arrow-Debreu stock market prices (p_j^*), the budget set in (2) is the same as that in (1). Also, because C_i is spanned by R', every feasible consumption vector in the complete market model will be feasible in the incomplete market model. Since c_i^* is feasible and preferred, it is also optimal in the incomplete market model. The consumption c_{ik}^+ of consumer i in state k for any proposed production plan (r_j) will be $\sum_j s_{ij}^+ r_{jk} = c_{ik}^+$. Summing over i and noting that $r_{jk} = y_{jk}$ yields

$$\sum_i \sum_j s_{ij}^+ r_{jk} = \sum_j (\sum_i s_{ij}^+) y_{jk} = \sum_j y_{jk} = \sum_i c_{ik}, \; k=1,\ldots,K,$$

so the markets clear in each state when the (s_{ij}^+) satisfy $\sum_i s_{ij}^+ = 1$ for all j.

10. Ekern and Wilson make the same observation, p.173.

11. Spanning in many of the incomplete market models also requires a risk-free security.

12. The firm is a public "good" to shareholders ($\bar{s}_j^i > 0$) and public "bad" to short-sellers ($\bar{s}_j^i < 0$).

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